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Research Article

When every ideal is ϕ-P-flat

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Abstract

Let R be a commutative ring with nonzero identity. An R-module M is called ϕ-P-flat if $x \in \mathrm{Ann}(s)M$ for every non-nilpotent element $s \in R$ and $x \in M$ such that $sx = 0$. In this paper, we introduce and study the class of ϕ-PF-rings, i.e., rings in which all ideals are ϕ-P-flat. Among other results, the transfer of the ϕ-PF-ring to the amalgamation is investigated. Several examples which delineate the concepts and results are provided.

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1. Introduction

Throughout this paper, all rings considered are assumed to be commutative with the identity element and all modules are unitary.

Let R be a ring. Denote by $\mathrm{Nil}(R)$ and $Z(R)$ the ideal of all nilpotent elements of R and the set of all zero-divisors of R respectively. A ring R is called an *NP-ring* (resp., a *ZN-ring*) if $\mathrm{Nil}(R)$ is a prime ideal (resp., $Z(R) = \mathrm{Nil}(R)$). An ideal I of R is called a *nonnil ideal* if $I \nsubseteq \mathrm{Nil}(R)$. Let R be a PN-ring and M an R-module. Set

$$\phi\text{-}\operatorname{tor}(M) := \{x \in M \mid sx = 0 \text{ for some } s \in R \setminus \mathrm{Nil}(R)\}.$$

Then M is called a *ϕ-torsion* (resp., *ϕ-torsion-free*) module if $\phi\text{-}\operatorname{tor}(M) = M$ (resp., $\phi\text{-}\operatorname{tor}(M) = 0$). Recall from [22, 23] that an R-module F is said to be *ϕ-flat* if for any R-monomorphism $f : A \to B$ with $\mathrm{Coker}(f)$ a ϕ-torsion R-module, $1_F \otimes_R f : F \otimes_R A \to F \otimes_R B$ is an R-monomorphism, equivalently $\mathrm{Tor}_1^R(P, F) = 0$ for any ϕ-torsion R-module P.

An R-module M is said to be *P-flat* if $x \in \mathrm{Ann}(s)M$ for any $(s, x) \in R \times M$ such that $sx = 0$. If M is flat, then M is naturally P-flat. When R is a domain, M is P-flat if and only if it is torsion-free. When R is an arithmetic ring, any P-flat module is flat by [8, p. 236]. Also every P-flat cyclic module is flat by [8, Proposition 1(2)]. A ring R is called a *PF-ring* if all principal ideals of R are flat. Recall that R is a PF-ring if and only if every

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ideal of R is P-flat; if and only if for any element $(s, x) \in R^2$ with $sx = 0$, there exists an $\alpha \in \text{Ann}(s)$ such that $x = \alpha x$ by [7, Theorem 2.1].

Let A and B be two rings, J be an ideal of B and let $f : A \to B$ be a ring homomorphism. In this setting, we consider the following subring of $A \times B$:

$$A \bowtie^f J = \{(a, f(a) + j) \mid a \in A \text{ and } j \in J\}$$

is called the *amalgamation* of A and B along J with respect to f. This construction is a generalization of the amalgamated duplication of a ring along an ideal, denoted by $A \bowtie I$ (introduced and studied by D'Anna and Fontana in [9, 13, 14]). The interest of amalgamation resides partly in its ability to cover several basic constructions in commutative algebra including pullbacks and trivial ring extensions. See for instance [10, 11, 15].

Let A be a ring and let M be an R-module. Then $R \propto M$, the set of pairs (r, m) with componentwise addition and multiplication defined by: $(r, m)(b, f) = (rb, rf + bm)$, is a unitary commutative ring, called the *trivial extension* (or *idealization*) of R by M. Recall that prime (resp., maximal) ideals of R have the form $\mathfrak{p} \propto E$, where $\mathfrak{p}$ is a prime (resp., maximal) ideal of A. The basic properties of the trivial ring extension are summarized in [2, 5, 17, 18].

In this paper, we introduce and investigate a new class of rings, called "ϕ-PF-rings", in which every ideal is ϕ-P-flat. Examples of such rings are the ϕ-Prüfer rings, the PF-rings, and the ϕ-von Neumann regular rings. Thereby some properties and new examples are provided.

For any undefined terminology and notation the reader is referred to [16, 17, 20, 21].

2. Main results

An R-module M is said to be *ϕ-P-flat* if $x \in \text{Ann}(s)M$ for any $s \in R \setminus \text{Nil}(R)$ and $x \in M$ such that $sx = 0$.

Now we state our definition of ϕ-PF-rings.

Definition 2.1. A ring R is called a *ϕ-PF-ring* if every ideal of R is ϕ-P-flat.

Recall from [7, Theorem 2.1] that every ideal of R is P-flat if and only if every principal ideal of R is P-flat; if and only if R is a PF-ring (i.e., every principal ideal of R is flat); if and only if for any element $(s, x) \in R^2$ with $sx = 0$ there exists $\alpha \in \text{Ann}(s)$ such that $x = \alpha x$.

Now we have an analog of this characterization for the ϕ-PF-rings.

Theorem 2.2. *The following conditions are equivalent for a ring R.*

(1) R is a ϕ-PF-ring.
(2) Every principal ideal of R is ϕ-P-flat.
(3) Every submodule of any ϕ-P-flat R-module is ϕ-P-flat.
(4) $\text{Tor}_2^R(N, R/Ra) = 0$ for every R-module N and any $a \in R \setminus \text{Nil}(R)$.
(5) Every nonnil principal ideal of R is flat.
(6) For any element $x \in R$ and $s \in R \setminus \text{Nil}(R)$ with $sx = 0$, there exists $\alpha \in \text{Ann}(x)$ such that $s = \alpha s$.
(7) For any element $x \in R$ and $s \in R \setminus \text{Nil}(R)$ with $sx = 0$, there exists $\alpha \in \text{Ann}(s)$ such that $x = \alpha x$.

Proof. (1) $\Rightarrow$ (2) Straightforward.

(2) $\Rightarrow$ (5) Let $I = Ra$ be a nonnil principal ideal of R and J a principal ideal of R. Consider the map $1 \otimes \lambda_a : J \otimes aR \to J \otimes R$, where $\lambda_a : aR \to R$ is the inclusion. If $m \otimes a \in \text{Ker}\,(1 \otimes \lambda_a)$, where $m \in J$, then $m \otimes a = 0$ in $J \otimes R$; hence $am = 0$ in J. By hypothesis, $m = \Sigma_j s_j m_j$, where $s_j \in \text{Ann}(a)$ and $m_j \in J$. Thus $m \otimes a = \Sigma_j s_j m_j \otimes a = \Sigma_j (m_j \otimes s_j a) = 0$. Hence $\text{Ker}\,(1 \otimes \lambda_a) = \{0\}$. So $\text{Tor}_1^R(J, R/aR) = 0$. Then

$$\text{Tor}_1^R(R/J, I) \cong \text{Tor}_2^R(R/I, R/J) \cong \text{Tor}_1^R(R/I, J) = 0$$

for any principal ideal J of R, and hence I is P-flat. As I is principal, it is flat by [8, Proposition 1].

(5) $\Rightarrow$ (3) Let N be a submodule of a ϕ-P-flat R-module M and $a \in R \setminus \mathrm{Nil}(R)$. Then Ra is flat. Consider the following commutative diagram:

$$\begin{array}{ccc} N \otimes_R Ra & \xrightarrow{\mu} & N \otimes_R R \\ \downarrow{\scriptstyle\alpha} & & \downarrow \\ M \otimes_R Ra & \xrightarrow{\beta} & M \otimes_R R \end{array}$$

Since Ra is flat, α is a monomorphism. Our claim is to show that β is injective. For this, let $m \otimes a \in \mathrm{Ker}\,\beta$. Then $ma = 0$. Since M is a ϕ-P-flat R-module, there exist $(\beta_i)_{i=1,\dots,n} \in \mathrm{Ann}(a)^n$ and $(m_i)_{i=1,\dots,n} \in M^n$ such that $m = \sum_{i=1}^n \beta_i m_i$. Consequently

$$m \otimes a = \sum_{i=1}^n \beta_i m_i \otimes a = \sum_{i=1}^n m_i \otimes \beta_i a = 0.$$

So β and α are monomorphisms, and hence μ is a monomorphism. Next we must demonstrate that if $na = 0$ where $n \in N$ and $a \in R \setminus \mathrm{Nil}(R)$, then $n \in \mathrm{Ann}(a)M$. So $n \otimes a = 0$ since $\beta(n \otimes a) = na = 0$. Consider the map $f : R \to Ra$ defined by $f(1) = a$. Since $0 \to \mathrm{Ker}(f) \overset{i}{\to} R \overset{f}{\to} Ra \to 0$ is an exact sequence, we get the following exact sequence:

$$\mathrm{Ker}(f) \otimes N \overset{i\otimes 1_N}{\to} R \otimes N \overset{f\otimes 1_N}{\to} Ra \otimes N \to 0.$$

As $(f \otimes 1_N)(1 \otimes n) = a \otimes n = 0$, we have $(1 \otimes n) \in \mathrm{Ker}(f \otimes 1_N) = \mathrm{Im}(i \otimes 1_N)$. So there exist $(y_j, n_j)_{1 \leqq j \leqq k} \in \mathrm{Ker}(f) \times N$ such that:

$$\begin{aligned} 1 \otimes n &= (i \otimes 1_N)\left(\sum_{1\leqq j\leqq k} (y_j \otimes n_j)\right) \\ &= \sum_{1\leqq j\leqq k} (i(y_j) \otimes n_j) \\ &= 1 \otimes \sum_{1\leqq i\leqq k} i(y_j)\, n_j. \end{aligned}$$

Therefore $n = \sum_{1\leqq i\leqq k} i(y_i)\, n_i$. Since $i(y_j)\, a = i(y_j a) = i(f(y_j)) = i(0) = 0$ for all $j \in \{1, \dots, k\}$, we get $i(y_i) \in \mathrm{Ann}(a)$. Thus N is ϕ-P-flat.

(3) $\Rightarrow$ (4) For any R-module N, there exists an exact sequence $0 \to K \to F \to N \to 0$ with F a free R-module. Then K is ϕ-P-flat by (3), and so as in (5) $\Rightarrow$ (3) we have $\mathrm{Tor}_1^R(K, R/Ra) = 0$ for any $a \in R \setminus \mathrm{Nil}(R)$. Consider the induced exact sequence:

$$0 = \mathrm{Tor}_2^R(F, R/Ra) \to \mathrm{Tor}_2^R(N, R/Ra) \to \mathrm{Tor}_1^R(K, R/Ra) = 0.$$

Hence $\mathrm{Tor}_2^R(N, R/Ra) = 0$.

(4) $\Rightarrow$ (1) Let I be an ideal of R and $a \in R \setminus \mathrm{Nil}(R)$. Then $\mathrm{Tor}_2^R(\,R/I, R/Ra) = 0$ by (4). On the other hand, the exact sequence $0 \to I \to R \to R/I \to 0$ induces the exact sequence:

$$0 = \mathrm{Tor}_2^R(R/I, R/Ra) \to \mathrm{Tor}_1^R(I, R/Ra) \to \mathrm{Tor}_1^R(R, R/Ra) = 0.$$

Hence $\mathrm{Tor}_1^R(I, R/Ra) = 0$. Thus the map $I \otimes aR \to I$ defined by $m \otimes a \mapsto am$ is injective for every $a \in R \setminus \mathrm{Nil}(R)$. So we have the following exact sequence of R-modules:

$$0 \to (0 : a) \overset{i}{\to} R \overset{f}{\to} aR \to 0$$

with $f(1) = a$. It is clear that $1 \otimes m \in \mathrm{Ker}(f \otimes 1_I) = \mathrm{Im}(\iota \otimes 1_I)$. Hence $1 \otimes m = \Sigma_j s_j \otimes m_j$ where $s_j \in (0 : a)$ and $m_j \in I$. Thus $1 \otimes m = 1 \otimes (\Sigma_j s_j m_j)$, and so $m = \Sigma_j s_j m_j$. Consequently I is a ϕ-P-flat module.

(5) $\Rightarrow$ (6) Let $x \in R$ and $s \in R \setminus \mathrm{Nil}(R)$ such that $sx = 0$. Since $I = sR$ is a nonnil principal ideal of R, we get I is P-flat, which implies that $s \in (0 : x)I$. Therefore there exists $\alpha \in \mathrm{Ann}(x)$ such that $s = \alpha s$.

(6) $\Rightarrow$ (5) Let $I = sR$ be a nonnil principal ideal of R. If $y = as \in I$ and $x \in R$ such that $yx = 0$, then there exists $\alpha \in \mathrm{Ann}(ax)$ such that $s = \alpha s$. Hence $y = as = a\alpha s \in \mathrm{Ann}(x)I$.

(2) $\Rightarrow$ (7) Let $x \in R$ and $s \in R \setminus \mathrm{Nil}(R)$ such that $sx = 0$. Since $I = xR$ is ϕ-P-flat, we get $x \in \mathrm{Ann}(s)I$. So there exists $\alpha \in \mathrm{Ann}(s)$ such that $x = \alpha x$.

(7) $\Rightarrow$ (1) Let I be an ideal of R. Let $x \in I$ and $s \in R \setminus \mathrm{Nil}(R)$ such that $sx = 0$. Then there exists $\alpha \in \mathrm{Ann}(s)$ such that $x = \alpha x$, and so $x \in \mathrm{Ann}(s)I$. Thus I is ϕ-P-flat. □

Recall that an ideal I of a ring R is said to be *pure* if for every $x \in I$, there exists $y \in I$ such that $xy = x$.

Corollary 2.3. *A ring R is a ϕ-PF-ring if and only if* $\mathrm{Ann}(a)$ *is a pure ideal of R for every $a \in R \setminus \mathrm{Nil}(R)$.*

We next give some examples of ϕ-PF-rings.

Example 2.4. (1) Every PF-ring is a ϕ-PF-ring.

(2) Every ring R with $Z(R) = \mathrm{Nil}(R)$ is a ϕ-PF-ring.

Remark 2.5. In general, R being a ϕ-PF-ring does not imply that $Z(R) = \mathrm{Nil}(R)$. It suffices to consider $R := \mathbb{Z}/6\mathbb{Z}$. Then R is a ϕ-PF-ring since it is a PF-ring by [7, Theorem 2.7]. But $Z(R) = \{0, 2, 3, 4\} \neq \mathrm{Nil}(R) = 0$.

Recall that a ring R is said to be *présimplifiable* if for every $a, r \in R$, $ar = a$ implies $a = 0$ or r is a unit. It is easy to check that any local ring is présimplifiable.

The following corollary shows that if we assume that R is a présimplifiable ring or a PN-ring, we will have an equivalence between the ϕ-PF-rings and the rings R with $Z(R) = \mathrm{Nil}(R)$.

Corollary 2.6. *(1) If R is a PN-ring, then R is a ϕ-PF-ring if and only if $Z(R) = \mathrm{Nil}(R)$.*

(2) If R is présimplifiable, then R is a ϕ-PF-ring if and only if $Z(R) = \mathrm{Nil}(R)$.

Proof. (1) Assume that R is a PN-ring and let $x \in Z(R)$. Then there is a nonzero $s \in R$ such that $sx = 0$. If $x \notin \mathrm{Nil}(R)$, then $s = \alpha s$ for some $\alpha \in \mathrm{Ann}(x)$. Since $\alpha \in \mathrm{Ann}(x)$, we get $\alpha x = 0$. Hence $\alpha \in \mathrm{Nil}(R)$, and so $\alpha^n = 0$ for some $n \in \mathbb{N}$. Then

$$s = \alpha s = \alpha(\alpha s) = \alpha^2 s = \cdots = \alpha^n s = 0,$$

a contradiction. Thus x is nilpotent.

(2) Assume that R is présimplifiable and let $x \in R \setminus \mathrm{Nil}(R)$. It is only required to show that $x \notin Z(R)$. Let $s \in R$ such that $sx = 0$. Then there is $\alpha \in \mathrm{Ann}(s)$ such that $x = \alpha x$. Since R is présimplifiable and $x \neq 0$, we get α is a unit, and hence $s = 0$. Therefore $x \notin Z(R)$. □

Recall from [4] that a prime ideal P of R is said to be *divided* if it is comparable to every ideal of R. Set

$$\mathcal{H} := \{R \mid R \text{ is a commutative ring and } \mathrm{Nil}(R) \text{ is a divided prime ideal of } R\}.$$

Then R is called a ϕ-ring if $R \in \mathcal{H}$.

Following [23], a ϕ-ring R is said to be *ϕ-von Neumann regular* if every R-module is ϕ-flat. Thus a ϕ-von Neumann regular ring is naturally a ϕ-PF-ring, while the converse is not true in general, as the following example shows.

Example 2.7. Let D be a domain which is not a field and set $R := D \propto D$. Then R is a ϕ-PF-ring which is not a ϕ-von Neumann regular ring.

Recall that a ϕ-ring is called a *ϕ-Prüfer ring* if $R/\operatorname{Nil}(R)$ is a Prüfer domain by [1, Theorem 2.6].

Corollary 2.8. *Let R be a ϕ-ring. Then every ideal of R is ϕ-flat if and only if R is a ϕ-Prüfer ring with $Z(R) = \operatorname{Nil}(R)$.*

Proof. Assume that every ideal of R is ϕ-flat. Let $K/\operatorname{Nil}(R)$ be a nonzero ideal of $R/\operatorname{Nil}(R)$. Then K is a nonnil ideal of R. Thus as in the proof of (2) $\Rightarrow$ (5) in Theorem 2.2 we have

$$\operatorname{Tor}_1^R(R/I, K) \cong \operatorname{Tor}_2^R(R/I, R/K) \cong \operatorname{Tor}_1^R(R/K, I) = 0$$

for any ideal I of R. Hence K is a flat R-module. Note that $\operatorname{Nil}(R)K = \operatorname{Nil}(R)$ [19, Lemma 2.9(1)]. Therefore $K/\operatorname{Nil}(R)$ is a flat $R/\operatorname{Nil}(R)$-module by [21, Corollary 2.5.12(1)]. Thus all ideals of $R/\operatorname{Nil}(R)$ are flat. Hence $R/\operatorname{Nil}(R)$ is a Prüfer domain, and so R is a ϕ-Prüfer ring by [1, Theorem 2.6]. On the other hand, if every ideal of R is ϕ-flat, then every ideal of R is ϕ-P-flat, i.e., R is a ϕ-PF-ring. Since R is a ϕ-ring, R is a PN-ring, whence $Z(R) = \operatorname{Nil}(R)$. For the converse see [22, Theorem 4.3]. □

Recently Chang and Kim [6] introduced a new pullback. Let D be a domain with K its quotient field. Let $K[X]$ be the polynomial ring over K, $n \geq 2$ be an integer and $K[\theta] = K[X]/\langle X^n \rangle$, where $\theta = X + \langle X^n \rangle$. Denote by $i : D \hookrightarrow K$ the natural embedding map and $\pi : K[\theta] \twoheadrightarrow K$ a ring homomorphism satisfying $\pi(f) = f(0)$. Consider the pullback of i and π as follows:

$$\begin{array}{ccc} R_n := D + \theta K[\theta] & \longrightarrow & K[\theta] \\ \downarrow & & \downarrow \pi \\ D & \overset{i}{\hookrightarrow} & K \end{array}$$

Then $R_n = D + \theta K[\theta] = \{f \in K[\theta] \mid f(0) \in D\}$ is a subring of $K[\theta]$. Note that R_n is a ϕ-ring and $Z(R_n) = \operatorname{Nil}(R_n) = \theta K[\theta]$. Thus we have the following:

Corollary 2.9. *Let the notation be as above. Then every ideal of R_n is ϕ-flat if and only if R_n is a ϕ-Prüfer ring.*

Proposition 2.10. *Let R be a ϕ-ring and let I be a nonnil ideal of R. Then I is ϕ-flat over R if and only if $I/\operatorname{Nil}(R)$ is flat over $R/\operatorname{Nil}(R)$.*

Proof. Assume that I is ϕ-flat over R and let $K/\operatorname{Nil}(R)$ be a nonzero ideal of $R/Nil(R)$. Then K is a nonnil ideal of R. Thus R/K is ϕ-torsion, and so is $R/K \otimes_R R/\operatorname{Nil}(R)$. Consider the following exact sequence: $0 \to K \to R \to R/K \to 0$. Note that $R/\operatorname{Nil}(R)$ is ϕ-flat. So $0 \to K \otimes_R R/\operatorname{Nil}(R) \to R \otimes_R R/\operatorname{Nil}(R) \to R/K \otimes_R R/\operatorname{Nil}(R) \to 0$ is exact. Since I is ϕ-flat, we have the following exact sequence:

$$0 \to I \otimes_R K \otimes_R R/\operatorname{Nil}(R) \to I \otimes_R R \otimes_R R/\operatorname{Nil}(R) \to I \otimes_R R/K \otimes_R R/\operatorname{Nil}(R) \to 0.$$

Note that $I \otimes_R R/\operatorname{Nil}(R) = I/I\operatorname{Nil}(R) = I/\operatorname{Nil}(R)$ and $K \otimes_R R/\operatorname{Nil}(R) = K/K\operatorname{Nil}(R) = K/\operatorname{Nil}(R)$ as I and K are nonnil. Thus we have the following exact sequence:

$$\begin{aligned} 0 &\to (I \otimes_R R/\operatorname{Nil}(R)) \otimes_{R/\operatorname{Nil}(R)} (K \otimes_R R/\operatorname{Nil}(R)) \\ &\to (I \otimes_R R/\operatorname{Nil}(R)) \otimes_{R/\operatorname{Nil}(R)} (R \otimes_R R/\operatorname{Nil}(R)) \\ &\to (I \otimes_R R/\operatorname{Nil}(R)) \otimes_{R/\operatorname{Nil}(R)} (R/K \otimes_R R/\operatorname{Nil}(R)) \to 0. \end{aligned}$$

That is,

$$\begin{aligned} 0 &\to I/\operatorname{Nil}(R) \otimes_{R/\operatorname{Nil}(R)} K/\operatorname{Nil}(R) \to I/\operatorname{Nil}(R) \otimes_{R/\operatorname{Nil}(R)} R/\operatorname{Nil}(R) \\ &\to I/\operatorname{Nil}(R)) \otimes_{R/\operatorname{Nil}(R)} R/K \to 0 \end{aligned}$$

is exact. Therefore $I/\operatorname{Nil}(R)$ is flat over $R/\operatorname{Nil}(R)$. The converse follows from [23, Theorem 3.8]. □

Remark 2.11. (1) The necessity of Proposition 2.10 can be proved by using [23, Theorem 3.8] since for a domain the flat modules and the ϕ-flat modules coincide.
(2) The first part of the necessity of Corollary 2.8 can be proved by Proposition 2.10.

A ring R being a ϕ-PF-ring does not guarantee that it is also a ϕ-Prüfer ring as shown by the following example.

Example 2.12. Let A be a domain which is not a Prüfer domain and K its quotient field. Set $R = A \propto K$. Then:

(1) R is a ϕ-ring with $Z(R) = \mathrm{Nil}(R)$.
(2) Every ideal of R is ϕ-P-flat.
(3) R is not a ϕ-Prüfer ring, and hence there is an ideal of R which is not ϕ-flat.

Recall from Theorem 2.2 that a ring R is a ϕ-PF-ring if and only if every nonnil principal ideal of R is P-flat. However, this does not imply that any nonnil finitely generated ideal is P-flat as shown in the following remark.

Remark 2.13. If R is a ϕ-PF-ring, then it does not imply that every nonnil finitely generated ideal of R is P-flat.

Proof. Let $R = \mathbb{Z} \propto \mathbb{Z}$. Since $Z(R) = \mathrm{Nil}(R)$, we get R is a ϕ-PF-ring. Set $I = (2,0)R + (0,3)R$. Then I is a finitely generated nonnil ideal of R. Set $a = (0,1)$. Then $a(0,3) = 0$ and $\mathrm{Ann}(a) = 0 \propto \mathbb{Z}$. So $\mathrm{Ann}(a)I = 0 \propto 2\mathbb{Z}$, whence $(0,3) \notin \mathrm{Ann}(a)I$. Thus I is not P-flat. □

Note that a PF-ring is a ϕ-PF-ring, but the converse is not true in general. As an example, we may consider the ring $R = \mathbb{Z}/4\mathbb{Z}$. The following theorem gives a necessary and sufficient condition to have the converse. Recall that a ring R is said to be *reduced* if $\mathrm{Nil}(R) = 0$.

Theorem 2.14. *Let R be a ring. Then R is a PF-ring if and only if it is a reduced ϕ-PF-ring.*

Proof. Assume that R is a PF-ring. Then R is naturally a ϕ-PF-ring. It is known that a PF-ring is reduced [16, Theorem 4.2.2]. The converse is straightforward since if $\mathrm{Nil}(R) = 0$, then the notion of ϕ-P-flat rings is equivalent to that of P-flat rings. □

In [3, Theorem 2.3] Aritico and Marconi proved that a ring R is a PF-ring if and only if $\mathrm{Ann}(a) + \mathrm{Ann}(b) = R$, whenever $ab = 0$. We next give an analogous result for the ϕ-PF-rings.

Theorem 2.15. *The following conditions are equivalent for a ring R.*

(1) R is a ϕ-PF-ring.
(2) For every $a \in R \setminus \mathrm{Nil}(R)$ and $b \in R$ such that $ab = 0$, $\mathrm{Ann}(a) + \mathrm{Ann}(b) = R$.
(3) For every $a \in R \setminus \mathrm{Nil}(R)$ and $b \in R$, $\mathrm{Ann}(a) + \mathrm{Ann}(b) = \mathrm{Ann}(ab)$.

Proof. (1) $\Rightarrow$ (2) Let $a \in R \setminus \mathrm{Nil}(R)$. Then $\mathrm{Ann}(a)$ is a pure ideal of R. Let $b \in R$ such that $ab = 0$. We claim that $\mathrm{Ann}(a) + \mathrm{Ann}(b) = R$. Assume on the contrary that $\mathrm{Ann}(a) + \mathrm{Ann}(b) \neq R$. Then there exists a maximal ideal $\mathfrak{m}$ containing $\mathrm{Ann}(a) + \mathrm{Ann}(b)$. Since $ab = 0$, we have $b \in \mathrm{Ann}(a)$. Then by purity of $\mathrm{Ann}(a)$ there exists $c \in \mathrm{Ann}(a)$ such that $b = bc$. So $1 - c \in \mathrm{Ann}(b) \subseteq \mathfrak{m}$. But $c \in \mathrm{Ann}(a) \subseteq \mathfrak{m}$, and hence $1 \in \mathfrak{m}$, a contradiction. Thus $\mathrm{Ann}(a) + \mathrm{Ann}(b) = R$.

(2) $\Rightarrow$ (3) Let $a \in R \setminus \mathrm{Nil}(R)$ and $b \in R$. As $\mathrm{Ann}(a) \subseteq \mathrm{Ann}(ab)$ and $\mathrm{Ann}(b) \subseteq \mathrm{Ann}(ab)$, it follows that $\mathrm{Ann}(a)+\mathrm{Ann}(b) \subseteq \mathrm{Ann}(ab)$. For the other inclusion, let $x \in \mathrm{Ann}(ab)$. Then $x(ab) = a(xb) = 0$, and so $\mathrm{Ann}(a) + \mathrm{Ann}(xb) = R$. Hence there exist $y \in \mathrm{Ann}(xb)$ and $z \in \mathrm{Ann}(a)$ such that $1 = y + z$. Thus $x = xy + xz$ with $xy \in \mathrm{Ann}(b)$ and $xz \in \mathrm{Ann}(a)$. Therefore $\mathrm{Ann}(a) + \mathrm{Ann}(b) = \mathrm{Ann}(b)$.

(3) ⇒ (1) Let $a \in R\backslash \mathrm{Nil}(R)$ and $b \in \mathrm{Ann}(a)$ such that $ab = 0$. Then $\mathrm{Ann}(a)+\mathrm{Ann}(b) = \mathrm{Ann}(ab) = \mathrm{Ann}(0) = R$. In particular $1 = \alpha_1+\alpha_2$ for some $\alpha_1 \in \mathrm{Ann}(a)$ and $\alpha_2 \in \mathrm{Ann}(b)$. Multiplying by b, we get $b = \alpha_1 b$. Thus R is a ϕ-PF-ring. □

As a corollary to Theorem 2.15, we can provide another proof for [3, Theorem 2.3.]

Corollary 2.16. *The following conditions are equivalent for a ring R.*

(1) R is a PF-ring.
(2) For every $a, b \in R$ such that $ab = 0$, $\mathrm{Ann}(a) + \mathrm{Ann}(b) = R$.
(3) For every $a, b \in R$, $\mathrm{Ann}(a) + \mathrm{Ann}(b) = \mathrm{Ann}(ab)$.

Proof. (1) ⇒ (2) Assume that R is a PF-ring. Then R is a reduced ϕ-PF-ring by Theorem 2.14. So $\mathrm{Ann}(a) + \mathrm{Ann}(b) = R$ whenever $ab = 0$ by Theorem 2.15 .

(2) ⇒ (3) Assume that $\mathrm{Ann}(a)+\mathrm{Ann}(b) = R$ for every $a, b \in R$ such that $ab = 0$. Then $\mathrm{Ann}(a) + \mathrm{Ann}(b) = \mathrm{Ann}(ab)$ for every $a \in R \setminus \mathrm{Nil}(R)$ and $b \in R$ by the previous theorem. Let $a \in \mathrm{Nil}(R)$. Then $a^n = 0$ for some $n \in \mathbb{N}$. Hence $\mathrm{Ann}(a) + \mathrm{Ann}(a^{n-1}) = R$ since $a \cdot a^{n-1} = 0$. Thus $\mathrm{Ann}(a) = R$, and so R is a reduced. Therefore $\mathrm{Ann}(a) + \mathrm{Ann}(b) = R$ for every $a, b \in R$.

(3) ⇒ (1) Assume that for every $a, b \in R$, $\mathrm{Ann}(a) + \mathrm{Ann}(b) = \mathrm{Ann}(ab)$. Then R is a ϕ-PF-ring and we can easily verify that $\mathrm{Ann}(a) = \mathrm{Ann}(a^n)$ for every $n > 0$. Therefore R is a reduced ϕ-PF-ring, and hence by Theorem 2.14 R is a PF-ring. □

Note that every domain is présimplifiable, however the converse is not true in general. Examine the ring $\mathbb{Z}/4\mathbb{Z}$ for example. Similarly, any domain is a PF-ring, while the converse is not true, for this it is enough to consider the ring $\mathbb{Z}/6\mathbb{Z}$.

The following corollary shows that a présimplifiable PF-ring is a domain.

Corollary 2.17. *Let R be a ring. Then R is a domain if and only if it is a présimplifiable PF-ring.*

Proof. If R is a domain, then it is straightforward that R is a présimplifiable PF-ring. Conversely, assume that R is a présimplifiable PF-ring. Then R is a reduced ϕ-PF-ring by Theorem 2.14. By Corollary 2.6, we get $Z(R) = \mathrm{Nil}(R)$, and so $Z(R) = 0$. Thus R is a domain. □

Note that every ϕ-flat module is ϕ-P-flat, and any P-flat module is ϕ-P-flat. However the converse of the two statements may not be true. Now, our goal is to construct a class of ϕ-P-flat ideals which are neither ϕ-flat nor P-flat. For this we will start with the following proposition.

Proposition 2.18. *Let D be a domain which is not a field and let $R = D \propto D$. Set $J = (0, a)R$ to be the ideal generated by $(0, a)$ with a a nonunit of D. Then J is not ϕ-flat.*

Proof. Consider the exact sequence:

$$0 \to 0 \propto D \xrightarrow{i} R \xrightarrow{f} J \to 0,$$

where i is the inclusion and $f(x, y) = (x, y)(0, a)$ for every $(x, y) \in R$. Now consider a nonnil ideal $I := Da \propto D$ of R. Then

$$0 \propto D \cap RI = 0 \propto D \neq (0 \propto D)I = 0 \propto Da.$$

Thus J is not ϕ-flat by [23, Theorem 3.2]. □

Denote by $U(R)$ the set of all units of a ring R. Now we will give an example of an ideal which is ϕ-P-flat but which is neither ϕ-flat nor P-flat.

Example 2.19. Let D be a domain which is not a field, and set $R = D \propto D$. Then the ideal $J = (0, a)R$, generated by $(0, a)$ with $a \in D\backslash U(D)$, is ϕ-P-flat which is neither ϕ-flat nor P-flat.

Proof. Note that $\mathrm{Nil}(R) = 0 \propto D$ is a prime ideal of R, and so R is a PN-ring, and $Z(R) = \mathrm{Nil}(R) = 0 \propto D$. Then R is a ϕ-PF-ring.

Let $x, y \in D \setminus \{0\}$. Then $(0, x)(0, y) = 0$ and for every $(s_1, s_2) \in (0 : (0, y))$, we have $s_1 = 0$ and $(0, x) \neq (0, s_2)(0, x) = 0$. Then the ideal $(0, x)R$ is not P-flat for every $x \in D\setminus\{0\}$. In particular J is not P-flat. The ideal $J = (0, a)R$ is not ϕ-flat by Proposition 2.18. □

Remark 2.20. Recall that each P-flat cyclic R-module is flat according to [8, Proposition 1]. However, the above example shows that a ϕ-P-flat cyclic module is not always ϕ-flat.

Now we will study the transfer of the ϕ-PF-ring in the direct product.

Theorem 2.21. *Let $(R_i)_{i\in\Lambda}$ be a family of commutative rings, set $R = \prod_{i\in\Lambda} R_i$. Then R is a ϕ-PF-ring if and only if R_i is a PF-ring for all $i \in \Lambda$.*

Proof. Assume that R is a ϕ-PF-ring. Let $i_0 \in \Lambda$ and let $x_{i_0}, s_{i_0} \in R_{i_0}$ such that $x_{i_0}s_{i_0} = 0$. Set $x = (x_i)_{i\in\Lambda}$ where $x_i = 0$ if $i \neq i_0$ and $s = (s_i)_{i\in\Lambda}$ where $s_i = 1$ if $i \neq i_0$. Then we have $sx = 0$ and $s \in R \setminus \mathrm{Nil}(R)$. So there is $\alpha = (\alpha_i)_{i\in\Lambda} \in (0 : s)$ such that $\alpha x = x$. Therefore $\alpha_{i_0} \in (0 : s_{i_0})$ and $\alpha_{i_0}x_{i_0} = x_{i_0}$. So R_{i_0} is a PF-ring.

The converse follows from Theorem 2.14 and [7, Proposition 2.5]. □

Theorem 2.22. *Let R be a ϕ-PF-ring and S be a multiplicative subset of R. Then $S^{-1}R$ is a ϕ-PF-ring.*

Proof. Let $\frac{x}{t} \in S^{-1}R$ and $\frac{a}{s} \in S^{-1}R\setminus\mathrm{Nil}(S^{-1}R)$ such that $\frac{x}{t}\frac{a}{s} = 0$. Then $a \in R\setminus\mathrm{Nil}(R)$. As $\frac{ax}{st} = 0$, there is $s' \in S$ such that $s'xa = 0$. Since R is a ϕ-PF-ring, there is $\alpha \in \mathrm{Ann}(a)$ such that $s'x\alpha = s'x$. Therefore $\frac{\alpha}{1} \in \mathrm{Ann}(\frac{a}{s})$ and $\frac{x}{t}\frac{\alpha}{1} = \frac{x}{t}$. □

Let A and B be two rings. Then it is well known that the prime ideal of $A \times B$ has the form $P \times B$ with P a prime ideal of A or $A \times P$ with P a prime ideal of B. Note that if P is a prime ideal of A, then it is easy to verify that $(A \times B)_{P\times B}$ is isomorphic to A_P via the isomorphism $\frac{(a,b)}{(s,t)} \longmapsto \frac{a}{s}$.

Remark 2.23. The ϕ-PF-ring is not a local property.

Proof. Let $R = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Then R is not a ϕ-PF-ring since $\mathbb{Z}/4\mathbb{Z}$ is not a PF-ring by Theorem 2.21. On the other hand R has exactly two prime ideal $P_1 = 2\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $P_2 = \mathbb{Z}/4\mathbb{Z} \times 0$. Hence R_P is a ϕ-PF-ring for all prime ideal P of R since $R_{P_1} \cong \mathbb{Z}/4\mathbb{Z}$ and $R_{P_2} \cong \mathbb{Z}/2\mathbb{Z}$ are ϕ-PF-rings. □

The following theorem describes the localization of the ϕ-PF-rings.

Theorem 2.24. *The following conditions are equivalent for a ring R.*

(1) R is a ϕ-PF-ring.

(2) For every $a \in R \setminus \mathrm{Nil}(R)$ and any prime ideal $\mathfrak{p}$ of R, a is a nonzero divisor in $R_\mathfrak{p}$ or $a = 0$ in $R_\mathfrak{p}$.

(3) For every $a \in R \setminus \mathrm{Nil}(R)$ and any maximal ideal $\mathfrak{m}$ of R, a is a nonzero divisor in $R_\mathfrak{m}$ or $a = 0$ in $R_\mathfrak{m}$.

Proof. (1) $\Rightarrow$ (2) Let $a \in R \setminus \mathrm{Nil}(R)$ and $\mathfrak{p}$ be a prime ideal of R. Since Ra is a P-flat ideal of R, we get $aR_\mathfrak{p}$ is a flat $R_\mathfrak{p}$-module, and so it is free since $aR_\mathfrak{p}$ is a finitely generated $R_\mathfrak{p}$-module and $R_\mathfrak{p}$ is a local ring. Therefore $a = 0$ in $R_\mathfrak{p}$ or a is a nonzero divisor in $R_\mathfrak{p}$.

(2) $\Rightarrow$ (3) Straightforward.

(3) $\Rightarrow$ (1) Let $a \in R\setminus\mathrm{Nil}(R)$. We need to show that Ra is a flat R-module. Let $\mathfrak{m}$ be a maximal ideal of R. If $a = 0$ in $R_\mathfrak{m}$, then $aR_\mathfrak{m}$ is as an $R_\mathfrak{m}$-module flat since $aR_\mathfrak{m} = 0$. If a is a nonzero divisor in $R_\mathfrak{m}$, then $aR_\mathfrak{m}$ is a flat $R_\mathfrak{m}$-module since it is free. So $aR_\mathfrak{m}$ is an $R_\mathfrak{m}$-flat module for any maximal ideal $\mathfrak{m}$ of R. Since the flatness is a local property, aR is a flat R-module. Thus R is a ϕ-PF-ring by Theorem 2.2. □

Proposition 2.25. *Let R be a ϕ-PF-ring. Then $R/\operatorname{Nil}(R)$ is a PF-ring.*

Proof. Let J be a nonzero principal ideal of $R/\operatorname{Nil}(R)$. Then there is a principal nonnil ideal I of R such that $J = I/\operatorname{Nil}(R)$. Since R is a ϕ-PF-ring, I is a flat R-module. Hence J is a flat $R/\operatorname{Nil}(R)$-module by Proposition 2.10. Thus R is a PF-ring by [7, Theorem 2.1]. □

The converse of the previous proposition is not true in general as the following example shows.

Example 2.26. Let $R = \mathbb{Z} \propto \mathbb{Z}/2\mathbb{Z}$. Then $R/\operatorname{Nil}(R) \cong \mathbb{Z}$ is a PF-ring, but R is not a ϕ-PF-ring.

We next study the transfer of the ϕ-PF-ring to homomorphic images. The following example shows that the homomorphic image of a ϕ-PF-ring is not always a ϕ-PF-ring.

Example 2.27. Let $R = \mathbb{Z} \propto \mathbb{Z}$ and $I = 0 \propto 3\mathbb{Z}$. Then R is a ϕ-PF-ring and $R/I \cong \mathbb{Z} \propto \mathbb{Z}/3\mathbb{Z}$ is not a ϕ-PF-ring.

The following theorem shows that the class of ϕ-PF-rings is closed under the homomorphic images by pure ideals.

Theorem 2.28. *Let R be a ϕ-PF-ring. Then R/I is a ϕ-PF-ring for any pure ideal I of R.*

Proof. Let $a + I \in R/I \setminus \operatorname{Nil}(R/I)$. Then a is non-nilpotent, and hence $\operatorname{Ann}_R(a)$ is a pure ideal of R. Our claim now is to show that $\operatorname{Ann}_{R/I}(a + I)$ is a pure ideal of R/I. For this, consider $x + I \in \operatorname{Ann}_{R/I}(a + I)$. Then $xa \in I$. Since I is a pure ideal of R, there exists $y \in I$ such that $yxa = xa$, and so $a(yx - x) = 0$. Then $yx - x = z(yx - x)$ for some $z \in \operatorname{Ann}_R(a)$, and thus $xz - x \in I$. Therefore $(z+I)(a+I) = I$ and $(x+I)(z+I) = (x+I)$. Consequently R/I is a ϕ-PF-ring. □

Proposition 2.29. *(1) Let R be a ring and I be a primary ideal of R. Then R/I is a ϕ-PF-ring.*

(2) $\mathbb{Z}/n\mathbb{Z}$ is a ϕ-PF-ring if and only if $n = p^\alpha$ for some prime integer p or $n = p_1 \cdots p_{n_i}$, where $p_1, \ldots, p_{n_i}$ are the prime integers defined by n.

Proof. (1) As I is a primary ideal of R, then $Z(R/I) = Nil(R/I)$. Thus, R/I is a $\phi - PF-$ring.
(2) Assume that $n = p^\alpha q$ with $\alpha > 1$ and p and q are relatively prime to each other. Then $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/p^\alpha\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$ is not a ϕ-PF-ring by Theorem 2.21 since $\mathbb{Z}/p^\alpha$ is not a PF-ring. The converse is straightforward. □

Example 2.30. $\mathbb{Z}/p^n\mathbb{Z}$ is a ϕ-PF-ring for any prime number p and any integer $n \geq 2$.

Let I be an ideal of a ring R. Recall from [7, Theorem 2.7] that I is a primary ideal of R and R/I is a PF-ring if and only if I is a prime ideal of R.

Thus, to construct a ϕ-PF-ring which is not a PF-ring, it is sufficient to consider a primary ideal which is not prime. Then R/I is a ϕ-PF-ring which is not a PF-ring.

Example 2.31. (1) $\mathbb{Z}/4\mathbb{Z}$ is a ϕ-PF-ring which is not a PF-ring.

(2) Let D be a local domain whose maximal ideal $\mathfrak{m} = xD$ is principal. Let $M = D/\mathfrak{m}$ and $R = D \propto M$. Set $I = (x, 1)R$. Then R/I is a ϕ-PF-ring which is not a PF-ring.

Proof. (1) It is straightforward since $4\mathbb{Z}$ is a primary ideal of $\mathbb{Z}$ which is not prime.

(3) Note that $I = (x, 1)R$ is not a homogeneous ideal by [18, Example 2.5] (i.e., it is not of the form $J \propto N$, with J an ideal of D and N a submodule of M). On the other hand, $\sqrt{I} = \sqrt{xD} \propto M = \mathfrak{m} \propto M$ is a maximal ideal of R by [2, Theorem 3.2]. Then I

is a primary ideal of R (it is an example of a primary ideal which is not homogeneous in the trivial extension ring). So it is not prime (in fact, it is not a product of prime ideals. So we cannot apply the second result of the previous theorem). Thus R/I is a ϕ-PF-ring which is not a PF-ring. $\square$

Our next goal is to investigate the transfer of the ϕ-PF-ring to the amalgamation $A \bowtie^f J$. For this purpose, we will start with the following theorem which characterizes the case where the amalgamation $A \bowtie^f J$ is a PN-ring. We recall from [15, Proposition 2.20] that

$$\mathrm{Nil}(A \bowtie^f J) = \{(a, f(a) + j) \mid a \in \mathrm{Nil}(A), j \in J \cap \mathrm{Nil}(B)\}$$

Theorem 2.32. *Let A and B be rings, J a nonzero ideal of B, and $f : A \to B$ be a ring homomorphism.*

(1) Assume that $J \nsubseteq \mathrm{Nil}(B)$. Then $A \bowtie^f J$ is a PN-ring if and only if B is a PN-ring and $a \in \mathrm{Nil}(A)$ for every $a \in A$ such that $f(a) + j \in \mathrm{Nil}(B)$ for some $j \in J$.

(2) Assume that $J \subseteq \mathrm{Nil}(B)$. Then $A \bowtie^f J$ is a PN-ring if and only if so is A.

Proof. (1) Assume that $A \bowtie^f J$ is a PN-ring. Since $J \nsubseteq \mathrm{Nil}(B)$, there exists $j \in J$ such that $j \notin \mathrm{Nil}(B)$. Then $(0, j) \notin \mathrm{Nil}(A \bowtie^f J)$. So $0 \times J \nsubseteq \mathrm{Nil}(A \bowtie^f J)$. Since $\mathrm{Nil}(A \bowtie^f J)$ is a prime ideal of $A \bowtie^f J$,

$$\mathrm{Nil}(A \bowtie^f J) = \overline{Q}^f := \{(a, f(a) + j) \mid a \in A, j \in J, f(a) + j \in Q\}$$

for some $Q \in \mathrm{Spec}(B) \setminus V(J)$ by [12, Corollary 2.5]. Since for every $(a, f(a) + j) \in \mathrm{Nil}(A \bowtie^f J)$, we have $f(a) + j \in \mathrm{Nil}(B)$. Hence $Q = \mathrm{Nil}(B)$. Therefore B is a PN-ring. On the other hand, let $a \in A$ such that $f(a) + j \in \mathrm{Nil}(B)$ for some $j \in J$. Then $(a, f(a) + j) \in \overline{Q}^f = \mathrm{Nil}(A \bowtie^f J)$. So $a \in \mathrm{Nil}(A)$.

Conversely, assume that B is a PN-ring and $a \in \mathrm{Nil}(A)$ for every $a \in A$ such that $f(a) + j \in \mathrm{Nil}(B)$ for some $j \in J$. It is clear that $\mathrm{Nil}(A \bowtie^f J) \subseteq \overline{\mathrm{Nil}(B)}^f$. For the other inclusion, let $(a, f(a) + j) \in \overline{\mathrm{Nil}(B)}^f$. Then $f(a) + j \in \mathrm{Nil}(B)$, and so $a \in \mathrm{Nil}(A)$. Therefore $j = (f(a) + j) - f(a) \in J \cap \mathrm{Nil}(B)$, whence $(a, f(a) + j) \in \mathrm{Nil}(A \bowtie^f J)$. Thus $\mathrm{Nil}(A \bowtie^f J) = \overline{\mathrm{Nil}(B)}^f$ is a prime ideal of $A \bowtie^f J$.

(2) Assume that $J \subseteq \mathrm{Nil}(B)$. Then $\mathrm{Nil}(A \bowtie^f J) = \mathrm{Nil}(A) \bowtie^f J$ is a prime ideal of $A \bowtie^f J$ if and only if $\mathrm{Nil}(A)$ is a prime ideal of A. Hence $A \bowtie^f J$ is a PN-ring if and only if so is A. $\square$

Denote by $\mathrm{Jac}(R)$ the Jacobson radical of a ring R.

Theorem 2.33. *Let A and B be two rings, J a nonzero ideal of B, and $f : A \to B$ be a ring homomorphism.*

(1) If $A \bowtie^f J$ is a ϕ-PF-ring, then so is A.

(2) Assume that $J \nsubseteq \mathrm{Nil}(B)$, B is a PN-ring, $f^{-1}(J) \neq 0$, and $a \in \mathrm{Nil}(A)$ for every $a \in A$ such that $f(a) + j \in \mathrm{Nil}(B)$ for some $j \in J$. Then $A \bowtie^f J$ is not a ϕ-PF-ring.

(3) Assume that $J \subseteq \mathrm{Nil}(B)$ and A is a PN-ring. Then $A \bowtie^f J$ is a ϕ-PF-ring if and only if $Z(A) = \mathrm{Nil}(A)$ and $a \in \mathrm{Nil}(A)$ for every $a \in A$ such that $j'(f(a) + j) = 0$ for some $j' \in J \setminus \{0\}$ and $j \in J$.

(4) Assume that $J \subseteq \mathrm{Jac}(B)$, $f^{-1}(J) \neq 0$, and A is a local ring. Then $A \bowtie^f J$ is a ϕ-PF-ring if and only if $J \subseteq \mathrm{Nil}(B)$, $Z(A) = \mathrm{Nil}(A)$, and $a \in \mathrm{Nil}(A)$ for every $a \in A$ such that $j'(f(a) + j) = 0$ for some $j' \in J \setminus \{0\}$ and $j \in J$.

Before proving Theorem 2.33, we establish the following lemma.

Lemma 2.34. *Let R and S be rings and let $\varphi : R \to S$ be a ring homomorphism making R a module retract of S. If S is a ϕ-PF-ring, then so is R.*

Proof. Let $\Psi : S \to R$ be a ring homomorphism such that $\psi \circ \varphi = \mathrm{id}_R$. Let $(x, y) \in R \times (R \setminus \mathrm{Nil}(R))$ such that $xy = 0$. Then $\varphi(x)\varphi(y) = 0$ and $\varphi(y) \in S \setminus \mathrm{Nil}(S)$. Since S is a ϕ-PF-ring, there exists an element $\alpha \in \mathrm{Ann}_S(\varphi(x))$ such that $\varphi(y) = \alpha\varphi(y)$. So

$$y = \psi(\varphi(y)) = \psi(\alpha\varphi(y)) = \psi(\alpha)y$$

and $\psi(\alpha) \in \mathrm{Ann}(x)$ since

$$\psi(\alpha)x = \psi(\alpha)\psi(\varphi(x)) = \psi(\alpha\varphi(x)) = \psi(0) = 0.$$

Thus S is a ϕ-PF-ring. □

Proof of Theorem 2.33. (1) Assume that $A \bowtie^f J$ is a ϕ-PF-ring. As A is a retract of $A \bowtie^f J$, it follows by Lemma 2.34 that A is a ϕ-PF-ring.

(2) Assume that $J \nsubseteq \mathrm{Nil}(B)$, B is a PN-ring, and $a \in \mathrm{Nil}(A)$ for every $a \in A$ such that $f(a) + j \in \mathrm{Nil}(B)$ for some $j \in J$. Then by Theorem 2.32 $A \bowtie^f J$ is a PN-ring. Let $j \in J$ which is not in $\mathrm{Nil}(B)$. Choose any $0 \neq a \in f^{-1}(J)$. Then $(a, 0)(0, j) = 0$. Thus $(0, j) \in Z(A \bowtie^f J) \setminus \mathrm{Nil}(A \bowtie^f J)$. Therefore $A \bowtie^f J$ is not a ϕ-PF-ring by Corollary 2.6.

(3) Assume that $J \subseteq \mathrm{Nil}(B)$ and A is a PN-ring. Then by Theorem 2.32 $A \bowtie^f J$ is a PN-ring. Hence $A \bowtie^f J$ is a ϕ-PF-ring if and only if $Z(A \bowtie^f J) = \mathrm{Nil}(A \bowtie^f J)$ by Corollary 2.6.

Assume that $A \bowtie^f J$ is a ϕ-PF-ring and let $a \in Z(A)$. Then $(a, f(a)) \in Z(A \bowtie^f J) = \mathrm{Nil}(A \bowtie^f J)$. Hence $a \in \mathrm{Nil}(A)$, and so $Z(A) = \mathrm{Nil}(A)$. On the other hand, let $a \in A$ such that $j'(f(a) + j) = 0$ for some $j' \in J \setminus \{0\}$ and $j \in J$. Since $(a, f(a) + j)(0, j') = 0$, we have $(a, f(a) + j) \in Z(A \bowtie^f J) = \mathrm{Nil}(A \bowtie^f J)$. Therefore $a \in \mathrm{Nil}(A)$.

Conversely, assume that $Z(A) = \mathrm{Nil}(A)$ and $a \in \mathrm{Nil}(A)$ for every $a \in A$ such that $j'(f(a) + j) = 0$ for some $j' \in J \setminus \{0\}$ and $j \in J$. Let $(a, f(a) + j) \in Z(A \bowtie^f J)$. Since $(0, j) \in \mathrm{Nil}(A \bowtie^f J)$, $(a, f(a)) = (a, f(a) + j) - (0, j) \in Z(A \bowtie^f J)$. Hence there exists $(r, f(r) + j') \in A \bowtie^f J \setminus \{0\}$ such that $(a, f(a))(r, f(r) + j') = 0$, and so $ar = 0$ and $j'f(a) = 0$. If $r \neq 0$, then $a \in Z(A) = \mathrm{Nil}(A)$. If $r = 0$, then $j'f(a) = 0$, whence $a \in \mathrm{Nil}(A)$. So in the all cases $a \in \mathrm{Nil}(A)$. Thus $(a, f(a) + j) \in \mathrm{Nil}(A \bowtie^f J)$. Therefore $A \bowtie^f J$ is a ϕ-PF-ring.

(4) Assume that $J \subseteq \mathrm{Jac}(B)$, $f^{-1}(J) \neq 0$, and A is a local ring. Then $A \bowtie^f J$ is a local ring. Hence $A \bowtie^f J$ is a ϕ-PF-ring if and only if $Z(A \bowtie^f J) = \mathrm{Nil}(A \bowtie^f J)$.

Assume that $A \bowtie^f J$ is a ϕ-PF-ring. Let $j \in J$ and choose $0 \neq a \in f^{-1}(J)$. Then $(a, 0)(0, j) = 0$. So $(0, j) \in Z(A \bowtie^f J) = \mathrm{Nil}(A \bowtie^f J)$. Therefore $J \subseteq \mathrm{Nil}(B)$ and as in (3) we can easily deduce that $a \in \mathrm{Nil}(A)$ for every $a \in A$ such that $j'(f(a) + j) = 0$ for some $j' \in J \setminus \{0\}$ and $j \in J$.

The converse is analogous to (3). □

Corollary 2.35. *Let A be a ring and I be an ideal of A.*

(1) If $A \bowtie I$ is a ϕ-PF-ring, then so is A.
(2) If $I \nsubseteq \mathrm{Nil}(A)$, A is a PN-ring, and $a \in \mathrm{Nil}(A)$ for every $a \in A$ such that $a + i \in \mathrm{Nil}(A)$ for some $i \in I$, then $A \bowtie I$ is not a ϕ-PF-ring.
(3) Assume that $I \subseteq \mathrm{Nil}(A)$ and A is a PN-ring. Then $A \bowtie I$ is a ϕ-PF-ring if and only if $Z(A) = \mathrm{Nil}(A)$ and $a \in \mathrm{Nil}(A)$ for every $a \in A$ such that $i'(a + i) = 0$ for some $i' \in I \setminus \{0\}$ and $i \in I$.
(4) Assume that $(A, \mathfrak{m})$ is a local ring and $I \subseteq \mathfrak{m}$. Then $A \bowtie I$ is a ϕ-PF-ring if and only if $I \subseteq \mathrm{Nil}(A)$, $Z(A) = \mathrm{Nil}(A)$ and $a \in \mathrm{Nil}(A)$ for every $a \in A$ such that $i'(a + i) = 0$ for some $i' \in I \setminus \{0\}$ and $i \in I$.

Proof. If we set $f := \mathrm{id}_A$, the identity map on A, then $A \bowtie I = A \bowtie^f I$. Thus this follows immediately from Theorem 2.33. □

Corollary 2.36. *Let A be a ring and M an A-module. Set $R := A \propto M$.*

(1) If R is a ϕ-PF-ring, then so is A.

(2) *Assume that A is a PN-ring. Then R is a ϕ-PF-ring if and only if A is a ϕ-PF-ring and $Z_A(M) \subseteq \mathrm{Nil}(A)$.*

(3) *Assume that A is a local ring. Then R is a ϕ-PF-ring if and only if A is a ϕ-PF-ring and $Z_A(M) \subseteq \mathrm{Nil}(A)$.*

Proof. Consider a ring homomorphism

$$\begin{aligned} f : A &\hookrightarrow A \ltimes M \\ a &\mapsto f(a) = (a, 0) \end{aligned}$$

and a nonzero ideal $J := 0 \propto M$ of $A \propto M$. Then $A \bowtie^f J \cong A \propto M$ and $J \subseteq \mathrm{Nil}(A \propto M)$ since $J^2 = 0$.

(1) This follows immediately by Theorem 2.33.

(2) Assume that R is a ϕ-PF-ring. Then $Z(A) = \mathrm{Nil}(A)$ by Theorem 2.33. On the other hand, let $a \in Z_A(M)$. Then $am = 0$ for some $m \in M \setminus \{0\}$, and so $(a, 0)(0, m) = 0$. Then $a \in \mathrm{Nil}(A)$ by Theorem 2.33. Hence $Z(A) = \mathrm{Nil}(A)$ and $Z_A(M) \subseteq \mathrm{Nil}(A)$.

Conversely, assume that $Z(A) = \mathrm{Nil}(A)$ and $Z_A(M) \subseteq \mathrm{Nil}(A)$. Let $a \in A$ such that $j'(f(a)+j) = 0$ for some $j' \in J\backslash\{0\}$ and $j \in J$. Since $J^2 = 0$, we have $j'f(a) = (0, am') = 0$ with $j' = (0, m')$. Hence $a \in Z_A(M) \subseteq \mathrm{Nil}(A)$. Therefore R is a ϕ-PF-ring by Theorem 2.33.

(3) Assume that A is a local ring. Then R is also a local ring, and hence R is présimplifiable. Therefore R is a ϕ-PF-ring if and only if $Z(R) = \mathrm{Nil}(R)$, if and only if A is a ϕ-PF-ring and $Z_A(M) \subseteq \mathrm{Nil}(A)$. □

Corollary 2.37. *Let D be a domain and M be a D-module. Then $R = D \propto M$ is a ϕ-PF-ring if and only if M is a torsion-free D-module.*

Example 2.38. (1) $\mathbb{Z} \propto n\mathbb{Z}$ is a ϕ-PF-ring for any $n \in \mathbb{N}$.

(2) Let $M := \bigoplus_{p \in \mathcal{P}} \mathbb{Z}/p\mathbb{Z}$ and $\mathcal{P}$ is the set of all prime numbers. Then $\mathbb{Z} \propto M$ is not a ϕ-PF-ring.

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