Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 52 (6) (2023), 1567 – 1577
DOI : 10.15672/hujms.1148250

Special Issue Paper

# Algorithm analysis of solving fixed point of nonexpansive mappings based on runge-kutta method

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## Abstract

In order to solve the fixed point of nonexpansive mappings, we propose two iterative algorithms based on runge-kutta method. The first algorithm is focused on solving the fixed point problem of a single nonexpansive mapping, and weak convergence has been proved. we suggest the second algorithm by dynamic string-averaging rule. It can be used to find a common fixed point of a family of finite nonexpansive mappings. We show that the second algorithm is bounded perturbations resilient, and it is strongly convergent.

**Mathematics Subject Classification (2020).** Primary 53C15, 53C25; Secondary 53D10, 53D15

**Keywords.** nonexpansive mapping, common fixed point, runge-kutta method

## 1. Introduction

Fixed point theory and method is an important branch of nonlinear functional analysis, it is widely applied in signal processing, image recovery, variational inequality, equilibrium problems, etc. (see [6, 16, 17] and references therein). The origin of these theory can be traced back to last century. Since 1922 Banach proposed the contraction mapping principle, fixed point theory has attracted the attention of many scholars. With the development of science, researchers are no longer limited to the existence of fixed point, and more and more people begin to focus on the study of iterative algorithms (see [2–5, 7, 9, 12–14]). Up to now, scholars have made remarkable achievements. For example, Mann introduced the so-called one-step method [15] in 1953, it iteration format as follows:

$$x_0 \in H,\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n,\ \forall n \geq 0.$$

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Received: 25.07.2022; Accepted: 22.12.2022

In 1974, Ishikawa introduced two-step Mann method[15]:

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\ x_{n+1} = (1-\beta_n)x_n + \beta_n T y_n. \end{cases}$$

In 2000, Moudafi introduced viscosity algorithm[15]:

$$x_0 \in H,\ x_{n+1} = (1-\alpha_n)f(x_n) + \alpha_n T x_n,\ \forall n \geq 0,$$

where $f$ is a contractive mapping.

Fixed point theory and differential equations are closely linked. On the one hand, fixed point method is an effective tool for solving differential equations. On another hand, the iterative method of numerical solution of differential equations also provides help for the development of fixed point theory. In recent years, many scholars began to connect the differential equation theory with the fixed point iterative methods(see [1, 8, 10, 11]). Motivated by these articles, we utilize runge-kutta method to propose two algorithms to solve the fixed point problem.

For the ordinary differential equation, it is the following form:

$$y'(x) = f(x, y(x)) = f_x(y(x)), y(0) = y_0. \tag{1.1}$$

Where $f : [0, \bar{x}] \times R^n \longrightarrow R^n$, and $f_x(\cdot) = f(x, \cdot)$. Let $h > 0$ be a step-size. Using the runge-kutta method, we can get

$$\begin{cases} y[(n+1)h] = y(nh) + h(b_1k_1 + b_2k_2 + \cdots + b_mk_m) \\ k_1 = f(nh, y(nh)) \\ k_2 = f(nh + \lambda_2 h, y(nh) + \mu_2 h) \\ \vdots \\ k_m = f(nh + \lambda_m h, y(nh) + \mu_m h) \end{cases}$$

where $\lambda_j \in [0,1]$, $y(nh) + \mu_j h$ is the approximate value of $y(nh + \lambda_j h)$. Under some mild smoothness conditions on $f$, the above method converges uniformly to the exact solution of (1.1) as $h \to 0$ over $x$ in any fixed finite time interval $[0, \bar{x}]$.

If we set the function $f_x(y) = g_x(y) - y$, the differential equation (1.1) will become

$$y'(x) = g_x(y(x)) - y(x), y(0) = y_0. \tag{1.2}$$

Thus, differential equation (1.2) is related to the equilibrium problem by a common fixed point

$$y = g_x(y).$$

This encouraged us to apply the runge-kutta method to solve fixed point equation

$$Tx = x.$$

Note that the above process can be rewritten as

$$\begin{aligned} y_{n+1} &= y_n + h[b_1 g(nh, y(nh)) + \cdots + b_m g(x_n + \lambda_m h, y(nh) + \mu_m h)] \\ &- h(\sum_{i=1}^m b_i y_n + b_2\mu_2 h + \cdots + b_m \mu_m h). \end{aligned} \tag{1.3}$$

Based on the local truncation error, when $\sum_{i=1}^m b_i = 1$ the runge-kutta method has first-order precision.

Set $\lambda_1 = \mu_1 = 0$, then the equation becomes

$$y_{n+1} = (1-h)y_n + h\sum_{i=1}^m b_i g(nh + \lambda_i h, y(nh) + \mu_i h) - h^2 \sum_{i=1}^m b_i \mu_i. \tag{1.4}$$

Due to $h \to 0$, so we have

$$y_{n+1} \approx (1-h)y_n + h\sum_{i=1}^m b_i g(nh + \lambda_i h, y(nh) + \mu_i h). \tag{1.5}$$

Denote $T_\lambda$ the relaxation of $T$. Since $h \to 0$, the difference between $g(nh, y(nh))$ and $g(nh + \lambda_i h, y(nh) + \mu_i h)$ is very small. Meanwhile, $(1-\lambda)y(nh) + \lambda g(nh, y(nh))$ is approximately equal to $g(nh, y(nh))$ when $\lambda \to 1$. If we substitute the relaxation of $g$ for $g(nh + \lambda_i h, y(nh) + \mu_i h)$, then we can construct the following iteration of a nonexpansive mapping $T$ :

$$\begin{cases} y_{n+1} = (1-h)y_n + hu_n \\ u_n = \sum_{k=1}^{m} \omega(k) T_{\lambda_k} y_n. \end{cases} \tag{1.6}$$

Where $\sum_{k=1}^{m} \omega(k) = 1$, and$\{\lambda_k\} \subset [0,1]$.

The structure of this work is as follows. In section 2, we review some of the concepts and preliminary results used below. In section 3, we deal with analyzing the weak convergence of the first algorithm. In section 4, we propose the second algorithm, and in section 5 we prove the strong convergence of the second algorithm and that it is bounded perturbations resilient.

## 2. Preliminaries

Denote by $H$ a Hilbert space endowed with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. Let $C \subset H$ is a nonempty closed convex subset, and $Fix(T)$ the set of fixed points of $T$. The sequence $\{x_n\}$ converges weakly to $x$ is denoted by $x_n \rightharpoonup x$ as $n \to \infty$. $\omega_w(x_n)$ denotes the weak limit set of $\{x_n\}$, i.e.

$$\omega_n(x_n) = \{x \in H : \text{there exists a subsequence } \{n_j\} \text{ of } \{n\} \text{ such that } x_{n_j} \rightharpoonup x\}.$$

For any $u, v \in H$ and $\lambda \in [0,1]$, we have the following equality.

$$\|\lambda u + (1-\lambda)v\|^2 = \lambda\|u\|^2 + (1-\lambda)\|v\|^2 - \lambda(1-\lambda)\|u-v\|^2. \tag{2.1}$$

We also have

$$\|\sum_{i=1}^{m} \omega_i x_i\|^2 = \sum_{i=1}^{m} \omega_i \|x_i\|^2 - \frac{1}{2}\sum_{i,j=1}^{m} \omega_i\omega_j \|x_i - x_j\|^2, \tag{2.2}$$

where $\sum_{i=1}^{m} \omega_i = 1$, $x_i \in H$.

**Definition 2.1.** Let $T : H \to H$ be an operator. Then

1. the operator $T$ is called nonexpansive if
$$\|Tx - Ty\| \le \|x - y\|$$
for all $x, y$ in $H$ .
2. the operator $T$ is called strongly quasi-nonexpansive if exists $\alpha > 0$
$$\|Tx - z\|^2 \le \|x - z\|^2 - \alpha\|Tx - x\|^2$$
for all $x$ in $H$ and $z$ in $Fix(T)$.
3. denote $T_\lambda$ the relaxation of $T$,
$$T_\lambda = (1-\lambda)I + \lambda T,$$
where $I$ is the identity operator and $\lambda \in [0,2]$. If $\lambda \in (0,1)$ and $T$ is nonexpansive, then $T_\lambda$ is averaged and denote by $S$. Clearly, an averaged operator is both nonexpansive and strongly quasi-nonexpansive.

**Lemma 2.2.** ([11]). *Let $T$ be a nonexpansive self-mapping on $C$ with $Fix(T) \neq \emptyset$. Assume that $x_n$ converges weakly to $x$ as $n \to \infty$, and $(I-T)x_n \to 0$. Then $x \in Fix(T)$.*

**Lemma 2.3.** ([11]). *Let $C$ be a nonempty closed convex subset of a Hilbert space $H$, and $\{y_n\}$ represents a bounded sequence in $H$ such that:*

(1) $\lim_{n\to\infty} \|y_n - u\|$ *exists for all $u \in C$;*
(2) $\omega_w(y_n) \subset C$.

*Then $\{y_n\}$ converges to a point in $C$ weakly.*

## 3. The first algorithm

In this section, a new algorithm is introduced to solve the fixed point problem of a nonexpansive mapping. In order to obtain the convergence of the method, we make the following assumptions. Assume that an operator $T : C \to C$ is nonexpansive, and $\omega(k) \in (0,1), k = 1,2,\cdots,m$, such that $\sum_{k=1}^{m} \omega_k = 1$.

**Algorithm 3.1**

$$\begin{cases} y_{n+1} = (1-s_n)y_n + s_n u_n \\ u_n = \displaystyle\sum_{k=1}^{m} \omega(k) T_{\lambda_k} y_n, \end{cases} \tag{3.1}$$

where $\{s_n\} \subset (0,1)$, $\{\lambda_k\} \subset [0,1]$ and there exists $k \in \{1,2,\cdots,m\}$ such that $\lambda_k \neq 0$.

**Proposition 3.1.** *Let $\{y_n\}$ be the sequence generated by* (3.1)*, then we have for all $x^* \in Fix(T)$,*

$$\|y_{n+1} - x^*\| \leq \|y_n - x^*\|. \tag{3.2}$$

***Proof.*** Using (2.1), we have

$$\begin{aligned} \|y_{n+1} - x^*\|^2 &= \|(1-s_n)(y_n - x^*) + s_n(u_n - x^*)\|^2 \\ &= (1-s_n)\|(y_n - x^*)\|^2 + s_n\|(u_n - x^*)\|^2 - (1-s_n)s_n\|y_n - u_n\| \\ &= (1-s_n)\|(y_n - x^*)\|^2 \\ &+ s_n(\sum_{k=1}^{m} \omega(k)\|T_{\lambda_k} y_n - x^*\|^2 - \frac{1}{2}\sum_{i,j=1}^{m} \omega(i)\omega(j)\|T_{\lambda_i} y_i - T_{\lambda_j} y_j\|^2) \\ &- (1-s_n)s_n\|y_n - u_n\| \\ &\leq (1-s_n)\|(y_n - x^*)\|^2 \\ &+ s_n \sum_{k=1}^{m} \omega(k)\|T_{\lambda_k} y_n - x^*\|^2 - (1-s_n)s_n\|y_n - u_n\|^2 \\ &\leq \|(y_n - x^*)\|^2 - (1-s_n)s_n\|y_n - u_n\|^2. \end{aligned} \tag{3.3}$$

We get immediately that

$$\|y_{n+1} - x^*\| \leq \|y_n - x^*\|.$$

□

Further, we also have

$$\sum_{n=1}^{\infty} (1-s_n)s_n\|y_n - u_n\| < +\infty. \tag{3.4}$$

Based on iterative parameter $\{s_n\}$ satisfies the condition that $\{s_n\} \subset (0,1)$, then we can deduce

$$\lim_{n\to\infty} \|y_n - u_n\| = 0. \tag{3.5}$$

**Theorem 3.2.** *Let $H$ be a Hilbert space, and an operator $T : C \to C$ be nonexpansive. Assume that $\{y_n\}$ is a sequence generated by* (3.1)*, then $\{y_n\}$ converges weakly to a point in $Fix(T)$.*

***Proof.*** According to the norm triangle inequality, we have

$$\begin{aligned}\|y_n - Ty_n\| &\leq \|u_n - Ty_n\| + \|y_n - u_n\| \\ &= \|y_n - u_n\| + \|\sum_{k=1}^{m}(1-\lambda_k)\omega(k)(y_n - Ty_n)\| \\ &\leq \|y_n - u_n\| + \sum_{k=1}^{m}(1-\lambda_k)\omega(k)\|(y_n - Ty_n)\|.\end{aligned} \tag{3.6}$$

By (3.5), we can deduce immediately that

$$\lim_{n\to\infty} \|y_n - Ty_n\| = 0. \tag{3.7}$$

It follows from Lemma 2.2, we have $\omega_w(y_n) \subset Fix(T)$. Based on (3.2) we get that $\lim_{n\to\infty} \|y_n - x^*\|$ exists for all $x^* \in Fix(T)$. So, we can apply Lemma 2.3 to derive the weak convergence of $\{y_n\}$ to a point in $Fix(T)$. □

## 4. The second algorithm

Now, we extend our result to solve the common fixed point problem for a family of finite nonexpansive mappings. Set $\{T_i\}_{i=1}^m$ is a family of nonexpansive self-mapping on $H$. Denote by $T_{i,\lambda} = I + \lambda(T_i - I)$, where $I$ is the identity operator. For each $x_0 \in H$ and any $r > 0$, we mean

$$B(x_0, r) = \{x \in H : \|x - x_0\| \leq r\}.$$

For any positive number $\epsilon > 0$ and every $i \in \{1, 2, \cdots, m\}$, assume that

$$F = \bigcap_{i=1}^{m} Fix(T_i) \tag{4.1}$$

$$F_\epsilon(T_i) = \{x \in H : \|x - T_i(x)\| \leq \epsilon\} \tag{4.2}$$

$$\tilde{F}_\epsilon(T_i) = F_\epsilon(T_i) + B(0, \epsilon) \tag{4.3}$$

$$F_\epsilon = \bigcap_{i=1}^{m} F_\epsilon(T_i) \tag{4.4}$$

$$\tilde{F}_\epsilon = \bigcap_{i=1}^{m} \tilde{F}_\epsilon(T_i) \tag{4.5}$$

By an index vector $t = (t_1, t_2, \cdots, t_p)$ such that $t_i \in \{1, 2, \cdots, m\}$ for all $i = 1, 2, \cdots, p$. For an index vector $t = (t_1, t_2, \cdots, t_q)$, we mean

$$T[t] = T_{t_1,\lambda_{t_1}^{(n+1)}} T_{t_2,\lambda_{t_2}^{(n+1)}} \cdots T_{t_q,\lambda_{t_q}^{(n+1)}},\ p(t) = q. \tag{4.6}$$

Fix a number $\theta \in (0, \frac{1}{2}]$, for all $n \geq 0$, and $i \in \{1, 2, \cdots, m\}$,we define $\lambda_i^{(n)} \in [\theta, 1-\theta]$.

It is obvious for every index vector $t$ and for all $z \in F, x, y \in H$ that $T[t]$ is nonexpansive and

$$T[t](z) = z. \tag{4.7}$$

Let $A$ be the collection of all pairs $(\Omega, \omega)$ and $\omega$ satisfies

$$\omega : \Omega \to (0, 1), \sum_{t\in\Omega} \omega(t) = 1, \tag{4.8}$$

where $\Omega$ represents a finite collection of index vectors.

Let $(\Omega, \omega) \in A$ and set

$$T_{\Omega,\omega}(x) = \sum_{t\in\Omega} \omega(t)T[t](x), x \in H. \tag{4.9}$$

It is clearly to see $T_{\Omega,\omega}$ is nonexpansive, and for all $z \in F$ we have $T_{\Omega,\omega}(z) = z$.

In the present, we propose a dynamic string-averaging algorithm.

**Algorithm 4.1**

$$\begin{cases} y_{n+1} = (1-s_n)y_n + s_n T_{\Omega_{n+1},\omega_{n+1}}(y_n) \\ T_{\Omega_{n+1},\omega_{n+1}}(y_n) = \Sigma_{t\in\Omega_{n+1}}\omega(t)T[t](y_n). \end{cases}$$

Where $\{s_n\} \subset [a, 1-a], a \in (0, \frac{1}{2}]$ is a fixed number.

Note that (1.5), we will prove that algorithm 4.1 is bounded perturbations resilient. In other words, set a sequence $\{\epsilon_n\} \subset [0, \infty)$ satisfies $\sum_{n=1}^{\infty} \epsilon_n < \infty$, and $\{y_n\}$ is a sequence generated by algorithm 4.1. If for each $n \geq 0$

$$\|y_{n+1} - (1-s_n)y_n - s_n T_{\Omega_{n+1},\omega_{n+1}}(y_n)\| \leq \epsilon_{n+1}, \tag{4.10}$$

then $\{y_n\}$ converges strongly to a point in $\tilde{F}_\epsilon$.

Fix an integer $\bar{q}$ and a number $\delta$ such that

$$\bar{q} \geq m, \tag{4.11}$$

$$\delta \in (0, m^{-1}]. \tag{4.12}$$

Denote by $A^*$ the subset of $A$ such that for each $(\Omega, \omega) \in A^*$ and $t \in \Omega$,

$$p(t) \leq \bar{q}, \tag{4.13}$$

$$\omega(t) \geq \delta. \tag{4.14}$$

Set $\bar{N}$ a fixed natural number and $Card(A)$ the cardinality of a set $A$.

**Theorem 4.1.** *Let $R > 0$ satisfy*

$$B(0, R) \bigcap F \neq \emptyset, \tag{4.15}$$

*Fix a positive number $\epsilon$ and $\{\epsilon_n\} \subset [0, \infty)$ satisfy*

$$\Lambda := \sum_{n=1}^{\infty} \epsilon_n < \infty. \tag{4.16}$$

*Let $n_0$ be a natural number such that for every $n > n_0$*

$$\epsilon_n < \epsilon(\bar{N}+1)^{-1}(\bar{q}+1)^{-1}. \tag{4.17}$$

*Assume that*

$$\{(\Omega_n, \omega_n)\}_{n=1}^{\infty} \subset A^*, \tag{4.18}$$

*satisfy for any natural number $j$*

$$\{1, 2, \cdots, m\} \subset \bigcup_{n=j}^{j+\bar{N}-1} \Big(\bigcup_{t\in\Omega_n} \{t_1, t_2, \cdots, t_{p(t)}\}\Big), \tag{4.19}$$

$$y_0 \in B(0, R). \tag{4.20}$$

*Then a sequence $\{y_n\}$ generated by algorithm 4.1 is bounded perturbations resilient and*

$$Card(\{n \in \{1, 2, \cdots\} : y_n \notin \tilde{F}_\epsilon\})$$

$$\leq n_0 + ((2R+\Lambda)^2 + 2\Lambda(2R+\Lambda))\bar{N}(1+\bar{N})^2(1+\bar{q})^2\epsilon^{-2}\delta^{-1}c,$$

*where $c = [a\theta(1-\theta)]^{-1}$.*

## 5. Proof of Theorem 4.1

**Proposition 5.1.** *For all $n \geq 0$,*

$$\|x^* - y_n\| \leq 2R + \sum_{i=0}^{n} \epsilon_i, \ \forall x^* \in B(0, R) \bigcap F. \tag{5.1}$$

***Proof.*** Based on (4.15), there is a $x^* \in B(0, R) \bigcap F$. By (4.20) we have

$$\|x^* - y_0\| \leq 2R. \tag{5.2}$$

Set $\epsilon_0 = 0$,inequality (5.1) clearly holds for $n = 0$. Assume that $n \geq 0$ inequality (5.1) holds, by (4.10)

$$\begin{aligned} \|y_{n+1} - x^*\| &\leq \|(1 - s_n)y_n + s_n T_{\Omega_{n+1},\omega_{n+1}}(y_n) - x^*\| \\ &\quad + \|y_{n+1} - (1 - s_n)y_n - s_n T_{\Omega_{n+1},\omega_{n+1}}(y_n)\| \\ &\leq \|x^* - y_n\| + \epsilon_{n+1} \leq 2R + \sum_{i=0}^{n+1} \epsilon_i. \end{aligned} \tag{5.3}$$

Above all, proposition 5.1 is proved. □

Set $\gamma_0 = \epsilon(\bar{q} + 1)^{-1}(\bar{N} + 1)^{-1}$, by (4.17) for all $n > n_0$

$$\epsilon_n < \gamma_0. \tag{5.4}$$

Let $n > 0$, for every $t = (t_1, t_2, \cdots, t_{p(t)}) \in \Omega_{n+1}$, we can find a finite sequence $\{x_j^{(n,t)}\}_{j=0}^{p(t)} \subset H$ such that

$$y_n = x_0^{(n,t)} \tag{5.5}$$

$$T_{t_j}(y_{j-1}^{(n,t)}) = x_j^{(n,t)}, \ j = 1, 2, \cdots, p(t). \tag{5.6}$$

$$x_{p(t)}^{(n,t)} = x_{n,t}. \tag{5.7}$$

Set

$$\beta_{n,t} = \max\{\|x_{j+1}^{(n,t)} - x_j^{(n,t)}\| : j = 0, 1, \cdots, p(t) - 1\}. \tag{5.8}$$

Then we get

$$T_{\Omega_{n+1},\omega_{n+1}}(y_n) = \sum_{t \in \Omega_{n+1}} \omega_{n+1}(t) x_{n,t}. \tag{5.9}$$

Set

$$\mu_{n+1} = \max\{\beta_{n,t} : t \in \Omega_{n+1}\}. \tag{5.10}$$

**Proposition 5.2.** *For each natural number $n \geq n_0$ satisfies $\mu_{n+1} < \gamma_0$, then we have*

$$y_n \in \tilde{F}_{\bar{q}\gamma_0}(T_s), s \in \bigcup_{t \in \Omega_{n+1}} \{t_1, t_2, \cdots, t_{p(t)}\}, \|y_{n+1} - y_n\| \leq \gamma_0(\bar{q} + 1). \tag{5.11}$$

***Proof.*** Let $t = (t_1, t_2, \cdots, t_{p(t)}) \in \Omega_{n+1}$, based on $T_{t_j}, j = t_1, t_2, \cdots, t_{p(t)}$ is averaged , for every $0 \leq j < p(t)$

$$\begin{aligned} &\|x_j^{(n,t)} - x^*\|^2 - \|x_{j+1}^{(n,t)} - x^*\|^2 \\ &= \|x_j^{(n,t)} - x^*\|^2 - \|T_{t_{j+1},\lambda_{t_{j+1}}^{(n+1)}}(x_j^{(n,t)}) - x^*\|^2 \\ &\geq (1 - \lambda_{t_{j+1}}^{(n+1)})\lambda_{t_{j+1}}^{(n+1)}\|x_j^{(n,t)} - T_{t_{j+1},\lambda_{t_{j+1}}^{(n+1)}}(x_j^{(n,t)})\|^2 \\ &\geq (1 - \theta)\theta\|x_j^{(n,t)} - x_{j+1}^{(n,t)}\|^2. \end{aligned} \tag{5.12}$$

Moreover, we have

$$\begin{aligned} &\|y_n - x^*\|^2 - \|x_{n,t} - x^*\|^2 = \|x_0^{(n,t)} - x^*\|^2 - \|x_{p(t)}^{(n,t)} - x^*\|^2 \\ &= \sum_{j=0}^{p(t)-1} (\|x_j^{(n,t)} - x^*\|^2 - \|x_{j+1}^{(n,t)} - x^*\|^2) \\ &\geq (1-\theta)\theta \sum_{j=0}^{p(t)-1} \|x_j^{(n,t)} - x_{j+1}^{(n,t)}\|^2 \\ &\geq (1-\theta)\theta\beta_{n,t}^2. \end{aligned} \tag{5.13}$$

It follows from (4.8), (4.13), (4.18), (5.10) and (5.13)

$$\begin{aligned} &\|(1-s_n)y_n + s_n T_{\Omega_{n+1},\omega_{n+1}}(y_n) - x^*\|^2 \\ &\leq (1-s_n)\|y_n - x^*\|^2 + s_n \sum_{t\in\Omega_{n+1}} \omega_{n+1}(t)\|x_{n,t} - x^*\|^2 \\ &\leq (1-s_n)\|y_n - x^*\|^2 + s_n \sum_{t\in\Omega_{n+1}} \omega_{n+1}(t)(\|y_n - x^*\|^2 - (1-\theta)\theta\beta_{n,t}^2) \\ &= \|y_n - x^*\|^2 - s_n(1-\theta)\theta \sum_{t\in\Omega_{n+1}} \omega_{n+1}(t)\beta_{n,t}^2 \\ &\leq \|y_n - x^*\|^2 - a\delta\theta(1-\theta)\mu_{n+1}^2. \end{aligned} \tag{5.14}$$

Further, by (4.10) and (5.1)

$$\begin{aligned} &|\|y_{n+1} - x^*\|^2 - \|(1-s_n)y_n + s_n T_{\Omega_{n+1},\omega_{n+1}}(y_n) - x^*\|^2| \\ &= |\|y_{n+1} - x^*\| - \|(1-s_n)y_n + s_n T_{\Omega_{n+1},\omega_{n+1}}(y_n) - x^*\|| \\ &\times \|y_{n+1} - x^*\| + \|(1-s_n)y_n + s_n T_{\Omega_{n+1},\omega_{n+1}}(y_n) - x^*\|| \\ &\leq |\|y_{n+1} - x^*\| - \|(1-s_n)y_n + s_n T_{\Omega_{n+1},\omega_{n+1}}(y_n) - x^*\|| \\ &\times (\|y_n - x^*\| + \|y_{n+1} - x^*\|) \\ &\leq 2\epsilon_{n+1}(2R+\Lambda). \end{aligned} \tag{5.15}$$

Therefore, we have

$$\|y_{n+1} - x^*\|^2 \leq \|y_n - x^*\|^2 - a\delta\theta(1-\theta)\mu_{n+1}^2 + 2\epsilon_{n+1}(2R+\Lambda). \tag{5.16}$$

For each $n > n_0$,

$$\begin{aligned} &(2R+\Lambda)^2 \geq \|y_{n_0} - x^*\|^2 \\ &\geq \|y_{n_0} - x^*\|^2 - \|y_n - x^*\|^2 = \sum_{i=n_0}^{n-1} (\|y_i - x^*\|^2 - \|y_{i+1} - x^*\|^2) \\ &\geq \sum_{i=n_0}^{n-1} [a\delta\theta(1-\theta)\mu_{i+1}^2 - 2\epsilon_{n+1}(2R+\Lambda)]. \end{aligned} \tag{5.17}$$

So, we have

$$\begin{aligned} &(2R+\Lambda)^2 + 2\Lambda(2R+\Lambda) \geq a\delta\theta(1-\theta) \sum_{i=n_0}^{n-1} \mu_{i+1}^2 \\ &\geq a\delta\theta(1-\theta)\gamma_0^2 Card(\{k \in \{n_0, \cdots, n-1\} : \mu_{k+1} \geq \gamma_0\}). \end{aligned} \tag{5.18}$$

This implies

$$Card(\{k \in \{n_0, \cdots, n-1\} : \mu_{k+1} \geq \gamma_0\} \leq (a\delta\theta(1-\theta)\gamma_0^2)^{-1}((2R+\Lambda)^2 + 2\Lambda(2R+\Lambda)). \tag{5.19}$$

Assume that $n \geq n_0$ such that $\mu_{n+1} < \gamma_0$. Set $t = (t_1, \cdots, t_{p(t)}) \in \Omega_{n+1}$, we have

$$\gamma_0 > \|x_{j+1}^{(n,t)} - x_j^{(n,t)}\| = \|T_{t_{j+1}}(x_j^{(n,t)}) - x_j^{(n,t)}\|, \tag{5.20}$$

therefore

$$x_j^{(n,t)} \in F_{\gamma_0}(T_{t_{j+1}}). \tag{5.21}$$

Moreover, $y_n = x_0^{(n,t)}$, so we have

$$\|y_n - x_j^{(n,t)}\| \leq j\gamma_0 \leq \bar{q}\gamma_0. \tag{5.22}$$

If $j < p(t)$, then

$$y_n \in \tilde{F}_{\bar{q}\gamma_0}(T_{t_{j+1}}), \tag{5.23}$$

so we have

$$y_n \in \tilde{F}_{\bar{q}\gamma_0}(T_{t_s}), s = 1, 2, \cdots, p(t). \tag{5.24}$$

For all $t \in \Omega_{n+1}$,

$$\|y_n - x_{n,t}\| \leq \bar{q}\gamma_0, \tag{5.25}$$

therefore

$$y_n \in \bigcap\{\tilde{F}_{\bar{q}\gamma_0}(T_s) : s \in \bigcup_{t\in\Omega_{n+1}} \{t_1, \cdots, t_{p(t)}\}\}. \tag{5.26}$$

Further, we have

$$\begin{aligned}
&\|y_{n+1} - y_n\| \\
&\leq \|(1 - s_n)y_n + s_n T_{\Omega_{n+1},\omega_{n+1}}(y_n) - y_{n+1}\| \\
&+ \|(1 - s_n)y_n + s_n T_{\Omega_{n+1},\omega_{n+1}}(y_n) - y_n\| \\
&\leq \epsilon_{n+1} + s_n\|T_{\Omega_{n+1},\omega_{n+1}}(y_n) - y_n\| \\
&\leq \epsilon_{n+1} + s_n\bar{q}\gamma_0 < (\bar{q} + 1)\gamma_0.
\end{aligned} \tag{5.27}$$

Above all, proposition 5.2 is proved. □

Set $E_0 = \{i \in \{n_0, n_0 + 1, \cdots\} : \mu_{i+1} \geq \gamma_0\}$, based on (5.19)

$$Card(E_0) \leq (a\delta\theta(1 - \theta)\gamma_0^2)^{-1}((2R + \Lambda)^2 + 2\Lambda(2R + \Lambda)). \tag{5.28}$$

Set $E_1 = \{i \in \{n_0, n_0 + 1, \cdots\} : [i, i + \bar{N} - 1] \bigcap E_0 \neq \emptyset\}$, then

$$\begin{aligned}
&Card(E_1) \leq \bar{N}Card(E_0) \\
&\leq \bar{N}(a\delta\theta(1 - \theta)\gamma_0^2)^{-1}((2R + \Lambda)^2 + 2\Lambda(2R + \Lambda)).
\end{aligned} \tag{5.29}$$

Let $j \geq n_0$ and $j \notin E_1$, then $[j, j + \bar{N} - 1] \bigcap E_0 = \emptyset$. By proposition 5.2, for each $n \in [j, j + \bar{N} - 1]$, we can get $\mu_{n+1} < \gamma_0$.

For any $n_1, n_2 \in \{j, j + 1, \cdots, j + \bar{N} - 1\}$,

$$\|y_{n_1} - y_{n_2+1}\| \leq (\bar{q} + 1)\bar{N}\gamma_0. \tag{5.30}$$

Therefore, for all $n \in \{j, j + 1, \cdots, j + \bar{N} - 1\}$,

$$\begin{aligned}
&y_n \in \tilde{F}_{(\bar{q}+1)\gamma_0(\bar{N}+1)}(T_s), \\
&s \in \bigcup_{n=j}^{j+\bar{N}-1} \bigcup\{\{t_1, \cdots, t_{p(t)}\} : t \in \Omega_{n+1}\} = \{1, 2, \cdots, m\}.
\end{aligned} \tag{5.31}$$

So, $y_n \in \tilde{F}_\epsilon, n \geq n_0$. This implies theorem 4.1 true.

**Remark 5.3.** If $T_i, i = 1, 2, \cdots m$ is quasi-nonexpansive, then theorem 4.1 is also true. Because we do not use the nonexpansibility of the operator in the proof. If $T_i, i = 1, 2, \cdots m$ is $\alpha$-strongly quasi-nonexpansive, then it is not difficult to get that $T_{i,\lambda}, i = 1, 2, \cdots m$ is also strongly quasi-nonexpansive with the coefficient $(\lambda(1-\lambda) + \lambda\alpha)$. Meanwhile, $\lambda \in (0, 1+\alpha)$ instead of in $(0, 1)$. In this case, $c = [a\theta(1-\theta)]^{-1}$ in theorem 4.1 becomes $c = [a\theta(1-\theta) + a\alpha\theta]^{-1}$. Obviously, it is an acceleration of algorithm 4.1.

**Corollary 5.4.** *If $T_i : C \to C, i = 1, 2, \cdots m$, where $C \subset H$ is compact. Then the sequence $\{y_n\}$ defined by algorithm 4.1 converges to a point in $F$.*

***Proof.*** Obviously, $T$ is nonexpansive and it follows that $T$ is continuous. Based on theorem 4.1, for each $\epsilon > 0$, there is a common $y(\epsilon)$ such that

$$\|T_i y(\epsilon) - y(\epsilon)\| < \epsilon, \ i = 1, 2, \cdots, m.$$

Because $C$ is compact, there is a sequence $\{\epsilon_n\}$ such that $y(\epsilon_n) \to y^*$. Let $\epsilon_n \to 0$, for all $i = 1, 2, \cdots, m$

$$\|T_i y(\epsilon_n) - y^*\| \leq \|T_i y(\epsilon_n) - y(\epsilon_n)\| + \|y(\epsilon_n) - y^*\| \leq \epsilon_n + \|y(\epsilon_n) - y^*\|.$$

This implies that $T_i y(\epsilon_n) \to y^*, \ i = 1, 2, \cdots, m.$

So, we have

$$T_i y^* = T_i \lim_{n\to\infty} y(\epsilon_n) = \lim_{n\to\infty} T_i y(\epsilon_n) = y^*, \ i = 1, 2, \cdots, m.$$

Above all, corollary 5.4 is proved. □

## 6. Conclusion

In this article, we propose two algorithms based on runge-kutta method. The first algorithm is weak convergent. The second algorithm turns out to be bounded perturbations resilient and convergent strongly.

**Acknowledgment.** This paper was supported by the National Natural Science Foundation of China(grant number 11861003), the Natural Science Foundation of Ningxia province(grant number NZ17015, NXYLXK2017B09), the construction project of first-class subjects in NingXia higher education(grant number 213170023), Major Scientific and Technological Innovation Projects of YinChuan (grant numbers 2022RKX03 and NXYLXK2017B09), and the key project of science and technology of Ningxia (grant number 2021BEG03049).

## References

[1] M. A. Alghamdi, M. A. Alghamdi, N. Shahzad and H. K. Xu, *The implicit midpoint rule for nonexpansive mappings*, Fixed Point Theory and Applications, **96**, 1-9, 2014.

[2] L. C. Ceng, D. R. Sahu and J. C. Yao, *Implicit iterative algorithms for asymptotically nonexpansive mappings in the intermediate sense and Lipschitz-continuous monotone mappings*, J. Comput. Appl. **11**, 2902-2915, 2010.

[3] L. C. Ceng, N. C. Wong and J. C. Yao, *Strong and weak convergence theorems for an infinite family of nonexpansive mappings and applications*, Fixed Point Theory and Applications, **117**, 2012.

[4] Y. Censor, R. Davidi and G. T. Herman, *Perturbation resilience and superiorization of iterative algorithms*, Inverse Probl. **26**, 065008, 2010.

[5] Y. Censor and A. J. Zaslavski, *Convergence and perturbation resilience of dynamic string-averaging projection methods* , Comput. Optim. Appl. **54**, 65-76, 2013.

[6] P. Deuflhard, *Recent progress in extrapolation methods for ordinary differential equations*, SIAM Rev. **27**, 505-535, 1985.

[7] B. Halpern, *Fixed points of nonexpanding maps*, Bull. Amer. Math. Soc. **73**, 957-961, 1967.

[8] L. He, L. J. Zhu and Y. M. Fu, *An iterative algorithm based on simpson methods for solving fixed point problem of nonexpansive mappings*, UPB Sci. Bull. A: Appl. Math. Phys. **83**, 13-20, 2021.

[9] S. Reich, *Weak convergence theorems for nonexpansive mappings in Banach spaces* , J. Math. Anal. Appl. **67**, 274-276, 1979.

[10] H. K. Xun and R. G. Ori, *An implicit iteration process for nonexpansive mappings* , Numer. Funct. Anal. Optim. **22**, 767-773, 2001.

[11] Y. H. Yao, Y. C. Liou, T. L. Lee and N. C. Wong, *An iterative algorithm based on the implicit midpoint rule for nonexpansive mappings*, J. Nonlinear Convex Anal. **17**, 655-668, 2016.

[12] A. J. Zaslavski *Approximate solutions of common fixed point problems*, Commun. Appl. Nonlinear Anal. **22**, 80-89, 2015.

[13] A. J. Zaslavski, *Asymptotic behavior of two algorithms for solving common fixed point problems*, Inverse Probl. **33**, 044004, 2017.

[14] A. J. Zaslavski, *Algorithms for Solving Common Fixed Point Problems*, Springer Science and Business Media LLC. 2018.

[15] H. Y. Zhou, X. L. Qin, *Fixed Points of Nonlinear Operators*, Walter de Gruyter GmbH. 2020.

[16] L. J. Zhu, M. Postolache and Y. Y. She, *Convergence of self-adaptive projection methods with linear search for pseudomonotone variational inequalities and fixed point problems* , J. Nonlinear Convex Anal. **22**, 1541-1554, 2021.

[17] L. J. Zhu, Y. H. Yao and M. Postolache, *Projection methods with linesearch technique for pseudomonotone equilibrium problems and fixed point problems*, UPB Sci. Bull. A: Appl. Math. Phys. **83**, 3-14, 2021.