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ON THE EXTENDED WRIGHT HYPERGEOMETRIC MATRIX FUNCTION AND ITS PROPERTIES

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Abstract. Recently, Bakhet *et al.* [9] presented the Wright hypergeometric matrix function ${}_2\mathbf{R}_1^{(\tau)}(A,B;C;z)$ and derived several properties. Abdalla [6] has since applied fractional operators to this function. In this paper, with the help of the generalized Pochhammer matrix symbol $(A;B)_n$ and the generalized beta matrix function $\mathcal{B}(P,Q;\mathbb{X})$, we introduce and study an extended form of the Wright hypergeometric matrix function, ${}_2\mathbf{R}_1^{(\tau)}((A;\mathbb{A}),B;C;z;\mathbb{X})$. We establish several potentially useful results for this extended form, such as integral representations and fractional derivatives. We also derive some properties of the corresponding incomplete extended Wright hypergeometric matrix function.

1. Introduction

Let $\mathbb{C}^{r\times r}$ be the vector space of r-square matrices with complex entries. A square matrix $P\in\mathbb{C}^{r\times r}$ is said to be positive stable if $\Re(\lambda)>0$ for all $\lambda\in\sigma(P)$, where $\Re(\lambda)$ denotes the real part of a complex number λ and $\sigma(P)$ is the set of all eigenvalues of P.

Let P and Q be positive stable matrices in $\mathbb{C}^{r\times r}$. The gamma matrix function $\Gamma(P)$ and the beta matrix function $\mathfrak{B}(P,Q)$ were defined by Jódar and Cortés [12] as follows:

$$\Gamma(P)=\int_0^\infty e^{-t}t^{P-I}dt,\qquad t^{P-I}=\exp((P-I)\ln t)$$

and

$$\mathfrak{B}(P,Q)=\int_0^1 t^{P-I}(1-t)^{Q-I}dt, \tag{1}$$

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respectively. The shifted factorial matrix function $(P)_n$ for $P \in \mathbb{C}^{r\times r}$, given by [13], is

$$(P)_n = \begin{cases} I, & n = 0, \\ P(P+I)\dots(P+(n-1)I), & n \geq 1. \end{cases}$$

Let $P \in \mathbb{C}^{r\times r}$, and suppose that

$$P + nI \text{ is invertible for all integers } n, \tag{2}$$

then the reciprocal gamma matrix function [12] is given by

$$\Gamma^{-1}(P) = (P)_n\Gamma^{-1}(P + nI).$$

Over the past two decades, several generalizations of the well-known special matrix functions have been studied by various authors (for example, [5], [7] and [10]). In particular, in 2015, Abul-Dahab *et al.* [7] introduced a generalized Pochhammer matrix symbol $(A;B)_n$. Let A and B be positive stable matrices in $\mathbb{C}^{r\times r}$ that satisfy the condition (2). Then

$$(A;B)_n = \begin{cases} \Gamma^{-1}(A)\Gamma(A+nI,B) & \big(B \neq \mathbf{0}\big) \\ \\ (A)_n & (B \doteq \mathbf{0}), \end{cases} \tag{3}$$

where $\mathbf{0} \in \mathbb{C}^{r\times r}$ is the zero matrix and $\Gamma(A,B)$ is the generalized gamma matrix function given by (see [7])

$$\Gamma(A,B) = \int_0^\infty t^{A-I}\, e^{-\left(It+\frac{B}{t}\right)}\mathrm{d}t,$$

so that the integral representation for the generalized Pochhammer matrix symbol is

$$(A;B)_n = \Gamma^{-1}(A) \int_0^\infty t^{A+(n-1)I}\, e^{-\left(It+\frac{B}{t}\right)}\mathrm{d}t, \tag{4}$$

where B and $A+nI$ are positive stable for all $n \geq 0$. Let A and B be positive stable matrices in $\mathbb{C}^{r\times r}$ that satisfy the condition (2). Then $\Gamma(A,B)$ is invertible; let its inverse be denoted by $\Gamma^{-1}(A,B)$.

Subsequently, in 2016, Abdalla and Bakhet [5] introduced the following extension of the beta matrix function:

$$\mathcal{B}(P,Q;\mathbb{X}) = \int_0^1 t^{P-1}(1-t)^{Q-1}\, \exp\left(-\frac{\mathbb{X}}{t(1-t)}\right)\, \mathrm{d}t, \tag{5}$$

where the matrices P, Q and $\mathbb{X}$ are positive stable and commutative matrices in $\mathbb{C}^{r\times r}$ satisfying the spectral condition (2).

The special case of (5) when $\mathbb{X} = \mathbf{0}$ gives the beta matrix function $\mathfrak{B}(P,Q)$ defined in (1) (see also [13]), that is,

$$\mathcal{B}(P,Q;\mathbf{0}) = \mathfrak{B}(P,Q).$$

Furthermore, under the given conditions we have the following identity (see [5]):

$$\mathcal{B}(P,Q;\mathbb{X}) = \Gamma(P,\mathbb{X})\Gamma(Q,\mathbb{X})\Gamma^{-1}(P+Q,\mathbb{X}).$$

In recent years, the generalized Pochhammer matrix symbol $(A;B)_n$ and the generalized beta matrix function $\mathcal{B}(P,Q;\mathbb{X})$ were used to introduce and investigate several extensions of hypergeometric matrix functions (see, for example, [4], [14]; see also the recent paper [20]).

On the other hand, Bakhet *et al.* [9] presented the Wright Kummer hypergeometric matrix function ${}_1\mathbf{R}_1^{(\tau)}$ and the Wright hypergeometric matrix function ${}_2\mathbf{R}_1^{(\tau)}$ as follows: let A, B and C be positive stable matrices in $\mathbb{C}^{r\times r}$ satisfying the condition (2). Then the Wright Kummer and Wright hypergeometric matrix functions are defined as

$${}_1\mathbf{R}_1^{(\tau)}(A;C;z) = \Gamma^{-1}(A)\Gamma(C)\sum_{n=0}^{\infty}\Gamma^{-1}(C+\tau nI)\Gamma(A+\tau nI)\,\frac{z^n}{n!},$$

and

$${}_2\mathbf{R}_1^{(\tau)}(A,B;C;z) = \Gamma^{-1}(B)\Gamma(C)\sum_{n=0}^{\infty}(A)_n\Gamma^{-1}(C+\tau nI)\Gamma(B+\tau nI)\,\frac{z^n}{n!},$$

where $\tau \in (0,\infty)$. In [9], the integral representations, differential formulas and fractional calculus of the Wright hypergeometric matrix function were studied. Furthermore, the incomplete Wright hypergeometric matrix function was defined and some of its properties were established. We remark in passing that the *incomplete* extension of the Pochhammer matrix symbol, which was also considered by Bakhet *et al.* [9], has also been used rather widely in the current literature on hypergeometric functions (see, for example, [2], [8], [18] and [19], and references therein). On the other hand, very recently, the authors (see [1, 3, 11]) introduced the extensions of the $(k;\tau)$-Gauss hypergeometric matrix function and obtained their various properties. Also, they used these functions to find the solutions of the generalization of fractional kinetic equation.

The goal of this paper is to introduce an extended form of ${}_2\mathbf{R}_1^{(\tau)}(A,B;C;z)$, which involves the Pochhammer matrix symbol $(A;B)_n$ defined by (3) and the extended beta matrix function $\mathcal{B}(P,Q;\mathbb{X})$ given by (5). The remainder of the paper is organized as follows. In Section 2, we define an extended form of the Wright hypergeometric matrix function,

$${}_2\mathbf{R}_1^{(\tau)}((A;\mathbb{A}),B;C;z;\mathbb{X}),$$

and obtain some useful results such as integral representations. In Section 3, we introduce the incomplete extended Wright hypergeometric matrix function with the help of the incomplete extended beta matrix function $\mathcal{B}_y(P,Q;\mathbb{X})$, and investigate some of its properties. In Section 4, we evaluate the Riemann–Liouville fractional derivative of this extended hypergeometric function. In Section 5, we make concluding remarks.

2. Extended Wright Hypergeometric Matrix Function

In this section, we introduce the extended Wright hypergeometric matrix function (EWHMF) ${}_2\mathbf{R}_1^{(\tau)}((A;\mathbb{A}),B;C;z;\mathbb{X})$ in terms of the generalized beta matrix function $\mathcal{B}(P,Q;\mathbb{X})$ defined by (5) and the generalized Pochhammer matrix symbol $(A;B)_n$ defined by (3).

Suppose that A, $\mathbb{A}$, B, C, $C-B$ and $\mathbb{X}$ are positive stable matrices in $\mathbb{C}^{r\times r}$ satisfying the condition (2), and suppose that B, C and $\mathbb{X}$ commute with each other. Then we introduce the EWHMF and the extended Wright Kummer hypergeometric matrix function (EWKHMF) as follows:

$$ {}_2\mathbf{R}_1^{(\tau)}((A;\mathbb{A}),B;C;z;\mathbb{X}) = \Gamma\binom{C}{B,C-B}\sum_{n=0}^{\infty}(A;\mathbb{A})_n\mathcal{B}(B+\tau nI,C-B;\mathbb{X})\,\frac{z^n}{n!}, \quad (6) $$

$$ {}_1\mathbf{R}_1^{(\tau)}(B;C;z;\mathbb{X}) = \Gamma\binom{C}{B,C-B}\sum_{n=0}^{\infty}\mathcal{B}(B+\tau nI,C-B;\mathbb{X})\,\frac{z^n}{n!}, \quad (7) $$

$$ |z|<1, \quad \tau\in(0,\infty), $$

where $\Gamma\binom{C}{B,C-B} = \Gamma(C)\Gamma^{-1}(B)\Gamma^{-1}(C-B)$.

Remark 1. *In the particular case when $\mathbb{A}=\mathbb{X}=\mathbf{0}$, the definition* (6) *gives the Wright hypergeometric matrix function ${}_2\mathbf{R}_1^{(\tau)}(A,B;C;z)$ studied in [9], and the case with $\mathbb{A}=\mathbf{0}$ and $\tau=1$ gives the extended Gauss hypergeometric matrix function $F^{(\mathbb{X})}(A,B;C;z)$ given in [4]. Moreover, if we set $\mathbb{A}=\mathbb{X}=\mathbf{0}$ and $\tau=1$, the unification given in* (6) *reduces to the familiar Gauss hypergeometric matrix function ${}_2F_1(A,B;C;z)$ defined in [13]. On the other hand, if we consider $\mathbb{X}=\mathbf{0}$ in the definition* (7)*, we get the Wright Kummer hypergeometric matrix function given in [9].*

We start with the following theorem.

Theorem 1. *Let A, $\mathbb{A}$, B, C, $C-B$ and $\mathbb{X}$ be positive stable matrices in $\mathbb{C}^{r\times r}$ satisfying the condition* (2)*, and suppose that B, C and $\mathbb{X}$ commute with each other. Then the EWHMF ${}_2\mathbf{R}_1^{(\tau)}((A;\mathbb{A}),B;C;z;\mathbb{X})$ can be given in integral form as follows:*

$$ \begin{aligned} {}_2\mathbf{R}_1^{(\tau)}((A;\mathbb{A}),B;C;z;\mathbb{X}) &= \Gamma\binom{C}{B,C-B,A}\int_0^\infty\int_0^1 u^{A-I}e^{-(Iu+\frac{\mathbb{A}}{u})}t^{B-I} \\ &\quad \times(1-t)^{C-B-I}\exp\left(-\frac{\mathbb{X}}{t(1-t)}\right)e^{zut^\tau}\,dtdu. \end{aligned} $$

Proof. Using the integral representations (4) and (5), we get

$$ {}_2\mathbf{R}_1^{(\tau)}((A;\mathbb{A}),B;C;z;\mathbb{X}) = \Gamma\binom{C}{B,C-B,A}\int_0^\infty\int_0^1 u^{A-I}e^{-(Iu+\frac{\mathbb{A}}{u})}t^{B-I} $$

$$\begin{aligned}&\times(1-t)^{C-B-I}\exp\left(-\frac{\mathbb{X}}{t(1-t)}\right)\sum_{n=0}^{\infty}\frac{(zut^{\tau})^{n}}{n!}dtdu\\ =\ &\Gamma\binom{C}{B,C-B,A}\int_0^{\infty}\int_0^1 u^{A-I}e^{-(Iu+\frac{\mathbb{A}}{u})}t^{B-I}\\ &\times(1-t)^{C-B-I}\exp\left(-\frac{\mathbb{X}}{t(1-t)}\right)e^{zut^{\tau}}dtdu.\end{aligned}$$

This completes the proof. □

Theorem 2. *Under the same conditions as Theorem 1, we have the following relation:*

$$_2\mathbf{R}_1^{(\tau)}((A;\mathbb{A}),B;C;z;\mathbb{X})=\Gamma^{-1}(A)\int_0^{\infty}t^{A-I}e^{-(It+\frac{\mathbb{A}}{t})}{}_1\mathbf{R}_1^{(\tau)}(B;C;zt;\mathbb{X})dt.$$

Proof. Substituting the integral representation (4) into the definition (6), we have

$$\begin{aligned}{}_2\mathbf{R}_1^{(\tau)}((A;\mathbb{A}),B;C;z;\mathbb{X}) &= \Gamma\binom{C}{B,C-B,A}\int_0^{\infty}t^{A-I}e^{-(It+\frac{\mathbb{A}}{t})}\\ &\quad\times\sum_{n=0}^{\infty}\mathcal{B}(B+\tau nI,C-B;\mathbb{X})\frac{(zt)^n}{n!}dt.\end{aligned}$$

Now, using the definition (7) gives the result. □

Theorem 3. *Under the same conditions as Theorem 1, we have the following integral representation for the EWHMF:*

$$\begin{aligned}{}_2\mathbf{R}_1^{(\tau)}((A;\mathbb{A}),B;C;z;\mathbb{X}) &= \Gamma\binom{C}{B,C-B}\int_0^1 t^{B-I}(1-t)^{C-B-I}\exp\left(-\frac{\mathbb{X}}{t(1-t)}\right)\\ &\quad\times{}_1F_0[(A;\mathbb{A}),-,zt^{\tau}]dt.\end{aligned}$$

Proof. Substituting the integral representation (5) into the definition (6), we get

$$\begin{aligned}&{}_2\mathbf{R}_1^{(\tau)}((A;\mathbb{A}),B;C;z;\mathbb{X})\\ &=\Gamma\binom{C}{B,C-B}\int_0^1 t^{B-I}(1-t)^{C-B-I}\exp\left(-\frac{\mathbb{X}}{t(1-t)}\right)\sum_{n=0}^{\infty}(A;\mathbb{A})_n\frac{(zt^{\tau})^n}{n!}dt.\end{aligned}$$

Since

$${}_1F_0[(A;\mathbb{A}),-,zt^{\tau}]=\sum_{n=0}^{\infty}(A;\mathbb{A})_n\frac{(zt^{\tau})^n}{n!},$$

the result follows. □

3. Incomplete Extended Wright Hypergeometric Matrix Function

In this section, motivated by [21], we introduce the incomplete extended Wright hypergeometric matrix function (IEWHMF) with the help of the incomplete extended beta matrix function defined in (8). Let B, C and $\mathbb{X}$ be positive stable matrices in $\mathbb{C}^{r\times r}$ satisfying the condition (2), and suppose B, C and $\mathbb{X}$ commute with each other. The incomplete extended beta matrix function $\mathcal{B}_y(B,C;\mathbb{X})$ is defined as follows:

$$\mathcal{B}_y(B,C;\mathbb{X}) := \int_0^y t^{B-I}(1-t)^{C-I}\exp\left(-\frac{\mathbb{X}}{t(1-t)}\right)dt,\ 0\le y<1. \tag{8}$$

Let B, $C-B$ and $\mathbb{X}$ be positive stable matrices in $\mathbb{C}^{r\times r}$ satisfying the condition (2), and suppose B, C and $\mathbb{X}$ commute with each other. Then we introduce the incomplete extended beta matrix functions $[B,C;\mathbb{X};y]_n^{(\tau)}$ and $\{B,C;\mathbb{X};y\}_n^{(\tau)}$ as

$$[B,C;\mathbb{X};y]_n^{(\tau)} = \mathcal{B}_y(B+n\tau I, C-B;\mathbb{X}),$$

and

$$\{B,C;\mathbb{X};y\}_n^{(\tau)} = \mathcal{B}_{1-y}(C-B, B+n\tau I;\mathbb{X}),$$

where $0\le y<1$, respectively. It can be shown that

$$[B,C;\mathbb{X};y]_n^{(\tau)} + \{B,C;\mathbb{X};y\}_n^{(\tau)} = \mathcal{B}(B+n\tau I, C-B;\mathbb{X}).$$

Suppose that A, $\mathbb{A}$, B, C, $C-B$ and $\mathbb{X}$ are positive stable matrices in $\mathbb{C}^{r\times r}$ satisfying the condition (2), and that B, C and $\mathbb{X}$ commute with each other. Then we define the IEWHMFs as follows:

$$\begin{aligned} {}_2\mathbf{R}_1((A;\mathbb{A});[B,C;\mathbb{X};y]_n^{(\tau)};z;\mathbb{X}) &= \Gamma\binom{C}{B,C-B} \\ &\quad\times\sum_{n=0}^{\infty}(A;\mathbb{A})_n\mathcal{B}_y(B+n\tau I,C-B;\mathbb{X})\frac{z^n}{n!}, \end{aligned} \tag{9}$$

and

$$\begin{aligned} {}_2\mathbf{R}_1((A;\mathbb{A});\{B,C;\mathbb{X};y\}_n^{(\tau)};z;\mathbb{X}) &= \Gamma\binom{C}{B,C-B} \\ &\quad\times\sum_{n=0}^{\infty}(A;\mathbb{A})_n\mathcal{B}_{1-y}(C-B,B+n\tau I;\mathbb{X})\frac{z^n}{n!}. \end{aligned}$$

It can be seen that the IEWHMFs satisfy the following relation:

$$\begin{aligned} {}_2\mathbf{R}_1^{(\tau)}((A;\mathbb{A}),B;C;z;\mathbb{X}) &= {}_2\mathbf{R}_1((A;\mathbb{A});[B,C;\mathbb{X};y]_n^{(\tau)};z;\mathbb{X}) \\ &\quad+{}_2\mathbf{R}_1((A;\mathbb{A});\{B,C;\mathbb{X};y\}_n^{(\tau)};z;\mathbb{X}). \end{aligned}$$

Theorem 4. *Let A, $\mathbb{A}$, B, C, $C-B$ and $\mathbb{X}$ be positive stable matrices in $\mathbb{C}^{r\times r}$ satisfying the condition* (2)*, and suppose that B, C and $\mathbb{X}$ commute with each other.*

Then we have the following integral representation for ${}_2\mathbf{R}_1((A;\mathbb{A});[B,C;\mathbb{X};y]_n^{(\tau)};z;\mathbb{X})$*:*

$$\begin{aligned}{}_2\mathbf{R}_1((A;\mathbb{A});[B,C;\mathbb{X};y]_n^{(\tau)};z;\mathbb{X}) &= \Gamma\binom{C}{B, C-B, A} y^B \\ &\times \int_0^\infty \int_0^1 u^{A-I} e^{-(Iu+\frac{\mathbb{A}}{u})} v^{B-I}(1-yv)^{C-B-I} \\ &\times \exp\left(-\frac{\mathbb{X}}{yv(1-yv)}\right) e^{uz(yv)^\tau} dvdu.\end{aligned}$$

Proof. From the definitions (3) and (8), straightforward calculations show that

$$\begin{aligned}{}_2\mathbf{R}_1((A;\mathbb{A});[B,C;\mathbb{X};y]_n^{(\tau)};z;\mathbb{X}) &= \Gamma\binom{C}{B, C-B, A} y^B \\ &\times \int_0^\infty \int_0^1 u^{A-I} e^{-(Iu+\frac{\mathbb{A}}{u})} v^{B-I}(1-yv)^{C-B-I} \\ &\times \exp\left(-\frac{\mathbb{X}}{yv(1-yv)}\right) \sum_{n=0}^\infty \frac{(uz(yv)^\tau)^n}{n!} dvdu,\end{aligned}$$

which proves the theorem. ☐

Theorem 5. *Under the conditions given in Theorem 4, let* $|z(uy)^\tau| < 1$*. Then we have the following integral representation:*

$$\begin{aligned}{}_2\mathbf{R}_1((A;\mathbb{A});[B,C;\mathbb{X};y]_n^{(\tau)};z;\mathbb{X}) &= \Gamma\binom{C}{B, C-B} y^B \\ &\times \int_0^1 u^{B-I}(1-uy)^{C-B-I} \exp\left(-\frac{\mathbb{X}}{uy(1-uy)}\right) \\ &\times {}_1F_0((A;\mathbb{A}), -, z(uy)^\tau)du.\end{aligned}$$

Proof. Using the integral representation (8) and applying similar calculations as in Theorem 3 proves the theorem. ☐

Next, we give a derivative formula for the IEWHMF.

Theorem 6. *Let* ${}_2\mathbf{R}_1((A;\mathbb{A});[B,C;\mathbb{X};y]_n^{(\tau)};z;\mathbb{X})$ *be defined in* (9)*. Then we have the following derivative formula:*

$$\begin{aligned}&\frac{d^n}{dz^n}({}_2\mathbf{R}_1((A;\mathbb{A});[B,C;\mathbb{X};y]_n^{(\tau)};z;\mathbb{X})) \\ &= \Gamma\binom{C}{B, C-B} \Gamma\binom{C-B, B+\tau I}{C+\tau I} \Gamma\binom{C-B, B+2\tau I}{C+2\tau I} \ldots \Gamma\binom{C-B, B+\tau nI}{C+\tau nI} \\ &\times (A)_n \quad {}_2\mathbf{R}_1((A+nI;\mathbb{A});[B+\tau nI, C+\tau nI;\mathbb{X};y]_n^{(\tau)};z;\mathbb{X}).\end{aligned}$$

Proof. It is straightforward to obtain that

$$\frac{d}{dz}({}_2\mathbf{R}_1((A;\mathbb{A});[B,C;\mathbb{X};y]_n^{(\tau)};z;\mathbb{X}))$$

$$
\begin{aligned}
= \quad & \Gamma\binom{C}{B, C-B}\Gamma\binom{C-B, B+\tau I}{C+\tau I}A \\
& \times {}_2\mathbf{R}_1((A+I;\mathbb{A}); [B+\tau I, C+\tau I; \mathbb{X}; y]_n^{(\tau)}; z; \mathbb{X}).
\end{aligned}
$$

Repeating this n times proves the result. □

4. Fractional Derivative

In this section, we study the extended Riemann–Liouville fractional derivative of the EWHMF defined by (6). Let $\mathbb{X}$ be a positive stable matrix in $\mathbb{C}^{r\times r}$ and $\mu \in \mathbb{C}$. The extended Riemann–Liouville fractional derivative of order μ is given by [20]

$$
D_z^{\mu,\mathbb{X}} f(z) = \frac{1}{\Gamma(-\mu)} \int_0^z f(t)(z-t)^{-\mu-1} \ \exp\left(-\frac{\mathbb{X}z^2}{t(z-t)}\right) \ \mathrm{d}t, \tag{10}
$$

$$
\big(\Re(\mu) < 0\big).
$$

The particular case $\mathbb{X} = p$, $p \in \mathbb{C}^{1\times 1}$, such that $\Re(p) \geq 0$, gives the extended Riemann–Liouville fractional derivative given in [16] (see also [17]). Moreover, $\mathbb{X} = 0$ yields the classical Riemann–Liouville fractional derivative operator D_z^{μ} (for details, see [15]).

In [20], the authors presented the extended Riemann-Liouville fractional derivative of the function $f(z) = z^A$.

Theorem 7. *([20]) Let A be a positive stable matrix in $\mathbb{C}^{r\times r}$ and $\Re(\mu) < 0$. Then*

$$
D_z^{\mu,\mathbb{X}}\{z^A\} = \frac{\mathcal{B}(A+I, -\mu I; \mathbb{X})}{\Gamma(-\mu)} \ z^{A-\mu I}.
$$

Proof. According the definition (10), it is clear that

$$
D_z^{\mu,\mathbb{X}}\{z^A\} = \frac{1}{\Gamma(-\mu)} \int_0^z t^A (z-t)^{-\mu-1} \ \exp\left(-\frac{\mathbb{X}z^2}{t(z-t)}\right) \ \mathrm{d}t. \tag{11}
$$

Upon setting $t = zu$ and $\mathrm{d}t = z\mathrm{d}u$ in (11) gives

$$
\begin{aligned}
D_z^{\mu,\mathbb{X}}\{z^A\} &= \frac{1}{\Gamma(-\mu)} \int_0^1 (uz)^A (z-uz)^{-\mu-1} \ \exp\left(-\frac{\mathbb{X}z^2}{uz(z-uz)}\right) \ z\mathrm{d}u \\
&= \frac{1}{\Gamma(-\mu)} z^{A-\mu I} \int_0^1 u^A (1-u)^{(-\mu-1)I} \ \exp\left(-\frac{\mathbb{X}}{u(1-u)}\right) \ \mathrm{d}u \\
&= \frac{\mathcal{B}(A+I, -\mu I; \mathbb{X})}{\Gamma(-\mu)} z^{A-\mu I},
\end{aligned}
$$

which completes the proof. □

We now prove the following theorem.

Theorem 8. *Let A, $\mathbb{A}$ and $\mathbb{X}$ be positive stable matrices in $\mathbb{C}^{r\times r}$ satisfying the condition* (2) *and let $\Re(\mu) > \Re(\lambda) > 0$. Then for $|z^\tau| < 1$, the following relation holds true for the EWHMF:*

$$D_z^{\lambda-\mu,\mathbb{X}}\left\{z^{(\lambda-1)I}\,{}_1F_0[(A;\mathbb{A});\,\text{---}\,;z^\tau]\right\} = \frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{(\mu-1)I}\,{}_2\boldsymbol{R}_1^{(\tau)}((A;\mathbb{A}),\lambda I;\mu I;z^\tau;\mathbb{X}).$$

Proof. According to the extended fractional derivative formula (10), we have

$$\begin{aligned} &D_z^{\lambda-\mu,\mathbb{X}}\left\{z^{(\lambda-1)I}\,{}_1F_0[(A;\mathbb{A});\,\text{---}\,;z^\tau]\right\} \\ &= \frac{1}{\Gamma(\mu-\lambda)}\int_0^z t^{(\lambda-1)I}\,{}_1F_0[(A;\mathbb{A});\,\text{---}\,;t^\tau] \\ &\quad\times (z-t)^{\mu-\lambda-1}\,\exp\left(-\frac{\mathbb{X}z^2}{t(z-t)}\right)\,\mathrm{d}t. \end{aligned} \tag{12}$$

Let $t = zu$ in (12), then if we consider Theorem 3 we obtain the result asserted by Theorem 8. □

Theorem 9. *Suppose that A, $\mathbb{A}$, B, C, $C-B$ and $\mathbb{X}$ are positive stable matrices in $\mathbb{C}^{r\times r}$ satisfying the condition* (2) *and that B, C and $\mathbb{X}$ commute with each other. Let $\Re(\mu) < 0$, then for $|xz^\tau| < 1$, we have*

$$\begin{aligned} &D_z^{\mu,\mathbb{X}}\left\{z^{C-I}\,{}_2\boldsymbol{R}_1^{(\tau)}((A;\mathbb{A}),B;C;xz^\tau;\mathbb{X})\right\} \\ &= \Gamma\begin{pmatrix} C, C-B-\mu I \\ C-\mu I, C-B, -\mu I\end{pmatrix}\Gamma\begin{pmatrix} C-B, -\mu I;\mathbb{X} \\ C-B-\mu I\end{pmatrix} \\ &\quad\times {}_2\mathbf{R}_1^{(\tau)}((A;\mathbb{A}),B;C-\mu I;xz^\tau;\mathbb{X})z^{C-\mu I-I}, \end{aligned} \tag{13}$$

where $\Gamma\binom{C-B,-\mu I;\mathbb{X}}{C-B-\mu I} = \Gamma(C-B;\mathbb{X})\Gamma(-\mu I;\mathbb{X})\Gamma^{-1}(C-B-\mu I;\mathbb{X})$.

Proof. Consider the definitions (6) and (10) and let the left-hand side of (13) be denoted by $\mathfrak{D}$. Direct calculations yield that

$$\begin{aligned} \mathfrak{D} &= \frac{1}{\Gamma(-\mu)}\int_0^z t^{C-I}\,{}_2\boldsymbol{R}_1^{(\tau)}((A;\mathbb{A}),B;C;xt^\tau;\mathbb{X})(z-t)^{-\mu-1}\,\exp\left(-\frac{\mathbb{X}z^2}{t(z-t)}\right)\,\mathrm{d}t \\ &= \Gamma\begin{pmatrix} C \\ B, C-B\end{pmatrix}\frac{1}{\Gamma(-\mu)}\sum_{n=0}^{\infty}(A;\mathbb{A})_n\mathcal{B}(B+\tau nI, C-B;\mathbb{X})\,\frac{x^n}{n!} \\ &\quad\times\int_0^z t^{\tau nI+C-I}(z-t)^{-\mu-1}\,\exp\left(-\frac{\mathbb{X}z^2}{t(z-t)}\right)\,\mathrm{d}t. \end{aligned}$$

Then we have

$$\begin{aligned} \mathfrak{D} &= \Gamma\begin{pmatrix} C \\ B, C-B\end{pmatrix}\frac{1}{\Gamma(-\mu)}\sum_{n=0}^{\infty}(A;\mathbb{A})_n\mathcal{B}(B+\tau nI, C-B;\mathbb{X}) \\ &\quad\times\mathcal{B}(\tau nI+C,-\mu I;\mathbb{X})\frac{(z^\tau x)^n}{n!}z^{C-\mu I-I} \end{aligned}$$

$$\begin{aligned}
&= \Gamma\binom{C}{B, C-B} \frac{1}{\Gamma(-\mu)} \Gamma(C-B;\mathbb{X})\Gamma(-\mu I;\mathbb{X}) \\
&\quad \times \sum_{n=0}^{\infty} (A;\mathbb{A})_n \Gamma(B+\tau n I;\mathbb{X})\Gamma^{-1}(C+\tau n I-\mu I;\mathbb{X})\, \frac{(z^{\tau}x)^n}{n!} z^{C-\mu I-I} \\
&= \Gamma\binom{C}{B, C-B} \Gamma\binom{C-B, -\mu I;\mathbb{X}}{C-B-\mu I} \frac{1}{\Gamma(-\mu)} \\
&\quad \times \sum_{n=0}^{\infty} (A;\mathbb{A})_n \mathcal{B}(B+\tau n I, C-B-\mu I;\mathbb{X})\, \frac{(z^{\tau}x)^n}{n!} z^{C-\mu I-I} \\
&= \Gamma\binom{C, C-B-\mu I}{C-\mu I, C-B, -\mu I} \Gamma\binom{C-B, -\mu I;\mathbb{X}}{C-B-\mu I} \\
&\quad \times \Gamma(C-\mu I)\Gamma^{-1}(C-B-\mu I)\Gamma^{-1}(B) \\
&\quad \times \sum_{n=0}^{\infty} (A;\mathbb{A})_n \mathcal{B}(B+\tau n I, C-B-\mu I;\mathbb{X})\, \frac{(z^{\tau}x)^n}{n!} z^{C-\mu I-I}.
\end{aligned}$$

Thus the result follows by the definition (6) of the EWHMF. □

Theorem 10. *Suppose that A, $\mathbb{A}$, B, C, $C-B$, $\mathbb{X}_1$ and $\mathbb{X}_2$ are positive stable matrices in $\mathbb{C}^{r\times r}$ satisfying the condition (2) and that B, C and $\mathbb{X}_1$ commute with each other. Let $\Re(\mu) > \Re(\lambda) > 0$ and $\left|\frac{x}{1-z^{\tau_2}}\right| < 1$, then we have*

$$\begin{aligned}
&D_z^{\lambda-\mu,\mathbb{X}_2}\left\{ z^{(\lambda-1)I}(1-z^{\tau_2})^{-A}\; {}_2\boldsymbol{R}_1^{(\tau_1)}\left((A;\mathbb{A}), B; C; \frac{x}{1-z^{\tau_2}};\mathbb{X}_1\right)\right\} \\
&= \Gamma^{-1}(\mu I)\Gamma(\lambda I) z^{(\mu-1)I} F_2^{(\tau_1,\tau_2)}(A, B, \lambda I; C, \mu I; x, z^{\tau_2};\mathbb{X}_1, \mathbb{X}_2;\mathbb{A}),
\end{aligned}$$

where $\tau_1, \tau_2 \in (0,\infty)$ and $F_2^{(\tau_1,\tau_2)}(A, B, C; D, E; x, y;\mathbb{X}_1, \mathbb{X}_2;\mathbb{A})$ is a two-variable function defined by

$$\begin{aligned}
&F_2^{(\tau_1,\tau_2)}(A, B, C; D, E; x, y;\mathbb{X}_1, \mathbb{X}_2;\mathbb{A}) \\
&\quad = \Gamma\binom{D, E}{B, D-B, C, E-C} \sum_{m,n=0}^{\infty} (A;\mathbb{A})_m (A+mI)_n \\
&\qquad \times \mathcal{B}(B+\tau_1 m I, D-B;\mathbb{X}_1)\mathcal{B}(C+\tau_2 n I, E-C;\mathbb{X}_2)\, \frac{x^m}{m!}\, \frac{y^n}{n!}.
\end{aligned} \tag{14}$$

Proof. Considering the definition (6) and Theorem 7, we get

$$\begin{aligned}
&D_z^{\lambda-\mu,\mathbb{X}_2}\left\{ z^{(\lambda-1)I}(1-z^{\tau_2})^{-A}\; {}_2\boldsymbol{R}_1^{(\tau_1)}\left((A;\mathbb{A}), B; C; \frac{x}{1-z^{\tau_2}};\mathbb{X}_1\right)\right\} \\
&\quad = D_z^{\lambda-\mu,\mathbb{X}_2}\{\Gamma\binom{C}{B, C-B} z^{(\lambda-1)I}(1-z^{\tau_2})^{-A}
\end{aligned}$$

$$\times \sum_{m=0}^{\infty}(A;\mathbb{A})_m\mathcal{B}(B+\tau_1 mI, C-B;\mathbb{X}_1)\ \frac{\left(\frac{x}{1-z^{\tau_2}}\right)^m}{m!}\}$$

$$= \Gamma\binom{C}{B, C-B}\sum_{m,n=0}^{\infty}(A;\mathbb{A})_m(A+mI)_n\mathcal{B}(B+\tau_1 mI, C-B;\mathbb{X}_1)$$

$$\times D_z^{\lambda-\mu,\mathbb{X}_2}\{z^{\tau_2 nI+(\lambda-1)I}\}\ \frac{x^m}{n!m!}$$

$$= \Gamma\binom{C}{B, C-B}\frac{z^{(\mu-1)I}}{\Gamma(\mu-\lambda)}\sum_{m,n=0}^{\infty}(A;\mathbb{A})_m(A+mI)_n\mathcal{B}(B+\tau_1 mI, C-B;\mathbb{X}_1)$$

$$\times \mathcal{B}(\tau_2 nI+\lambda I, (\mu-\lambda)I;\mathbb{X}_2)\ \frac{(z^{\tau_2})^n x^m}{n!m!}$$

$$= \Gamma^{-1}(\mu I)\Gamma(\lambda I)z^{(\mu-1)I}F_2^{(\tau_1,\tau_2)}(A, B, \lambda I; C, \mu I; x, z^{\tau_2};\mathbb{X}_1,\mathbb{X}_2;\mathbb{A}).$$

□

Remark 2. *Note that, when $\tau_1, \tau_2 = 1$, $\mathbb{A} = \mathbf{0}$ and $\mathbb{X}_1 = \mathbb{X}_2$, the definition* (14) *gives the extended Appell hypergeometric matrix function $F_2(A, B, C; D, E; x, y;\mathbb{X})$ introduced in [20].*

5. Concluding Remarks

In our investigation here, we have introduced and studied the EWHMF

$${}_2\mathbf{R}_1^{(\tau)}((A;\mathbb{A}), B; C; z;\mathbb{X}).$$

We have presented various potentially useful properties of this family of extended hypergeometric matrix functions. Many of the results derived in this paper can be shown to reduce to known or new results about functions previously defined in the literature. For instance, in some particular cases, Theorems 8-10 yield new fractional-derivative formulas for various known families of hypergeometric functions.

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