INTERNATIONAL ELECTRONIC JOURNAL OF GEOMETRY
VOLUME 16 NO. 1 PAGE 417–434 (2023)
DOI: HTTPS://DOI.ORG/10.36890/IEJG.1148612

Normalized Null Hypersurfaces of Indefinite Kähler Manifolds

Amrinder Pal Singh, Cyriaque Atindogbe, Rakesh Kumar,* Varun Jain

(Dedicated to the memory of Prof. Dr. Krishan Lal DUGGAL (1929 - 2022))

ABSTRACT

We study null hypersurfaces of indefinite Kähler manifolds and by taking the advantages of the almost complex structure J, we select a suitable rigging ζ, which we call the $J-$rigging, on the null hypersurface. This suitable rigging enables us to build an associated Hermitian metric $\breve{g}$ on the ambient space and which is restricted into a non-degenerated metric $\widetilde{g}$ on the normalized null hypersurface. We derive Gauss-Weingarten type formulae for null hypersurface M of an indefinite Kähler manifold $\overline{M}$ with a fixed closed Killing $J-$rigging for M. Later, we establish some relations linking the curvatures, null sectional curvatures, Ricci curvatures, scalar curvatures etc. of the ambient manifold and normalized null hypersurface.

Keywords: *Indefinite Kähler manifolds; Null hypersurfaces; Rigging vector field; Induced Ricci tensor; Null sectional curvature.*
AMS Subject Classification (2020): *Primary: 53C50 ; Secondary: 53C55.*

1. Introduction

The geometry of non-degenerate submanifolds of semi-Riemannian manifolds has many similarities with Riemannian submanifolds, see [8]. However, when the induced metric on a submanifold is degenerate then its geometry is remarkably difficult and different as the known techniques result in a failure.

In [3], Bejancu and Duggal initiated the general study of arbitrary null submanifolds of semi-Riemannian manifolds. To overcome the anomaly due to the degeneracy of the induced metric, their approach consists in fixing geometric data formed by a null section and a screen distribution (or equivalently by a null section and a null transversal section) on a null hypersurface. This technique, even if it has great success, has the disadvantage to depend on two independent and arbitrary choices and seems not appropriate to study intrinsic geometry of the null hypersurfaces.

Gutierrez and Olea [6] presented a rigging technique to study the geometry of null hypersurfaces in Lorentzian spaces (see [1, 7, 11] for further works). The major advantages of rigging technique on null hypersurface are that it allows the construction of a *Riemannian metric* on the null hypersurface and the geometric data depends on a rigging vector field ζ. Thus, the geometry of null hypersurfaces can be handled with Riemannian structure coupled with rigging vector field, that is, can be studied using the well known Riemannian geometry. Recently, we studied null hypersurfaces of a Lorentzian manifold with a closed rigging for the hypersurface and derived inequalities involving Ricci tensors, scalar curvature, squared mean curvatures for a null hypersurface in [10].

In the present paper, we extend the concept of the rigging technique from a null hypersurface of Lorentzian manifolds to a null hypersurface of indefinite almost Hermitian manifolds. In Section 2, we recall basic facts about null hypersurfaces of semi-Riemannian manifolds to fix up the terminology. We construct an associated Hermitian metric $\breve{g}$ on an indefinite almost Hermitian manifold with a fixed $J-$rigging ζ and then derive an induce non-degenerate $J-$rigged metric $\widetilde{g}$ on its normalized null hypersurface in Section 3. Further, since the structures $(\overline{g}, J)$ and $(\breve{g}, J)$ are not simultaneous Kählerian (Theorem 3.2), we obtain a non-existence result for globally defined closed Killing transverse vector field in a neighborhood of the null hypersurface (Theorem

Received : *28–07–2022,* ***Accepted :*** *21–10–2022*
*** Corresponding author***

3.3). We derive some fundamental formulas and equations for the rigged connection $\widetilde{\nabla}$ of the non-degenerate $J-$rigged metric $\widetilde{g}$ in Theorem 3.6. The Levi-Civita connections $\overline{\nabla}$ and $\widetilde{\nabla}$ of $\overline{g}$ and $\widetilde{g}$, respectively are linked in Proposition 3.1. In Section 4, curvature relations and symmetries including sectional curvatures and null sectional curvatures with respect to induced degenerate metric g and induced non-degenerate $J-$rigged metric $\widetilde{g}$ are established. Finally in Section 5, we obtain a condition for a null hypersurface of an indefinite Kähler manifold with fixed $J-$rigging ζ to be a locally product manifold (Theorem 5.2).

2. Geometry of null hypersurfaces

Let M be a hypersurface of an $(m+2)-$dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$ of index $q \in \{1, \ldots, m+1\}$. Contrary to classical theory of non-degenerate hypersurfaces, the hypersurface M is said to be a null (lightlike) hypersurface of $\overline{M}$ if the normal vector bundle $TM^{\perp}$ of M is a rank one subbundle of the tangent bundle TM of M. Then, there exists a non-degenerate complementary vector bundle $S(TM)$ of $TM^{\perp}$ in TM, called the screen distribution of M such that $TM = S(TM) \oplus_{orth} TM^{\perp}$, where $\oplus_{orth}$ denotes the orthogonal direct sum. Let ξ be any null section of $TM^{\perp}$ on a coordinate neighborhood $\mathcal{U} \subset M$ then there exists a unique null section N, called the null transversal vector field of M, of a unique vector bundle $tr(TM)$, called the transversal vector bundle of M, of rank 1 in $S(TM)^{\perp}$ satisfying $\overline{g}(\xi, N) = 1$, $\overline{g}(N, N) = 0$ and $\overline{g}(N, X) = 0$, for any $X \in \Gamma(S(TM))$. Thus, $T\overline{M}$ of $\overline{M}$ is decomposed as $T\overline{M}\mid_M = TM \oplus tr(TM) = \{TM^{\perp} \oplus tr(TM)\} \oplus_{orth} S(TM)$. Since $TM^{\perp} \oplus tr(TM)$ is a Lorentz plane, then g restricted to $S(TM)$ is non-degenerate with index $q-1$.

Let $\overline{\nabla}$ be the Levi-Civita connection of $\overline{M}$ with respect to $\overline{g}$ and P be the projection morphism of $\Gamma(TM)$ onto $\Gamma(S(TM))$. Then local Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y)N \tag{2.1}$$

$$\overline{\nabla}_X N = -A_N X + \tau(X)N \tag{2.2}$$

$$\nabla_X PY = \nabla^{\star}_X PY + C(X, PY)\xi \tag{2.3}$$

$$\nabla_X \xi = -A^{\star}_{\xi} X - \tau(X)\xi \tag{2.4}$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(tr(TM))$, where ∇ and $\nabla^{\star}$ are the induced liner connections on M and $S(TM)$, respectively. B and C are the local second fundamental forms on TM and $S(TM)$, respectively. A_N and $A^{\star}_{\xi}$ are the shape operators of TM and $S(TM)$, respectively and τ is a 1-form on TM, defined by $\tau(X) = \overline{g}(\overline{\nabla}_X N, \xi)$, (for details, see [4]). The local second fundamental forms B and C are related to their shape operators as

$$B(X, Y) = g(A^{\star}_{\xi} X, Y), \quad \overline{g}(A^{\star}_{\xi} X, N) = 0, \tag{2.5}$$

$$C(X, PY) = g(A_N X, PY), \quad \overline{g}(A_N X, N) = 0. \tag{2.6}$$

Furthermore, it should be noted that the local second fundamental form B is symmetric and independent of the choice of the screen distribution and satisfies $B(X, \xi) = 0$, for any $X \in \Gamma(TM)$.

Next, in their celebrated paper, Barros and Romero [2] defined indefinite Kähler manifolds as below.

Definition 2.1. Let $(\overline{M}, \overline{g}, J)$ be an indefinite almost Hermitian manifold with a semi-Riemannian metric $\overline{g}$ and an almost complex structure J on $\overline{M}$. Let $\overline{\nabla}$ be the Levi-Civita connection on $\overline{M}$ with respect to $\overline{g}$. Then $\overline{M}$ is called an indefinite Kähler manifold if J is parallel with respect to $\overline{\nabla}$, that is $(\overline{\nabla}_X J)Y = 0$, for any $X, Y \in \Gamma(T\overline{M})$.

3. Rigging on almost Hermitian manifold

A *rigging* for M is a vector field ζ defined on an open set containing M such that $\zeta_p \notin T_p M$, for each $p \in M$. For such a rigging, let ξ denotes the unique null vector in $TM^{\perp}$ such that $\overline{g}(\zeta, \xi) = 1$. We consider the screen distribution given by $TM \cap \zeta^{\perp}$. Associated to a rigging ζ, a null transverse vector field N is given by

$$N = \zeta - \frac{1}{2}\overline{g}(\zeta, \zeta)\xi. \tag{3.1}$$

Let $(\overline{M}, \overline{g}, J)$ be a real $2m-$dimensional indefinite almost Hermitian manifold, with $\overline{g}$ a semi-Riemannian metric of index $2q$, where $0 < q < m$ and J an almost complex structure on $\overline{M}$. Let (M, g) be a null real hypersurface of

$(\overline{M}, \overline{g}, J)$, where g is the degenerate induced metric of M. The presence of an auxiliary structure J leads us to select a special rigging more adapted to it. A rigging ζ for M is said to be compatible with the structure $(\overline{g}, J)$ if $\overline{g}(\zeta, J\xi) = 0$. We will abbreviate this by writing a $J-$rigging and denote the set of such riggings by $\mathcal{R}(J)$. Then, for $\zeta \in \mathcal{R}(J)$ we have

$$\overline{g}(J\xi, \xi) = \overline{g}(J\xi, N) = \overline{g}(JN, N) = \overline{g}(\xi, JN) = 0, \quad \overline{g}(J\xi, JN) = 1.$$

We see that a $J-$rigging $\zeta \in \mathcal{R}(J)$ induces a screen distribution (we denote) $\mathcal{S}_\zeta^J$ including $J(TM^\perp)$ and $J(tr(TM))$ as rank-one subbundles. In particular, there exists a non-degenerate almost complex distribution $\mathcal{D}$ of M with respect to the almost complex structure J of $\overline{M}$, that is, $J(\mathcal{D}) = \mathcal{D}$ such that

$$T\overline{M} \mid_M = \Big(\big(J(TM^\perp) \oplus J(tr(TM)) \big) \oplus_{orth} \mathcal{D} \oplus_{orth} TM^\perp \Big) \oplus tr(TM). \tag{3.2}$$

It should be noted that $J(TM^\perp) \oplus J(tr(TM))$ is a Lorentz plane in $\mathcal{S}_\zeta^J$, since $J\xi \in \mathcal{S}_\zeta^J$ and $JN \in \mathcal{S}_\zeta^J$ are lightlike and $g(J\xi, JN) = 1$. From now on, unless otherwise stated, only $J-$riggings will be in consideration.

Example 3.1. Let the $6-$dimensional space $\overline{M} = \mathbb{R}^6$ be endowed with an indefinite almost Hermitian structure $(\overline{g}, J)$ be given by $\overline{g} = -dx_0^2 - dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2$ and $J(x_0, x_1, x_2, x_3, x_4, x_5) = (-x_1, x_0, -x_3, x_2, -x_5, x_4)$ in the natural rectangular coordinates $(x_0, x_1, x_2, x_3, x_4, x_5)$. Consider a Monge hypersurface M of $\mathbb{R}_2^6$ defined by

$$M = \{(x_0, x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}_2^6 \mid x_0 = x_1 + x_2 + x_3\}.$$

Then, a local frame of the tangent bundle TM is given by

$$\Big\{ V_1 = \partial x_0 + \partial x_1, V_2 = \partial x_0 + \partial x_2, V_3 = \partial x_0 + \partial x_3, V_4 = \partial x_4, V_5 = \partial x_5 \Big\}$$

and

$$TM^\perp = Span\Big\{ \xi = -\frac{1}{2}\{V_2 + V_3 - V_1\} = -\frac{1}{2}\{\partial x_0 - \partial x_1 + \partial x_2 + \partial x_3\} \Big\}.$$

Thus, M is a lightlike hypersurface of $(\overline{M}, \overline{g}, J)$ and $J\xi = -\frac{1}{2}\{\partial x_0 + \partial x_1 - \partial x_2 + \partial x_3\}$. Moreover, $\overline{g}(\xi, \partial x_0 - \partial x_1) = 1$ and $\overline{g}(\partial x_0 - \partial x_1, J\xi) = 0$. So, a (compatible) $J-$rigging for M can be selected by setting $\zeta = \partial x_0 - \partial x_1$. It induces a screen distribution $\mathcal{S}_\zeta^J = \mathrm{span}\{\partial x_0 + \partial x_1, \partial x_2 - \partial x_3, \partial x_4, \partial x_5\}$.

Let (M, g) be a null hypersurface of a real $2m-$dimensional indefinite almost Hermitian manifold $(\overline{M}, \overline{g}, J)$ and $i : M \hookrightarrow \overline{M}$ be the canonical inclusion map. Let $\zeta \in \mathcal{R}(J)$ be a $J-$rigging for M, α be a $1-$form $\overline{g}-$metrically equivalent to ζ, that is, $\alpha(.) = \overline{g}(\zeta, .)$. Then, we define an associated metric $\breve{g}$ on $\overline{M}$ as

$$\breve{g}(\overline{U}, \overline{V}) = \overline{g}(\overline{U}, \overline{V}) + \alpha(\overline{U})\alpha(\overline{V}) + \alpha(J\overline{U})\alpha(J\overline{V}), \tag{3.3}$$

for any $\overline{U}, \overline{V} \in \Gamma(T\overline{M})$.

Lemma 3.1. *For $p \in \overline{M}$, $\breve{g}_p$ is non-degenerate if and only if $\overline{g}(\zeta_p, \zeta_p) \neq -1$.*

Proof. Assume $\overline{g}(\zeta_p, \zeta_p) \neq -1$ and let $u \in T_p\overline{M}$, $p \in \overline{M}$ such that $\breve{g}_p(u, v) = 0$ for any $v \in T_p\overline{M}$. In particular, for $v = \zeta_p$, $\breve{g}_p(u, \zeta_p) = 0$ implies that $\overline{g}_p(u, \zeta_p)\{1 + \overline{g}_p(\zeta_p, \zeta_p)\} = 0$ and consequently, $\overline{g}_p(u, \zeta_p) = \alpha(u) = 0$. Similarly, $\breve{g}_p(u, J\zeta_p) = 0$ implies that $\alpha(Ju) = 0$. Hence, $\breve{g}_p(u, v) = 0$ implies $\overline{g}_p(u, v) = 0$, for any $v \in T_p\overline{M}$ and then the non-degeneracy of $\overline{g}_p$ implies that $u = 0$. Thus, $\breve{g}_p$ is non-degenerate on $\overline{M}$.

Conversely, assume $\overline{g}(\zeta_p, \zeta_p) = -1$. Then, for every $v \in T_p\overline{M}$

$$\breve{g}_p(\zeta_p, v) = \overline{g}(\zeta_p, v) + \alpha(\zeta_p)\alpha(v) + \alpha(J\zeta_p)\alpha(Jv) = \alpha(v) - \alpha(v) = 0.$$

Hence $\breve{g}_p$ is degenerate and $\zeta_p \in \ker(\breve{g}_p)$. □

Remark 3.1. (i) If $\breve{g}$ is Riemannian, then either ζ_p is spacelike or ζ_p is timelike and $\mid \zeta_p \mid > 1$.
(ii) $\breve{g}$ is degenerate if and only if ζ_p is timelike and unitary for $\overline{g}$.

For any $\overline{U}, \overline{V} \in \Gamma(T\overline{M})$, using (3.3), it is obvious that

$$\breve{g}(J\overline{U}, J\overline{V}) = \overline{g}(\overline{U}, \overline{V}) + \alpha(J\overline{U})\alpha(J\overline{V}) + \alpha(\overline{U})\alpha(\overline{V}) = \breve{g}(\overline{U}, \overline{V});$$

this implies that $(\breve{g}, J)$ is also an almost Hermitian structure on $\overline{M}$.

Let $\breve{g}$ be non-degenerate then for a fixed $J-$rigging $\zeta \in \mathcal{R}(J)$ for M. We consider the induced associated metric $\widetilde{g}$ on M as

$$\widetilde{g}(U, V) = i^{\star}\breve{g}(U, V) = g(U, V) + \omega(U)\omega(V) + \alpha(i_{\star}JU)\alpha(i_{\star}JV), \tag{3.4}$$

where $\omega = i^{\star}\alpha$ being $i : M \hookrightarrow \overline{M}$ the inclusion map and $U, V \in \Gamma(TM)$. Using the definition of ω it is obvious that $\omega(\xi) = 1$.

Lemma 3.2. *Let $\overline{g}(\zeta, \zeta) \neq 1$ then for each $p \in M$, $\widetilde{g}_p$ is non-degenerate.*

Proof. Let $p \in M$ and $u \in T_pM$ such that $\widetilde{g}_p(u, v) = 0$ for all $v \in T_pM$, that is

$$\widetilde{g}_p(u, v) = g(u, v) + \omega(u)\omega(v) + \overline{g}(\zeta, Ju)\overline{g}(\zeta, Jv) = 0,$$

for all $v \in T_pM$. Set $v = \xi$. Then, $0 = \omega(u) + \overline{g}(\zeta, Ju)\overline{g}(\zeta, J\xi) = \omega(u)$, since $\zeta \in \mathcal{R}(J)$ and $\overline{g}(\zeta, J\xi) = 0$. Hence $u \in \mathcal{S}^J_\zeta$.

Now, setting $v = J\zeta$ leads to

$$0 \quad = \quad g(u, J\zeta) - \overline{g}(\zeta, Ju)\overline{g}(\zeta, \zeta) = -\overline{g}(Ju, \zeta)\big[1 + \overline{g}(\zeta, \zeta)\big].$$

Then $\overline{g}(Ju, \zeta) = 0$. Combining above two facts, it follows that $g_p(u, v) = 0$ for all $v \in TM$, which implies that $u \in T_pM^{\perp}$.

Finally, we get $u \in \mathcal{S}^J_\zeta \cap T_pM^{\perp} = \{0\}$, that is, $u = 0$ and $\widetilde{g}_p$ is non-degenerate. □

From now on, we consider $\overline{g}(\zeta, \zeta) \neq 1$, unless otherwise stated. For the fixed $J-$rigging ζ for M, we call the associated non-degenerate metric $\widetilde{g}$ on M the *$J-$rigged metric* on the normalized null hypersurface (M, ζ). Let $p \in M$ and $u \in T_pM$, we have

$$\widetilde{g}(\xi, u) = g(\xi, u) + \omega(\xi)\omega(u) + \overline{g}(\zeta, J\xi)\overline{g}(\zeta, Ju) = \overline{g}(\zeta, \xi)\omega(u) = \omega(u).$$

Hence, ξ is the $\widetilde{g}-$metrically equivalent vector field to the $1-$form ω. For this, it is called the *rigged vector field* on (M, ζ). In particular, ξ is $\widetilde{g}-$unitary. Also, for $u \in T_pM$, we have

$$u \in \mathcal{S}^J_\zeta \;\Leftrightarrow\; 0 = \overline{g}(\zeta, u) = \alpha(u) = \omega(u) = \widetilde{g}(\xi, u).$$

Thus, $\mathcal{S}^J_\zeta = \ker \widetilde{g}(\xi, \cdot)$ meaning that the screen space $\mathcal{S}^J_{\zeta\,|p}$ at $p \in M$ is the $\widetilde{g}-$orthogonal to ξ_p in M.

Theorem 3.1. *Let $\overline{\nabla}$ and $\breve{\nabla}$ be the Levi-Civita connections on indefinite almost Hermitian manifolds $(\overline{M}, \overline{g}, J)$ and $(\overline{M}, \breve{g}, J)$, respectively. Then for any $\overline{U}, \overline{V}, \overline{W} \in \Gamma(T\overline{M})$, we have*

$$\begin{aligned}
2\breve{g}(\breve{\nabla}_{\overline{U}}\overline{V}, \overline{W}) \quad = \quad & 2\overline{g}(\overline{\nabla}_{\overline{U}}\overline{V}, \overline{W}) + \Big\{(L_\zeta\overline{g})(\overline{U}, \overline{V}) + 2\alpha(\overline{\nabla}_{\overline{U}}\overline{V})\Big\}\alpha(\overline{W}) \\
& + d\alpha(\overline{U}, \overline{W})\alpha(\overline{V}) + d\alpha(\overline{V}, \overline{W})\alpha(\overline{U}) \\
& + \Big\{\overline{U}\alpha(J\overline{V}) + \overline{V}\alpha(J\overline{U}) + \alpha(J[\overline{U}, \overline{V}])\Big\}\alpha(J\overline{W}) \\
& + \Big\{\overline{U}\alpha(J\overline{W}) - \overline{W}\alpha(J\overline{U}) - \alpha(J[\overline{U}, \overline{W}])\Big\}\alpha(J\overline{V}) \\
& + \Big\{\overline{V}\alpha(J\overline{W}) - \overline{W}\alpha(J\overline{V}) - \alpha(J[\overline{V}, \overline{W}])\Big\}\alpha(J\overline{U}),
\end{aligned} \tag{3.5}$$

where L_ζ is the Lie derivative along ζ.

Proof. By Koszul formula for the connection $\breve{\nabla}$ and using equation (3.3), we obtain

$$\begin{aligned}
2\breve{g}(\breve{\nabla}_{\overline{U}}\overline{V}, \overline{W}) \quad = \quad & 2\overline{g}(\overline{\nabla}_{\overline{U}}\overline{V}, \overline{W}) + \{\overline{U}\alpha(\overline{V}) + \overline{V}\alpha(\overline{U}) + \alpha([\overline{U}, \overline{V}])\}\alpha(\overline{W}) \\
& + \{\overline{U}\alpha(\overline{W}) - \overline{W}\alpha(\overline{U}) - \alpha([\overline{U}, \overline{W}])\}\alpha(\overline{V}) \\
& + \{\overline{V}\alpha(\overline{W}) - \overline{W}\alpha(\overline{V}) - \alpha([\overline{V}, \overline{W}])\}\alpha(\overline{U}) \\
& + \{\overline{U}\alpha(J\overline{V}) + \overline{V}\alpha(J\overline{U}) + \alpha(J[\overline{U}, \overline{V}])\}\alpha(J\overline{W}) \\
& + \{\overline{U}\alpha(J\overline{W}) - \overline{W}\alpha(J\overline{U}) - \alpha(J[\overline{U}, \overline{W}])\}\alpha(J\overline{V}) \\
& + \{\overline{V}\alpha(J\overline{W}) - \overline{W}\alpha(J\overline{V}) - \alpha(J[\overline{V}, \overline{W}])\}\alpha(J\overline{U}).
\end{aligned} \tag{3.6}$$

It is known that

$$(L_{\zeta}\overline{g})(\overline{U},\overline{V}) = \overline{U}(\alpha(\overline{V})) - \alpha(\overline{\nabla}_{\overline{U}}\overline{V}) + \overline{V}(\alpha(\overline{U})) - \alpha(\overline{\nabla}_{\overline{V}}\overline{U}) \tag{3.7}$$

and

$$d\alpha(\overline{U},\overline{V}) = \overline{U}(\alpha(\overline{V})) - \overline{V}(\alpha(\overline{U})) - \alpha([\overline{U},\overline{V}]). \tag{3.8}$$

By using the above facts in (3.6), the proof is complete. □

Now, by replacing $\overline{V}$ by $J\overline{V}$ and $\overline{W}$ by $J\overline{W}$ in (3.5) and by adding the resulting expression with (3.5), we derive

$$\begin{aligned}
&2\breve{g}((\breve{\nabla}_{\overline{U}}J)\overline{V},\overline{W}) = 2\overline{g}((\overline{\nabla}_{\overline{U}}J)\overline{V},\overline{W}) + \Big\{(L_{\zeta}\overline{g})(\overline{U},J\overline{V}) + 2\alpha(\overline{\nabla}_{\overline{U}}J\overline{V}) - \overline{U}\alpha(J\overline{V}) - \overline{V}\alpha(J\overline{U})\\
&-\alpha(J[\overline{U},\overline{V}])\Big\}\alpha(\overline{W}) + \Big\{d\alpha(J\overline{V},\overline{W}) + d\alpha(\overline{V},J\overline{W})\Big\}\alpha(\overline{U})\\
&+\Big\{-\overline{U}\alpha(J\overline{W}) + \overline{W}\alpha(J\overline{U}) + \alpha(J[\overline{U},\overline{W}]) + d\alpha(\overline{U},J\overline{W})\Big\}\alpha(\overline{V})\\
&+\Big\{J\overline{V}\alpha(J\overline{W}) + \overline{W}\alpha(\overline{V}) - \alpha(J[J\overline{V},\overline{W}]) - \overline{V}\alpha(\overline{W}) - J\overline{W}\alpha(J\overline{V}) - \alpha(J[\overline{V},J\overline{W}])\Big\}\alpha(J\overline{U})\\
&+\Big\{d\alpha(\overline{U},\overline{W}) - \overline{U}\alpha(\overline{W}) - J\overline{W}\alpha(J\overline{U}) - \alpha(J[\overline{U},J\overline{W}])\Big\}\alpha(J\overline{V})\\
&+\Big\{-\overline{U}\alpha(\overline{V}) + J\overline{V}\alpha(J\overline{U}) + \alpha(J[\overline{U},J\overline{V}]) + (L_{\zeta}\overline{g})(\overline{U},\overline{V}) + 2\alpha(\overline{\nabla}_{\overline{U}}\overline{V})\Big\}\alpha(J\overline{W}).
\end{aligned} \tag{3.9}$$

Hence from (3.9), we have the following observation immediately.

Theorem 3.2. *The structures $(\overline{g},J)$ and $(\breve{g},J)$ are not simultaneously Kählerian.*

It is known that

$$(\overline{\nabla}_{\overline{U}}\alpha)\overline{V} = \overline{U}(\alpha(\overline{V})) - \alpha(\overline{\nabla}_{\overline{U}}\overline{V}) = \overline{g}(\overline{\nabla}_{\overline{U}}\zeta,\overline{V}). \tag{3.10}$$

Therefore, (3.7) and (3.8) can be respectively written as

$$(L_{\zeta}\overline{g})(\overline{U},\overline{V}) = (\overline{\nabla}_{\overline{U}}\alpha)\overline{V} + (\overline{\nabla}_{\overline{V}}\alpha)\overline{U} = \overline{g}(\overline{\nabla}_{\overline{U}}\zeta,\overline{V}) + \overline{g}(\overline{\nabla}_{\overline{V}}\zeta,\overline{U}), \tag{3.11}$$

$$d\alpha(\overline{U},\overline{V}) = (\overline{\nabla}_{\overline{U}}\alpha)\overline{V} - (\overline{\nabla}_{\overline{V}}\alpha)\overline{U} = \overline{g}(\overline{\nabla}_{\overline{U}}\zeta,\overline{V}) - \overline{g}(\overline{\nabla}_{\overline{V}}\zeta,\overline{U}). \tag{3.12}$$

Let $(\overline{M},\overline{g},\overline{\nabla},J)$ be an indefinite Kähler manifold and $\overline{U},\overline{V},\overline{W} \in \Gamma(T\overline{M})$. Using (3.10) to (3.12) in (3.9), we derive

$$\begin{aligned}
&2\breve{g}((\breve{\nabla}_{\overline{U}}J)\overline{V},\overline{W}) = \{(L_{\zeta}\overline{g})(\overline{U},J\overline{V}) + (L_{J\zeta}\overline{g})(\overline{U},\overline{V})\}\alpha(\overline{W})\\
&+\Big\{(L_{\zeta}\overline{g})(\overline{W},J\overline{V}) - (L_{\zeta}\overline{g})(J\overline{W},\overline{V}) + 2\{\overline{g}(\overline{\nabla}_{\overline{W}}J\zeta,\overline{V}) - \overline{g}(\overline{\nabla}_{\overline{V}}J\zeta,\overline{W})\}\Big\}\alpha(\overline{U})\\
&+\{\overline{g}(\overline{\nabla}_{J\overline{V}}\zeta,J\overline{U}) + \overline{g}(\overline{\nabla}_{\overline{V}}\zeta,\overline{U})\}\alpha(J\overline{W}) - \{(L_{\zeta}\overline{g})(\overline{U},J\overline{W}) + (L_{J\zeta}\overline{g})(\overline{U},\overline{W})\}\alpha(\overline{V})\\
&+\{d\alpha(J\overline{V},J\overline{W}) + d\alpha(\overline{W},\overline{V})\}\alpha(J\overline{U}) - \{\overline{g}(\overline{\nabla}_{J\overline{W}}\zeta,J\overline{U}) + \overline{g}(\overline{\nabla}_{\overline{W}}\zeta,\overline{U})\}\alpha(J\overline{V}).
\end{aligned} \tag{3.13}$$

Now,

$$(L_{J\zeta}\overline{g})(\overline{U},\overline{V}) = -\overline{U}\alpha(J\overline{V}) + \alpha(\overline{\nabla}_{\overline{U}}J\overline{V}) - \overline{V}\alpha(J\overline{U}) + \alpha(\overline{\nabla}_{\overline{V}}J\overline{U}). \tag{3.14}$$

Therefore from (3.7), (3.8) and (3.14), we can easily derive

$$(L_{\zeta}\overline{g})(\overline{U},J\overline{V}) + (L_{J\zeta}\overline{g})(\overline{U},\overline{V}) = (\overline{\nabla}_{J\overline{V}}\alpha)\overline{U} - (\overline{\nabla}_{\overline{V}}\alpha)\overline{JU}. \tag{3.15}$$

Using (3.11) and (3.12), we get

$$d\alpha(J\overline{V},\overline{W}) + d\alpha(\overline{V},J\overline{W}) = (L_{\zeta}\overline{g})(\overline{W},J\overline{V}) - (L_{\zeta}\overline{g})(J\overline{W},\overline{V}) + 2\{\overline{g}(\overline{\nabla}_{\overline{W}}J\zeta,\overline{V}) - \overline{g}(\overline{\nabla}_{\overline{V}}J\zeta,\overline{W})\}. \tag{3.16}$$

Using (3.10), we derive

$$\overline{g}(\overline{\nabla}_{J\overline{V}}\zeta,J\overline{U}) + \overline{g}(\overline{\nabla}_{\overline{V}}\zeta,\overline{U}) = (\overline{\nabla}_{J\overline{V}}\alpha)J\overline{U} + (\overline{\nabla}_{\overline{V}}\alpha)\overline{U}. \tag{3.17}$$

Hence, by using (3.12), (3.15), (3.16), (3.17) in (3.13), for an indefinite Kähler manifold $(\overline{M}, \overline{g}, \overline{\nabla}, J)$ with a fixed $J-$rigging ζ, we have the following expression

$$\begin{aligned}
&2\breve{g}((\breve{\nabla}_{\overline{U}}J)\overline{V}, \overline{W}) = \{(\overline{\nabla}_{J\overline{V}}\alpha)\overline{U} - (\overline{\nabla}_{\overline{V}}\alpha)J\overline{U}\}\alpha(\overline{W}) + \{(\overline{\nabla}_{J\overline{V}}\alpha)\overline{W} \\
&-(\overline{\nabla}_{\overline{W}}\alpha)J\overline{V} + (\overline{\nabla}_{\overline{V}}\alpha)J\overline{W} - (\overline{\nabla}_{J\overline{W}}\alpha)\overline{V}\}\alpha(\overline{U}) + \{(\overline{\nabla}_{J\overline{V}}\alpha)J\overline{U} \\
&+(\overline{\nabla}_{\overline{V}}\alpha)\overline{U}\}\alpha(J\overline{W}) - \{(\overline{\nabla}_{J\overline{W}}\alpha)\overline{U} - (\overline{\nabla}_{\overline{W}}\alpha)J\overline{U}\}\alpha(\overline{V}) \\
&+\{(\overline{\nabla}_{J\overline{V}}\alpha)J\overline{W} - (\overline{\nabla}_{J\overline{W}}\alpha)J\overline{V} + (\overline{\nabla}_{\overline{W}}\alpha)\overline{V} - (\overline{\nabla}_{\overline{V}}\alpha)\overline{W}\}\alpha(J\overline{U}) \\
&-\{(\overline{\nabla}_{J\overline{W}}\alpha)J\overline{U} + (\overline{\nabla}_{\overline{W}}\alpha)\overline{U}\}\alpha(J\overline{V}). \qquad (3.18)
\end{aligned}$$

Lemma 3.3. *The $1-$form α is parallel with respect to Levi-Civita connection $\overline{\nabla}$ of $\overline{M}$ if and only if ζ is a closed Killing vector field.*

Proof. Proof follows directly using (3.11) and (3.12). □

Theorem 3.3. *Let $(\overline{M}, \overline{g}, J)$ be an indefinite Kähler manifold. There exists no closed Killing vector field ζ which is (globally) transversal to a null hypersurface in $\overline{M}$ and whose image $J\zeta$ is tangent to it.*

Proof. Assume that there exists a closed Killing vector field ζ transversal to a null hypersurface with required properties. Then using Lemma 3.3 in (3.18), we get that $(\breve{g}, J)$ is a Kähler structure and since $(\overline{M}, \overline{g}, J)$ is also an indefinite Kähler structure, this contradicts Theorem 3.2. □

Furthermore, if $(\overline{M}, \overline{g}, J)$ is an indefinite Kähler manifold, then we can rewrite (3.5) as below

$$\begin{aligned}
2\breve{g}(\breve{\nabla}_{\overline{U}}\overline{V}, \overline{W}) \;=\; & 2\overline{g}(\overline{\nabla}_{\overline{U}}\overline{V}, \overline{W}) + \Big\{(\overline{\nabla}_{\overline{U}}\alpha)\overline{V} + (\overline{\nabla}_{\overline{V}}\alpha)\overline{U} + 2\alpha(\overline{\nabla}_{\overline{U}}\overline{V})\Big\}\alpha(\overline{W}) \\
&+\Big\{(\overline{\nabla}_{\overline{U}}\alpha)J\overline{V} + (\overline{\nabla}_{\overline{V}}\alpha)J\overline{U} + 2\alpha(\overline{\nabla}_{\overline{U}}J\overline{V})\Big\}\alpha(J\overline{W}) \\
&+\{(\overline{\nabla}_{\overline{U}}\alpha)\overline{W} - (\overline{\nabla}_{\overline{W}}\alpha)\overline{U}\}\alpha(\overline{V}) + \{(\overline{\nabla}_{\overline{U}}\alpha)J\overline{W} \\
&-(\overline{\nabla}_{\overline{W}}\alpha)J\overline{U}\}\alpha(J\overline{V}) + \{(\overline{\nabla}_{\overline{V}}\alpha)\overline{W} - (\overline{\nabla}_{\overline{W}}\alpha)\overline{V}\}\alpha(\overline{U}) \\
&+\{(\overline{\nabla}_{\overline{V}}\alpha)J\overline{W} - (\overline{\nabla}_{\overline{W}}\alpha)J\overline{V}\}\alpha(J\overline{U}). \qquad (3.19)
\end{aligned}$$

Hence on using (3.19), it yields the following result.

Theorem 3.4. *Let $(\overline{M}, \overline{g}, J)$ be an indefinite Kähler manifold with a fixed closed Killing $J-$rigging $\zeta \in \mathcal{R}(J)$ for a null hypersurface $M \subset \overline{M}$. Then*

$$\begin{aligned}
\breve{g}(\breve{\nabla}_{\overline{U}}\overline{V}, \overline{W}) &= \overline{g}(\overline{\nabla}_{\overline{U}}\overline{V}, \overline{W}) + \alpha(\overline{\nabla}_{\overline{U}}\overline{V})\alpha(\overline{W}) + \alpha(J\overline{\nabla}_{\overline{U}}\overline{V})\alpha(J\overline{W}) \\
&= \breve{g}(\overline{\nabla}_{\overline{U}}\overline{V}, \overline{W}), \qquad (3.20)
\end{aligned}$$

for any $\overline{U}, \overline{V}, \overline{W} \in \Gamma(T\overline{M})$.

Next, let $\overline{\nabla}$ and $\widetilde{\nabla}$ be the Levi-Civita connections of $\overline{g}$ and $\widetilde{g}$, respectively. Consider $\overline{D} = \overline{\nabla} - \widetilde{\nabla}$ is the difference tensor and which is symmetric tensor on TM. A straightforward computation leads to the following expression.

Proposition 3.1. *Let $U, V, W \in TM$; then it holds that*

$$\begin{aligned}
\overline{g}(\overline{D}(U,V), W) \;&= -\frac{1}{2}\Big\{(L_{\xi}\widetilde{g})(U,V)\omega(W) + d\omega(U,W)\omega(V) + d\omega(V,W)\omega(U)\Big\} \\
&+\frac{1}{2}\Big\{W(\alpha(i_{\star}JU)) - U(\alpha(i_{\star}JW)) - \alpha(i_{\star}J[W,U])\Big\}\alpha(i_{\star}JV) \\
&-\frac{1}{2}\Big\{V(\alpha(i_{\star}JW)) - W(\alpha(i_{\star}JV)) - \alpha(i_{\star}J[V,W])\Big\}\alpha(i_{\star}JU) \\
&-\frac{1}{2}\Big\{V(\alpha(i_{\star}JU)) + U(\alpha(i_{\star}JV)) + \alpha(i_{\star}J[V,U])\Big\}\alpha(i_{\star}JW) \\
&+\alpha(i_{\star}J\widetilde{\nabla}_{V}U)\alpha(i_{\star}JW). \qquad (3.21)
\end{aligned}$$

Take $D = \nabla - \widetilde{\nabla}$, where ∇ is a linear connection induced from $\overline{\nabla}$ on M and $\widetilde{\nabla}$ is the Levi-Civita connection of $\widetilde{g}$. It should be noted that D is symmetric and satisfies $\overline{D} - D = B \otimes N$. For any $U, V, W \in \Gamma(TM)$, we have

$$\overline{g}(D(U,V), W) = \overline{g}(\overline{D}(U,V), W) - B(U,V)\omega(W). \qquad (3.22)$$

Hence using (3.21) and (3.22), we have the following important corollary.

Corollary 3.1. *Let $U \in \Gamma(TM)$ and $X, Y, Z \in \Gamma(\mathcal{D})$; we have the following relations*

(1) $\widetilde{g}(D(X,U),X) = \overline{g}(\overline{D}(X,U),X) = 0.$

(2) $\widetilde{g}(D(X,Y),Z) = \overline{g}(\overline{D}(X,Y),Z) = 0.$

(3) $\overline{g}(\overline{D}(X,\xi),\xi) = 0$ *and* $\widetilde{g}(D(U,\xi),\xi) = -\tau(U) = -\overline{g}(\overline{\nabla}_U\zeta,\xi).$

(4) $-2C(U,X) = d\alpha(U,X) + (L_\zeta g)(U,X) + \overline{g}(\zeta,\zeta)B(U,X).$

A vector field W on $(\overline{M},\overline{g})$ is said to be concircular if it satisfies $\overline{\nabla}_U W = \lambda\, U$, for some smooth function λ on $\overline{M}$ and for any vector field U tangent to $\overline{M}$. Observe that concircular is equivalent to being closed and conformal.

Theorem 3.5. *Let $(\overline{M},\overline{g},J)$ be an indefinite Kähler manifold and $M \subset \overline{M}$ be a null hypersurface furnished with a concircular $J-$rigging ζ. Then*

(i) $\tau = 0.$

(ii) $B(J\zeta,U) = \lambda g(U,\mathcal{V})$ *with* $\mathcal{V} = -J\xi.$

(iii) $C(U,PV) = -\lambda\overline{g}(U,V) - \frac{1}{2}\overline{g}(\zeta,\zeta)B(U,V)$, *for* $U, V \in \Gamma(TM).$

Proof. The rigging ζ satisfies $\overline{\nabla}\zeta = \lambda \otimes I$, for $\lambda \in C^\infty(\overline{M})$, where I denotes the identity map on $\mathfrak{X}(\overline{M})$. In particular, $\overline{g}(\overline{\nabla}_U\zeta,V) = \lambda\overline{g}(U,V)$. Assertion (i) follows from item (3) in the Corollary 3.1. For (ii), note that from Kähler condition, we have

$$B(J\zeta,U) = \overline{g}(\overline{\nabla}_U J\zeta,\xi) = \overline{g}(J\overline{\nabla}_U\zeta,\xi) = \overline{g}(J(\lambda U),\xi) = -\lambda g(U,J\xi) = \lambda g(U\mathcal{V}).$$

Now, let $U \in \mathfrak{X}(M)$ and $X \in \mathcal{S}_\zeta^J$; then

$$\begin{aligned} C(U,X) &= \overline{g}(\nabla_U X,N) = \overline{g}(\overline{\nabla}_U X,N) = U\cdot\overline{g}(X,N) - \overline{g}(X,\overline{\nabla}_U N) \\ &= -\overline{g}(X,\overline{\nabla}_U(\zeta - \frac{1}{2}\overline{g}(\zeta,\zeta)\xi)) \\ &= -\lambda\overline{g}(X,U) + \frac{1}{2}\overline{g}(\zeta,\zeta)\overline{g}(\overline{\nabla}_U\xi,X) \\ &= -\lambda\overline{g}(U,X) - \frac{1}{2}\overline{g}(\zeta,\zeta)B(U,X). \end{aligned}$$

This completes the proof. ☐

Corollary 3.2. *Let $(\overline{M},\overline{g},J)$ be an indefinite Kähler manifold with a fixed closed Killing $J-$rigging $\zeta \in \mathcal{R}(J)$ for a null hypersurface $M \subset \overline{M}$. Then*

(i) $\tau = 0.$

(ii) $B(J\zeta,\cdot) = 0.$

(iii) $C(U,PV) = -\frac{1}{2}\overline{g}(\zeta,\zeta)B(U,V) = -\alpha(N)B(U,V)$, *for* $U, V \in \Gamma(TM)$. *Moreover,* $C(J\zeta,PU) = 0$ *and* $C(U,J\zeta) = 0$ *.*

Let $\widetilde{\nabla}$ be the Levi-Civita connection induced on null hypersurface M of an indefinite almost Hermitian manifold with a fixed $J-$rigging $\zeta \in \mathcal{R}(J)$. This connection is said to be an induced rigged connection on M by $\widetilde{g}$. In the following theorem, we derive Gauss and Weingarten type formulae for the screen space $\mathcal{S}_\zeta^J$ of the null hypersurface.

Theorem 3.6. *Let $(\overline{M},\overline{g},J)$ be an indefinite Kähler manifold with a fixed closed Killing $J-$rigging $\zeta \in \mathcal{R}(J)$ for a null hypersurface $M \subset \overline{M}$. Then for any $U \in \Gamma(TM)$ and $X \in \Gamma(\mathcal{S}_\zeta^J)$, we have*

$$\widetilde{\nabla}_U X = \nabla^\star_U X + B(U,X)\xi, \tag{3.23}$$

$$\widetilde{\nabla}_U\xi = -A^\star_\xi U. \tag{3.24}$$

Proof. Let $U, V \in \Gamma(TM)$ and $W \in \Gamma(\mathcal{S}^J_\zeta)$, then from (3.20), we have $\breve{g}(\widetilde{\nabla}_U V, W) = \breve{g}(\nabla_U V, W)$, which further implies

$$\widetilde{g}(\widetilde{\nabla}_U V, W) = \widetilde{g}(\nabla_U V, W). \tag{3.25}$$

Let $V = \xi$. Then using (2.4), we get $\widetilde{g}(\widetilde{\nabla}_U \xi, W) = \widetilde{g}(-A^\star_\xi U, W)$. Since $\widetilde{\nabla}$ is a Levi-Civita connection of $\widetilde{g}$, then the equality $(\widetilde{\nabla}_U \widetilde{g})(\xi, \xi) = 0$ implies that $\omega(\widetilde{\nabla}_U \xi) = \widetilde{g}(\widetilde{\nabla}_U \xi, \xi) = 0$, i.e., $\widetilde{\nabla}_U \xi \in \Gamma(\mathcal{S}^J_\zeta)$. Hence, the non-degeneracy of $\widetilde{g}$ on $\mathcal{S}^J_\zeta$, gives (3.24). Now, let $V = X \in \Gamma(\mathcal{S}^J_\zeta)$ in (3.25) then analogously, we obtain

$$\widetilde{\nabla}^\star_U X = \nabla^\star_U X. \tag{3.26}$$

Hence on using (3.26), we have $\widetilde{\nabla}_U X = \nabla^\star_U X + \gamma\xi$, for any $U \in \Gamma(TM)$ and $X \in \Gamma(\mathcal{S}^J_\zeta)$ and this further implies $\widetilde{\nabla}_U X = \nabla^\star_U X - \widetilde{g}(\widetilde{\nabla}_U \xi, X)\xi$. Using (2.5), (3.4), (3.24) and the Corollary 3.2, we obtain

$$\begin{aligned} \widetilde{g}(X, \widetilde{\nabla}_U \xi) &= -\widetilde{g}(A^\star_\xi U, X) = -B(U, X) - \alpha(i_\star J A^\star_\xi U)\alpha(i_\star JX) \\ &= -B(U, X) + B(U, J\zeta)\alpha(i_\star JX) = -B(U, X). \end{aligned}$$

Hence, the proof is complete. □

Corollary 3.3. *Let $(\overline{M}, \overline{g}, J)$ be an indefinite Kähler manifold with a fixed closed Killing $J-$rigging $\zeta \in \mathcal{R}(J)$ for a null hypersurface $M \subset \overline{M}$.*

1. *If M is totally geodesic, then the rigged vector field ξ is $\widetilde{g}-$parallel.*
2. *If M is totally geodesic, then the screen distribution $\mathcal{S}^J_\zeta$ is a parallel distribution on M with respect to the Levi-Civita connection $\widetilde{\nabla}$.*
3. *Let $\widetilde{\mathbb{I}}$ denote the second fundamental form of a generic leaf of $(\mathcal{S}^J_\zeta, \widetilde{g})$ in $(M, \widetilde{g})$. Then $\widetilde{\mathbb{I}} = B(U, X)\xi$. Hence, if M is totally geodesic, then each leaf of the screen distribution $\mathcal{S}^J_\zeta$ is totally geodesic as a hypersurface of $(M, \widetilde{g})$.*
4. *$\widetilde{\nabla}_\xi \xi = 0$.*

Theorem 3.7. *Let M be a null hypersurface of an indefinite almost Hermitian manifold $(\overline{M}, \overline{g}, J)$ with a fixed conformal $J-$rigging $\zeta \in \mathcal{R}(J)$ for M. Then $\tau(\xi) = 0$ and $\tau(X) = -\frac{1}{2}\widetilde{g}(\widetilde{\nabla}_\xi \xi, X)$, for any $X \in \mathcal{S}^J_\zeta$.*

Proof. Assume that $J-$rigging $\zeta \in \mathcal{R}(J)$ to be conformal, that is, $L_\zeta \overline{g} = 2\rho\overline{g}$, for some smooth function ρ in the domain of ζ. Then $L_\zeta \overline{g}(\xi, \xi) = 2\rho\overline{g}(\xi, \xi) = 0$, implies $\overline{g}(\overline{\nabla}_\xi \zeta, \xi) = 0$, consequently $\tau(\xi) = 0$ or $\nabla_\xi \xi = 0$. Let $X \in \mathcal{S}^J_\zeta$ then from (3.21), we have $\overline{g}(\overline{D}(\xi, \xi), X) = -d\omega(\xi, X) + \alpha(i_\star J\widetilde{\nabla}_\xi \xi)(i_\star JX)$. Since $\tau(\xi) = 0$ therefore $-\overline{g}(\widetilde{\nabla}_\xi \xi, X) = -d\omega(\xi, X) + \alpha(i_\star J\widetilde{\nabla}_\xi \xi)(i_\star JX)$, then using (3.4) and the fact that $d\omega(\xi, X) = d\alpha(\xi, X)$, as by definition $\omega = i^\star\alpha$, that is, $\widetilde{g}(\xi, .) = \overline{g}(\zeta, .)$, we obtain $\widetilde{g}(\widetilde{\nabla}_\xi \xi, X) = d\alpha(\xi, X)$. Also, $(L_\zeta \overline{g})(\xi, X) = 2\rho\overline{g}(\xi, X) = 0$ implies $\overline{g}(\overline{\nabla}_\xi \zeta, X) = -\overline{g}(\xi, \overline{\nabla}_X \zeta)$. Hence

$$\widetilde{g}(\widetilde{\nabla}_\xi \xi, X) = d\alpha(\xi, X) = -2\overline{g}(\xi, \overline{\nabla}_X \zeta) = -2\tau(X);$$

this completes the proof □

Remark 3.2. Let $X, Y \in \Gamma(\mathcal{D})$. Then from (3.21), we get $\overline{g}(\overline{D}(X, Y), \xi) = -\frac{1}{2}(L_\xi \widetilde{g})(X, Y)$, then on further using (3.22), $\overline{g}(D(X, Y), \xi) = -\frac{1}{2}(L_\xi \widetilde{g})(X, Y) - B(X, Y)$, this further gives $(L_\xi \widetilde{g})(X, Y) = -2B(X, Y)$.

4. Curvature relations with a closed Killing rigging

Let R, $\widetilde{R}$ and $R^\star$ be the curvature tensors associated to the induced connection ∇, the Levi-Civita connections $\widetilde{\nabla}$ and $\nabla^\star$, respectively. Let P be the projection morphism of $\Gamma(TM)$ onto $\Gamma(\mathcal{S}^J_\zeta)$. Then for any $U \in \Gamma(TM)$,

$$U = PU + \omega(U)\xi. \tag{4.1}$$

Lemma 4.1. *For any $U, V \in \Gamma(TM)$, we have*

$$B(U, A^\star_\xi V) = B(V, A^\star_\xi U), \; B(U, J\zeta)\alpha(i_\star J A^\star_\xi V) = B(V, J\zeta)\alpha(i_\star J A^\star_\xi U). \tag{4.2}$$

Proof. Assertions follow directly using (2.5). □

Lemma 4.2. *Let $(\overline{M}, \overline{g}, J)$ be an indefinite Kähler manifold with a fixed closed Killing $J-$rigging ζ for a null hypersurface $M \subset \overline{M}$. Then*

$$(\widetilde{\nabla}_U B)(V, W) = (\nabla_U B)(V, W) \tag{4.3}$$

for any $U, V, W \in \Gamma(TM)$.

Proof. Let $U, V \in \Gamma(TM)$ then from (3.23), (3.24) and (4.1), it follows that

$$\widetilde{\nabla}_U V = \nabla^\star_U PV + B(U, PV)\xi + (\widetilde{\nabla}_U \omega(V))\xi - \omega(V) A^\star_\xi U.$$

Then

$$(\widetilde{\nabla}_U B)(V, W) = U(B(V, W)) - B(\nabla^\star_U PV, W) - B(V, \nabla^\star_U PW) + \omega(V) B(A^\star_\xi U, W) + \omega(W) B(V, A^\star_\xi U).$$

Now, use (2.3), (2.4) and (4.1) to get the same expression for $(\nabla_U B)(V, W)$. □

Lemma 4.3. *Let $(\overline{M}, \overline{g}, J)$ be an indefinite Kähler manifold with a fixed closed Killing $J-$rigging $\zeta \in \mathcal{R}(J)$ for a null hypersurface $M \subset \overline{M}$. Then*

$$\widetilde{\nabla}_U(\alpha(i_\star JPV)) - \alpha(i_\star J \nabla^\star_U PV) = 0, \tag{4.4}$$

for any $U, V \in \Gamma(TM)$.

Proof. For a fixed $J-$rigging $\zeta \in \mathcal{R}(J)$, we have $\alpha(J\xi) = 0$ and $\alpha(JN) = 0$. Therefore using (2.1) and (2.3), we obtain

$$\widetilde{\nabla}_U(\alpha(i_\star JPV)) - \alpha(i_\star J \nabla^\star_U PV) = U(\alpha(i_\star JPV)) - \alpha(\overline{\nabla}_U JPV).$$

From (4.1), we can write $JV = JPV + \omega(V) J\xi$, on using this in the last equation, we have

$$\widetilde{\nabla}_U(\alpha(i_\star JPV)) - \alpha(i_\star J \nabla^\star_U PV) = (\overline{\nabla}_U \alpha)(JV) - \omega(V)(\overline{\nabla}_U \alpha)(J\xi).$$

Then, use of the Lemma 3.3 completes the proof. □

Furthermore, using (4.2) together with the Corollary 3.2 and Lemmas 4.2 and 4.3, we have the following relations.

Theorem 4.1. *Let $(\overline{M}, \overline{g}, J)$ be an indefinite Kähler manifold with a fixed closed Killing $J-$rigging $\zeta \in \mathcal{R}(J)$ for a null hypersurface $M \subset \overline{M}$. Then*

$$\widetilde{R}(U, V)PW = R^\star(U, V)PW + \Big\{(\widetilde{\nabla}_U B)(V, PW) - (\widetilde{\nabla}_V B)(U, PW)\Big\}\xi - A^\star_\xi U B(V, PW) + A^\star_\xi V B(U, PW), \tag{4.5}$$

$$\widetilde{R}(U, V)\xi = A^\star_\xi [U, V] + \nabla^\star_V A^\star_\xi U - \nabla^\star_U A^\star_\xi V, \tag{4.6}$$

$$\widetilde{R}(U, \xi)\xi = \nabla^\star_\xi A^\star_\xi U + A^\star_\xi [U, \xi], \tag{4.7}$$

where $(\widetilde{\nabla}_U B)(V, PW) = \widetilde{\nabla}_U(B(V, PW)) - B(\widetilde{\nabla}_U V, PW) - B(V, \nabla^\star_U PW)$, for any $U, V, W \in \Gamma(TM)$.

Proof. By straightforward calculations, assertions follow from (3.23), (3.24) and (4.1). □

Remark 4.1. By using (4.1), (4.5) and (4.6), we can derive

$$\widetilde{R}(U, V)W = \widetilde{R}(U, V)PW + \omega(W)\widetilde{R}(U, V)\xi. \tag{4.8}$$

By straightforward calculations, using (2.3) and (2.4), we have

$$\begin{aligned} R(U, V)PW \quad = \quad & R^\star(U, V)PW + \Big\{(\nabla_U C)(V, PW) - (\nabla_V C)(U, PW)\Big\}\xi \\ & - C(V, PW) A^\star_\xi U + C(U, PW) A^\star_\xi V \\ & - C(V, PW)\tau(U)\xi + C(U, PW)\tau(V)\xi, \end{aligned} \tag{4.9}$$

$$\begin{aligned} R(U,V)\xi &= A^{\star}_{\xi}[U,V] + \nabla^{\star}_{V}A^{\star}_{\xi}U - \nabla^{\star}_{U}A^{\star}_{\xi}V + C(V,A^{\star}_{\xi}U)\xi \\ &\quad -C(U,A^{\star}_{\xi}V)\xi - d\tau(U,V)\xi + \tau(V)A^{\star}_{\xi}U - \tau(U)A^{\star}_{\xi}V, \end{aligned} \tag{4.10}$$

$$\begin{aligned} R(U,\xi)\xi &= -\nabla_{U}\tau(\xi)\xi + \tau(\xi)A^{\star}_{\xi}U + \nabla^{\star}_{\xi}A^{\star}_{\xi}U + C(\xi,A^{\star}_{\xi}U)\xi \\ &\quad +\nabla_{\xi}\tau(U)\xi + A^{\star}_{\xi}([U,\xi]) + \tau([U,\xi])\xi, \end{aligned} \tag{4.11}$$

where $(\nabla_U C)(V,PW) = \nabla_U C(V,PW) - C(\nabla_U V,PW) - C(V,\nabla^{\star}_U PW)$.

Let $\overline{R}$ be the Riemannian curvature tensor associated with the Levi-Civita connection $\overline{\nabla}$ of a semi-Riemannian manifold $(\overline{M},\overline{g})$ and let $(M,g,S(TM))$ be a null hypersurface of $(\overline{M},\overline{g})$. Then from [4, p.94], we have Gauss-Codazzi structure equations as below

$$\overline{g}(\overline{R}(U,V)W,X) = g(R(U,V)W,X) + B(U,W)C(V,X) - B(V,W)C(U,X), \tag{4.12}$$

$$\overline{g}(\overline{R}(U,V)PW,N) = (\nabla_U C)(V,PW) - (\nabla_V C)(U,PW) + \tau(V)C(U,PW) - \tau(U)C(V,PW), \tag{4.13}$$

$$\overline{g}(\overline{R}(U,V)\xi,N) = C(V,A^{\star}_{\xi}U) - C(U,A^{\star}_{\xi}V) - d\tau(U,V), \tag{4.14}$$

$$\overline{g}(\overline{R}(U,V)PW,\xi) = (\nabla_U B)(V,PW) - (\nabla_V B)(U,PW) + B(V,PW)\tau(U) - B(U,PW)\tau(V), \tag{4.15}$$

for any $U,V,W \in \Gamma(TM)$ and $X \in \Gamma(S(TM))$.

Using the Corollary 3.2 with (4.3), (4.4), Theorem 4.1, (4.9), (4.10), (4.11), (4.12), (4.13), (4.14) and (4.15), we obtain following relations.

Theorem 4.2. *Let $(\overline{M},\overline{g},J)$ be an indefinite Kähler manifold with a fixed closed Killing $J-$rigging $\zeta \in \mathcal{R}(J)$ for a null hypersurface $M \subset \overline{M}$. Then*

$$\begin{aligned} R(U,V)X - \widetilde{R}(U,V)X &= \overline{g}(\overline{R}(U,V)X,N)\xi - \overline{g}(\overline{R}(U,V)X,\xi)\xi + C(U,X)A^{\star}_{\xi}V - C(V,X)A^{\star}_{\xi}U \\ &\quad +B(U,X)\nabla_V\xi - B(V,X)\nabla_U\xi, \\ R(U,V)\xi - \widetilde{R}(U,V)\xi &= \overline{g}(\overline{R}(U,V)\xi,N)\xi, \\ \widetilde{g}(R(X,\xi)\xi,X) &= \widetilde{g}(\widetilde{R}(X,\xi)\xi,X), \end{aligned} \tag{4.16}$$

for all $U,V \in \Gamma(TM)$ and $X \in \Gamma(\mathcal{D})$.

Let (M,g) be a normalized null hypersurface of $(\overline{M},\overline{g},J)$ with rigged vector field ξ and $\Pi^{null}_{\xi} = span\{X,\xi\}$ be a null plane at $p \in M$ directed by ξ, where X is a unitary vector tangent to $\mathcal{D}$. Then, the null sectional curvature $K^{null}(\Pi^{null}_{\xi})$ of Π^{null}_{ξ} is a real number defined by

$$K^{null}(\Pi^{null}_{\xi}) = \frac{g_p(R(X,\xi)\xi,X)}{g_p(X,X)} = g_p(R(X,\xi)\xi,X).$$

Let $\widetilde{K}(\Pi_{\xi})$ denote the $\widetilde{g}-$sectional curvature of the same plane $\Pi_{\xi} = span\{X,\xi\}$ containing ξ and a unitary vector X tangent to $\mathcal{D}$. Then from (4.16), the following observation is immediate.

Theorem 4.3. *Let $(\overline{M},\overline{g},J)$ be an indefinite Kähler manifold with a fixed closed Killing $J-$rigging $\zeta \in \mathcal{R}(J)$ for a null hypersurface $M \subset \overline{M}$. Then*

$$K^{null}(\Pi^{null}_{\xi}) - \widetilde{K}(\Pi_{\xi}) = \tau(\xi)B(X,X),$$

for any unitary vector field $X \in \Gamma(\mathcal{D})$. Moreover, $K^{null}(\Pi^{null}_{\xi}) = \widetilde{K}(\Pi_{\xi})$, if ζ is a conformal rigging vector field or M is a totally geodesic null hypersurface.

From (3.2), take an almost complex distribution $\overline{\mathcal{D}} = \{TM^{\perp} \oplus_{orth} J(TM^{\perp})\} \oplus_{orth} \mathcal{D}$. Then

$$TM = J(tr(TM)) \oplus \overline{\mathcal{D}}. \tag{4.17}$$

Consider local lightlike vector fields as $\mathcal{U} = -JN$ and $\mathcal{V} = -J\xi$. Let $X \in \Gamma(\mathcal{D})$ be a unitary vector field then we denote $\Pi^{null}_{\mathcal{U}} = span\{\mathcal{U},X\}$ and $\Pi^{null}_{\mathcal{V}} = span\{\mathcal{V},X\}$ the null planes directed by the null vector fields $\mathcal{U}$ and $\mathcal{V}$, respectively and denote the null sectional curvature associated to these null sections by $K^{null}(\Pi^{null}_{\mathcal{U}})$ and $K^{null}(\Pi^{null}_{\mathcal{V}})$, respectively. We denote the $\widetilde{g}-$sectional curvatures of the planes $\Pi_{\mathcal{U}} = span\{X,\mathcal{U}\}$ and $\Pi_{\mathcal{V}} = span\{X,\mathcal{V}\}$ by $\widetilde{K}(\Pi_{\mathcal{U}})$ and $\widetilde{K}(\Pi_{\mathcal{V}})$, respectively. Then using the Corollary 3.2 with (2.5), (4.5) and (4.9), we derive the following relations.

Theorem 4.4. *Let $(\overline{M}, \overline{g}, J)$ be an indefinite Kähler manifold with a fixed closed Killing $J-$rigging $\zeta \in \mathcal{R}(J)$ for a null hypersurface $M \subset \overline{M}$. Then*

$$K^{null}(\Pi^{null}_{\mathcal{U}}) - \widetilde{K}(\Pi_{\mathcal{U}}) = B(\mathcal{U}, X)C(X,\mathcal{U}) - B(X, X)C(\mathcal{U},\mathcal{U}) + B(X, X)B(\mathcal{U},\mathcal{U}) - B(X,\mathcal{U})^2,$$

$$K^{null}(\Pi^{null}_{\mathcal{V}}) - \widetilde{K}(\Pi_{\mathcal{V}}) = B(\mathcal{V}, X)C(X, \mathcal{V}) - B(X, X)C(\mathcal{V}, \mathcal{V}) + B(X, X)B(\mathcal{V}, \mathcal{V}) - B(X, \mathcal{V})^2.$$

Corollary 4.1. *If M is totally geodesic null submanifold of $\overline{M}$ then*

$$K^{null}(\Pi^{null}_{\mathcal{U}}) = \widetilde{K}(\Pi_{\mathcal{U}}), \quad K^{null}(\Pi^{null}_{\mathcal{V}}) = \widetilde{K}(\Pi_{\mathcal{V}}).$$

Let $\{\xi,\mathcal{U}, \mathcal{V}, e_i, e^\star_i\}$ be a quasi-orthogonal frame on M, where $\mathcal{D} = span\{e_i, e^\star_i\}^{m-2}_{i=1}$ and $e^\star_i = Je_i$. Hence $\{\xi, N,\mathcal{U}, \mathcal{V}, e_i, e^\star_i\}$ is the corresponding frame of fields on $\overline{M}$. Then, the Ricci tensor $\overline{\mathcal{R}}(U, V)$ on $\overline{M}$ and the induced Ricci tensor $\mathcal{R}(U, V)$ on M are respectively given by

$$\begin{aligned}\overline{\mathcal{R}}(U, V) \quad &= \quad \sum_{i=1}^{m-2} \varepsilon_i\Big\{\overline{g}(\overline{R}(U, e_i)V, e_i) + \overline{g}(\overline{R}(U, e^\star_i)V, e^\star_i)\Big\} + \overline{g}(\overline{R}(U,\mathcal{U})V, \mathcal{V}) \\ &+\overline{g}(\overline{R}(U, \mathcal{V})V,\mathcal{U}) + \overline{g}(\overline{R}(U, \xi)V, N) + \overline{g}(\overline{R}(U, N)V, \xi)\end{aligned}$$

and

$$\begin{aligned}\mathcal{R}(U, V) \quad &= \quad \sum_{i=1}^{m-2} \varepsilon_i\Big\{g(R(U, e_i)V, e_i) + g(R(U, e^\star_i)V, e^\star_i)\Big\} + g(R(U,\mathcal{U})V, \mathcal{V}) \\ &+g(R(U, \mathcal{V})V,\mathcal{U}) + \overline{g}(R(U, \xi)V, N).\end{aligned}$$

Now, by using Gauss-Codazzi equations, (4.12) leads to the following expression of the induced Ricci tensor on M:

$$\begin{aligned}\mathcal{R}(U, V) \quad &= \quad \overline{\mathcal{R}}(U, V) - \overline{g}(\overline{R}(U, N)V, \xi) + \sum_{i=1}^{m-2} \varepsilon_i\Big\{ - B(U, V)C(e_i, e_i) \\ &+B(e_i, V)C(U, e_i) - B(U, V)C(e^\star_i, e^\star_i) + B(e^\star_i, V)C(U, e^\star_i)\Big\} \\ &-B(U, V)C(\mathcal{V},\mathcal{U}) + B(\mathcal{V}, V)C(U,\mathcal{U}) - B(U, V)C(\mathcal{U}, \mathcal{V}) \\ &+B(\mathcal{U}, V)C(U.\mathcal{V}).\end{aligned}$$

Furthermore, observe that since we are dealing with an indefinite Kähler manifold, using (2.1), (2.2) and (2.6) lead to

$$C(U, \mathcal{V}) = g(A_N U, \mathcal{V}) = -g(J\overline{\nabla}_U N, \xi) = B(U,\mathcal{U}) \tag{4.18}$$

and similarly

$$C(V, \mathcal{V}) = B(V,\mathcal{U}). \tag{4.19}$$

Then

$$\begin{aligned}\mathcal{R}(U, V) \quad &-\mathcal{R}(V, U) = \overline{\mathcal{R}}(U, V) - \overline{\mathcal{R}}(V, U) - \overline{g}(\overline{R}(U, V)N, \xi) \\ &+ \sum_{i=1}^{m-2} \varepsilon_i\Big\{g(A^\star_\xi V, e_i)g(A_N U, e_i) + g(A^\star_\xi V, e^\star_i)g(A_N U, e^\star_i) \\ &+g(A^\star_\xi V, \mathcal{V})g(A_N U,\mathcal{U}) + g(A^\star_\xi V,\mathcal{U})g(A_N U, \mathcal{V})\Big\} \\ &- \sum_{i=1}^{m-2} \Big\{g(A^\star_\xi U, e_i)g(A_N V, e_i) + g(A^\star_\xi U, e^\star_i)g(A_N V, e^\star_i) \\ &+g(A^\star_\xi U, \mathcal{V})g(A_N V,\mathcal{U}) + g(A^\star_\xi U,\mathcal{U})g(A_N V, \mathcal{V})\Big\}.\end{aligned} \tag{4.20}$$

On the other side, it is easy to check that for arbitrary vector fields U and V in TM, we have

$$U = \sum_{i=1}^{m-2} \varepsilon_i \Big\{ g(U, e_i)e_i + g(U, e_i^*)e_i^* \Big\} + g(U, \mathcal{V})\mathcal{U} + g(U, \mathcal{U})\mathcal{V} + \overline{g}(U, N)\xi$$

and

$$g(U, V) = \sum_{i=1}^{m-2} \varepsilon_i \Big\{ g(U, e_i)g(V, e_i) + g(U, e_i^*)g(V, e_i^*) \Big\} + g(U, \mathcal{V})g(V, \mathcal{U}) + g(U, \mathcal{U})g(V, \mathcal{V}). \tag{4.21}$$

By using the fact that $\overline{\mathcal{R}}(U, V)$ is symmetric and (4.21) in (4.20), we have

$$\mathcal{R}(U, V) - \mathcal{R}(V, U) = -\overline{g}(\overline{R}(U, V)N, \xi) + g(A^\star_\xi V, A_N U) - g(A^\star_\xi U, A_N V). \tag{4.22}$$

Next, from straightforward calculations, we obtain

$$\begin{aligned} \overline{g}(\overline{R}(U, V)N, \xi) &= -B(U, A_N V) + B(V, A_N U) + d\tau(U, V) \\ &= -g(A^\star_\xi U, A_N V) + g(A^\star_\xi V, A_N U) + d\tau(U, V). \end{aligned} \tag{4.23}$$

On substituting (4.23) in (4.22), we finally have $\mathcal{R}(U, V) - \mathcal{R}(V, U) = -d\tau(U, V)$ and hence, we have following observation.

Theorem 4.5. *Let (M, ζ) be a normalized null hypersurface of an indefinite Kähler manifold $(\overline{M}, \overline{g}, J)$ with a fixed closed Killing $J-$rigging $\zeta \in \mathcal{R}(J)$ for M. Then the induced Ricci tensor on M is symmetric if and only if each $1-$form τ on $\mathcal{S}^J_\zeta$ is closed.*

Next, from (3.4), we know that $\widetilde{g}(\mathcal{U}, \mathcal{U}) = \alpha(N)^2$ and therefore $\mathcal{U}$ is not a $\widetilde{g}-$unitary. Then take $\mathcal{U}' = \frac{1}{\alpha(N)}\mathcal{U}$, which is clearly $\widetilde{g}-$unitary. Also $\widetilde{g}(\mathcal{V}, \mathcal{V}) = 1$, i.e, $\mathcal{V}$ is also $\widetilde{g}-$unitary. Hence $\{\xi, \mathcal{U}', \mathcal{V}, e_i, e_i^\star\}$ is a $\widetilde{g}-$frame on M then for any $U, V \in \Gamma(TM)$, the induced $\widetilde{g}-$Ricci tensor $\widetilde{\mathcal{R}}(U, V)$ is given by

$$\begin{aligned} \widetilde{\mathcal{R}}(U, V) &= \sum_{i=1}^{m-2} \varepsilon_i \Big\{ \widetilde{g}(\widetilde{R}(U, e_i)V, e_i) + \widetilde{g}(\widetilde{R}(U, e_i^\star)V, e_i^\star) \Big\} + \widetilde{g}(\widetilde{R}(U, \mathcal{U}')V, \mathcal{U}') \\ &\quad + \widetilde{g}(\widetilde{R}(U, \mathcal{V})V, \mathcal{V}) + \widetilde{g}(\widetilde{R}(U, \xi)V, \xi). \end{aligned} \tag{4.24}$$

Assume that the $J-$rigging $\zeta \in \mathcal{R}(J)$ is a closed Killing vector field. Then, using the Corollary 3.2 with (4.5), (4.7) and (4.8), we derive

$$\widetilde{g}(\widetilde{R}(U, e_i)V, e_i) = g(R^\star(U, e_i)PV, e_i) - B(U, e_i)B(e_i, V) + B(e_i, e_i)B(U, V) + \omega(V)g(R(U, e_i)\xi, e_i)$$

and taking into account of the Corollary 3.2 with (4.9) and (4.12) in the last equation, we obtain

$$\widetilde{g}(\widetilde{R}(U, e_i)V, e_i) = \overline{g}(\overline{R}(U, e_i)V, e_i) + (1 + \overline{g}(\zeta, \zeta))\big[B(U, V)B(e_i, e_i) - B(U, e_i)B(V, e_i)\big]. \tag{4.25}$$

A similar formula for $\widetilde{g}(\widetilde{R}(U, e_i^\star)V, e_i^\star)$ is obtained by interchanging e_i and $e_i^\star$. Since $N = \zeta - \alpha(N)\xi$, this further implies $J\zeta = -(\mathcal{U} + \alpha(N)\mathcal{V})$. Also, the Kähler condition together with the parallelism of ζ implies $\overline{g}(\overline{R}(U, \mathcal{U})V, J\zeta) = -\overline{g}(\overline{R}(U, \mathcal{U})J\zeta, V) = 0$. So

$$\widetilde{g}(\widetilde{R}(U, \mathcal{U}')V, \mathcal{U}') = -\frac{1}{\alpha(N)}\overline{g}(\overline{R}(U, \mathcal{U})V, \mathcal{V}) + (1 + \overline{g}(\zeta, \zeta))\big[B(U, V)B(\mathcal{U}', \mathcal{U}') - B(U, \mathcal{U}')B(V, \mathcal{U}')\big] \tag{4.26}$$

and

$$\widetilde{g}(\widetilde{R}(U, \mathcal{V})V, \mathcal{V}) = -\frac{1}{\alpha(N)}\overline{g}(\overline{R}(U, \mathcal{V})V, \mathcal{U}) + (1 + \overline{g}(\zeta, \zeta))\big[B(U, V)B(\mathcal{V}, \mathcal{V}) - B(U, \mathcal{V})B(V, \mathcal{V})\big], \tag{4.27}$$

$$\begin{aligned} \widetilde{g}(\widetilde{R}(U, \xi)V, \xi) &= -B([U, \xi], V) - \xi(B(U, V)) + B(U, \nabla^\star_\xi PV) \\ &= B(A^\star_\xi U, PV) - (\widetilde{\nabla}_\xi B)(U, V). \end{aligned} \tag{4.28}$$

Hence, using (4.25), (4.26), (4.27) and (4.28) in (4.24), we have the following important relationship between the intrinsic and extrinsic Ricci tensors for a null hypersurface with a closed Killing $J-$rigging $\zeta \in \mathcal{R}(J)$ of an indefinite Kähler manifold.

$$\begin{aligned}\widetilde{\mathcal{R}}(U,V) &= \overline{\mathcal{R}}(U,V) - \frac{(1+\alpha(N))}{\alpha(N)}\{\overline{g}(\overline{R}(U,\mathcal{U})V,\mathcal{V}) + \overline{g}(\overline{R}(V,\mathcal{U})U,\mathcal{V})\} - \overline{g}(\overline{R}(U,\xi)V,N) - \overline{g}(\overline{R}(V,\xi)U,N)\\ &+(1+\overline{g}(\zeta,\zeta))B(U,V)\Big\{\sum_{i=1}^{m-2}\epsilon_i\{B(e_i,e_i)+B(e_i^\star,e_i^\star)\} + B(\mathcal{U}',\mathcal{U}') + B(\mathcal{V},\mathcal{V})\Big\}\\ &-(1+\overline{g}(\zeta,\zeta))\Big\{\sum_{i=1}^{m-2}\epsilon_i\{B(U,e_i)B(V,e_i)+B(U,e_i^\star)B(V,e_i^\star)\} + B(U,\mathcal{U}')B(V,\mathcal{U}')\\ &+B(U,\mathcal{V})B(V,\mathcal{V})\Big\} + B(A_\xi^\star U,V) - (\widetilde{\nabla}_\xi B)(U,V). \end{aligned} \tag{4.29}$$

From the last expression, it is obvious that the intrinsic Ricci tensor for a null hypersurface with a closed Killing $J-$rigging $\zeta \in \mathcal{R}(J)$ of an indefinite Kähler manifold is a symmetric tensor and hence has significant applications in geometry and physics.

Since $\{e_i, e_i^\star, \mathcal{U}', \mathcal{V}, \xi\}$ is a $\widetilde{g}-$frame on M, where $i \in \{1,\ldots,m-2\}$ and $B(U,\xi)=0$, therefore the mean curvature μ of the null hypersurface is given by

$$\mu = \frac{1}{2m-2}\sum_{i=1}^{m-2}\epsilon_i\{B(e_i,e_i)+B(e_i^\star,e_i^\star)\} + B(\mathcal{U}',\mathcal{U}') + B(\mathcal{V},\mathcal{V}) \tag{4.30}$$

and using (2.5) and (4.21), we can write

$$B(A_\xi^\star U,V) = \sum_{i=1}^{m-2}\epsilon_i\{B(U,e_i)B(V,e_i)+B(U,e_i^\star)B(V,e_i^\star)\} + B(U,\mathcal{V})B(V,\mathcal{U}) + B(U,\mathcal{U})B(V,\mathcal{V}). \tag{4.31}$$

From the Corollary 3.2, it is known that $B(U,J\zeta)=0$; therefore $B(U,\mathcal{U}) = -\alpha(N)B(U,\mathcal{V})$ and using (4.30), (4.31) in (4.29), we derive

$$\begin{aligned}\widetilde{\mathcal{R}}(U,V) &= \overline{\mathcal{R}}(U,V) + (2+\overline{g}(\zeta,\zeta))\overline{g}(\overline{R}(U,\mathcal{V})V,\mathcal{V}) - \overline{g}(\overline{R}(U,\xi)V,N) - \overline{g}(\overline{R}(V,\xi)U,N)\\ &+(1+\overline{g}(\zeta,\zeta))B(U,V)(2m-2)\mu - \overline{g}(\zeta,\zeta)B(A_\xi^\star U,V)\\ &-(1+\overline{g}(\zeta,\zeta))(2+\overline{g}(\zeta,\zeta))B(U,\mathcal{V})B(V,\mathcal{V}) - (\widetilde{\nabla}_\xi B)(U,V). \end{aligned} \tag{4.32}$$

Next, using (4.32), we have

$$\begin{aligned}\widetilde{\mathcal{R}}(e_i) &= \overline{\mathcal{R}}(e_i) + (2+\overline{g}(\zeta,\zeta))\overline{g}(\overline{R}(e_i,\mathcal{V})e_i,\mathcal{V}) - 2\overline{g}(\overline{R}(e_i,\xi)e_i,N)\\ &+(1+\overline{g}(\zeta,\zeta))B(e_i,e_i)(2m-2)\mu - \overline{g}(\zeta,\zeta)B(A_\xi^\star e_i,e_i)\\ &-(1+\overline{g}(\zeta,\zeta))(2+\overline{g}(\zeta,\zeta))B(e_i,\mathcal{V})^2 - (\widetilde{\nabla}_\xi B)(e_i,e_i). \end{aligned} \tag{4.33}$$

Since $B(U,\mathcal{U}) = -\alpha(N)B(U,\mathcal{V})$, we get $B(\mathcal{U}',\mathcal{U}') = B(\mathcal{V},\mathcal{V})$ and $B(\mathcal{U}',\mathcal{V}) = -B(\mathcal{V},\mathcal{V})$ and consequently

$$\begin{aligned}\widetilde{\mathcal{R}}(\mathcal{U}') &= \overline{\mathcal{R}}(\mathcal{U}') - 2\overline{g}(\overline{R}(\mathcal{U}',\xi)\mathcal{U}',N) + (1+\overline{g}(\zeta,\zeta))B(\mathcal{V},\mathcal{V})(2m-2)\mu\\ &-\overline{g}(\zeta,\zeta)B(A_\xi^\star\mathcal{U}',\mathcal{U}') - (1+\overline{g}(\zeta,\zeta))(2+\overline{g}(\zeta,\zeta))B(\mathcal{V},\mathcal{V})^2 - (\widetilde{\nabla}_\xi B)(\mathcal{U}',\mathcal{U}'), \end{aligned} \tag{4.34}$$

$$\begin{aligned}\widetilde{\mathcal{R}}(\mathcal{V}) &= \overline{\mathcal{R}}(\mathcal{V}) - 2\overline{g}(\overline{R}(\mathcal{V},\xi)\mathcal{V},N) + (1+\overline{g}(\zeta,\zeta))B(\mathcal{V},\mathcal{V})(2m-2)\mu\\ &-\overline{g}(\zeta,\zeta)B(A_\xi^\star\mathcal{V},\mathcal{V}) - (1+\overline{g}(\zeta,\zeta))(2+\overline{g}(\zeta,\zeta))B(\mathcal{V},\mathcal{V})^2 - (\widetilde{\nabla}_\xi B)(\mathcal{V},\mathcal{V}), \end{aligned} \tag{4.35}$$

$$\widetilde{\mathcal{R}}(\xi) = \overline{\mathcal{R}}(\xi) + (2+\overline{g}(\zeta,\zeta))\overline{g}(\overline{R}(\xi,\mathcal{V})\xi,\mathcal{V}). \tag{4.36}$$

It is well known that an indefinite complex space form is a connected indefinite Kähler manifold of constant holomorphic sectional curvature $\overline{c}$ and is denoted by $\overline{M}(\overline{c})$. Then the curvature tensor field of $\overline{M}(\overline{c})$ is given by

$$\begin{aligned}\overline{g}(\overline{R}(X,Y)Z,W) &= \frac{\overline{c}}{4}\{\overline{g}(Y,Z)\overline{g}(X,W) - \overline{g}(X,Z)\overline{g}(Y,W) + \overline{g}(JY,Z)\overline{g}(JX,W)\\ &-\overline{g}(JX,Z)\overline{g}(JY,W) + 2\overline{g}(X,JY)\overline{g}(JZ,W)\}, \end{aligned} \tag{4.37}$$

for any $X, Y, Z, W \in \Gamma(T\overline{M})$.

Denote the scalar curvatures of null hypersurface M of an indefinite Kähler manifold $(\overline{M}, \overline{g}, J)$ with a fixed closed Killing $J-$rigging $\zeta \in \mathcal{R}(J)$ for $M \subset \overline{M}$ and of $(\overline{M}, \overline{g}, J)$ by $\widetilde{\tau}$ and $\overline{\tau}$, respectively. Then relationship between these intrinsic and extrinsic scalar curvatures is given in the next Theorem.

Theorem 4.6. *Let M be a null hypersurface of an indefinite complex space form $\overline{M}(\overline{c})$ with a fixed closed Killing $J-$rigging $\zeta \in \mathcal{R}(J)$ for $M \subset \overline{M}$. Then*

$$\begin{aligned}\widetilde{\tau} \;=\;& \overline{\tau}_{TM} + \widetilde{\mathcal{R}}(\xi) + \overline{c}(m-2) + (1+\overline{g}(\zeta,\zeta))(2m-2)^2\mu^2 - (1+\overline{g}(\zeta,\zeta))(2+\overline{g}(\zeta,\zeta)) \mid A^{\star}_{\xi}\mathcal{V} \mid^2 \\ & -(1+\overline{g}(\zeta,\zeta))(2+\overline{g}(\zeta,\zeta))^2 B(\mathcal{V},\mathcal{V})^2 - (1+\overline{g}(\zeta,\zeta))(tr(A^{\star}_{\xi})^2).\end{aligned}$$

Proof. By straightforward calculations, using (4.33), (4.34), (4.35), (4.36) and (4.37), we derive

$$\begin{aligned}\widetilde{\tau} \;=\;& \overline{\tau}_{TM} + \widetilde{\mathcal{R}}(\xi) + \overline{c}(m-2) + (1+\overline{g}(\zeta,\zeta))(2m-2)^2\mu^2 \\ & -(1+\overline{g}(\zeta,\zeta))(2+\overline{g}(\zeta,\zeta))\Big[\sum_{i=1}^{(m-2)} \epsilon_i\big\{B(e_i,\mathcal{V})^2 + B(e_i^{\star},\mathcal{V})^2\big\} + 2B(\mathcal{V},\mathcal{V})^2\Big] \\ & -(1+\overline{g}(\zeta,\zeta))\Big[\sum_{i=1}^{(m-2)} \epsilon_i\big\{B(A^{\star}_{\xi}e_i,e_i) + B(A^{\star}_{\xi}e_i^{\star},e_i^{\star})\big\} + B(A^{\star}_{\xi}\mathcal{U}',\mathcal{U}') + B(A^{\star}_{\xi}\mathcal{V},\mathcal{V})\Big]. \end{aligned} \tag{4.38}$$

From (4.31), we have

$$\sum_{i=1}^{m-2} \epsilon_i\{B(e_i,\mathcal{V})^2 + B(e_i^{\star},\mathcal{V})^2\} = B(A^{\star}_{\xi}\mathcal{V},\mathcal{V}) - 2B(\mathcal{U},\mathcal{V})B(\mathcal{V},\mathcal{V}) \tag{4.39}$$

and using (4.39) in (4.38), we obtain

$$\begin{aligned}\widetilde{\tau} \;=\;& \overline{\tau}_{TM} + \widetilde{\mathcal{R}}(\xi) + \overline{c}(m-2) + (1+\overline{g}(\zeta,\zeta))(2m-2)^2\mu^2 \\ & -(1+\overline{g}(\zeta,\zeta))(2+\overline{g}(\zeta,\zeta))B(A^{\star}_{\xi}\mathcal{V},\mathcal{V}) - (1+\overline{g}(\zeta,\zeta))(2+\overline{g}(\zeta,\zeta))^2 B(\mathcal{V},\mathcal{V})^2 \\ & -(1+\overline{g}(\zeta,\zeta))\Big[\sum_{i=1}^{(m-2)} \epsilon_i\big\{g((A^{\star}_{\xi})^2 e_i,e_i) + g((A^{\star}_{\xi})^2 e_i^{\star},e_i^{\star})\big\} + g((A^{\star}_{\xi})^2\mathcal{U}',\mathcal{U}') + g((A^{\star}_{\xi})^2\mathcal{V},\mathcal{V})\Big],\end{aligned}$$

this completes the proof. □

Corollary 4.2. *Let M be a null hypersurface of an indefinite complex space form $\overline{M}(\overline{c})$ with a fixed closed Killing $J-$rigging $\zeta \in \mathcal{R}(J)$ for $M \subset \overline{M}$. If M is totally geodesic then $\widetilde{\tau} = \overline{\tau}_{TM} + \overline{c}(m-2)$.*

5. Induced almost contact structure

Let Q be the projection morphism of TM on $\overline{\mathcal{D}}$; then using (4.17), any vector field U on M can be written as

$$U = QU + u(U)\mathcal{U}, \tag{5.1}$$

where u is a $1-$form locally defined on M by

$$u(U) = g(U,\mathcal{V}). \tag{5.2}$$

Denote $\widetilde{u}(U) = \widetilde{g}(U,\mathcal{V})$, using (3.4) and (5.2), we get

$$\widetilde{u}(U) = u(U) + \alpha(i_{\star}JU). \tag{5.3}$$

On applying J to (5.1), we obtain

$$JU = FU + u(U)N, \tag{5.4}$$

where F is a tensor field of type $(1,1)$ globally defined on M by $FU = JQU$. Further, it follows that

$$F^2U = -U + u(U)\mathcal{U}, \quad u(\mathcal{U}) = 1, \tag{5.5}$$

and $g(FU,FV) = g(U,V) - u(U)v(V) - u(V)v(U)$, for any $U,V \in \Gamma(TM)$ and where v is a $1-$form locally defined on M by $v(U) = g(U,\mathcal{U})$. Denote $\widetilde{v}(U) = \widetilde{g}(U,\mathcal{U})$ then

$$\widetilde{v}(U) = v(U) + \alpha(i_\star JU)\alpha(N). \tag{5.6}$$

It should be noted from (5.5) that $(F,u,\mathcal{U})$ defines an almost contact structure on M.

Lemma 5.1. *Let $(\overline{M},\overline{g},J)$ be an indefinite Kähler manifold with a fixed $J-$rigging $\zeta \in \mathcal{R}(J)$ for a null hypersurface $M \subset \overline{M}$. Then, for any $U,V \in \Gamma(TM)$, we have*

$$(\nabla_U\widetilde{v})(V) = \tau(U)\widetilde{v}(V) - \alpha(i_\star JV)\alpha(N)\tau(U) - \overline{g}(A_N U, i_\star JV) + (\overline{\nabla}_U\alpha)(i_\star JV)\alpha(N).$$

Proof. Let $U,V \in \Gamma(TM)$. Relations (2.2) and (5.6) together with the Kähler condition provide

$$\begin{aligned}(\nabla_U v)V &= U(v(V)) - v(\nabla_U V) = \tau(U)v(V) - \overline{g}(A_N U, i_\star JV)\\ &= \tau(U)\widetilde{v}(V) - \alpha(i_\star JV)\alpha(N)\tau(U) - \overline{g}(A_N U, i_\star JV).\end{aligned} \tag{5.7}$$

Next, using (5.6), we directly derive

$$(\nabla_U\widetilde{v})V = (\nabla_U v)V + \{U(\alpha(i_\star JV)) - \alpha(i_\star J\nabla_U V)\}\alpha(N) + U(\alpha(N))\alpha(i_\star JV). \tag{5.8}$$

Since the $J-$rigging $\zeta \in \mathcal{R}(J)$ is fixed then, the final expression follows from (5.7) and (5.8). □

Lemma 5.2. *Let $(\overline{M},\overline{g},J)$ be an indefinite Kähler manifold with a fixed $J-$rigging $\zeta \in \mathcal{R}(J)$ for a null hypersurface $M \subset \overline{M}$. Then*

$$(\nabla_U F)V = \widetilde{u}(V)A_N U - \alpha(i_\star JV)A_N U - B(U,V)\mathcal{U}, \tag{5.9}$$

$$(\nabla_U\widetilde{u})V = -B(U,FV) - \widetilde{u}(V)\tau(U) + U(\alpha(i_\star JV)) + \alpha(i_\star JV)\tau(U) - \alpha(i_\star J\nabla_U V), \tag{5.10}$$

where

$$(\nabla_U F)V = \overline{\nabla}_U FV - F\nabla_U V, \quad (\nabla_U\widetilde{u})V = \overline{\nabla}_U\widetilde{u}(V) - \widetilde{u}(\nabla_U V), \tag{5.11}$$

for any $U,V \in \Gamma(TM)$.

Proof. It is immediate by using the Gauss-Weingarten formulae and (5.3). □

The *type numbers* $t(p)$ and $t^\star(p)$ of M and of the screen distribution $\mathcal{S}^J_\zeta$ at a point p are the rank of the shape operators A_N and $A^\star_\xi$ at p, respectively.

Theorem 5.1. *Let M be a null hypersurface of an indefinite Kähler manifold $(\overline{M},\overline{g},J)$ with fixed $J-$rigging $\zeta \in \mathcal{R}(J)$. If F is parallel with respect to the induced connection ∇ on M, then the type number $t(p)$ of M satisfies $t(p) \le 1$, for any $p \in M$. Moreover, $B(U,V) = u(V)B(U,\mathcal{U})$, $B(U,\mathcal{V}) = 0$ and $B(U,X) = 0$, for any $U,V \in \Gamma(TM)$ and $X \in \Gamma(\mathcal{D})$.*

Proof. Let F be parallel with respect to the induced connection ∇ on M then on taking scalar product of (5.9) with $\mathcal{U}$ and X, we get $u(V)g(A_N U,\mathcal{U}) = 0$, and $u(V)g(A_N U, X) = 0$, respectively and then replace V by $\mathcal{U}$ in the above expressions, we obtain

$$g(A_N U,\mathcal{U}) = g(A_N U, X) = 0, \text{ that is, } C(U,\mathcal{U}) = C(U,X) = 0. \tag{5.12}$$

Take scalar product of (5.9) with $\mathcal{V}$ and by using (4.19), we have

$$B(U,V) = u(V)g(A_N U,\mathcal{V}) = u(V)C(U,\mathcal{V}) = u(V)B(U,\mathcal{U}).$$

On replacing V by $\mathcal{V}$ and X in the last expression, we get $B(U,\mathcal{V}) = 0$ and $B(U,X) = 0$, respectively. Furthermore, from $(2.6)_2$ and (5.12), it follows that the type number $t(p)$ of M satisfies $t(p) \le 1$, for any $p \in M$. □

From (4.17), it is known that $TM = J(tr(TM)) \oplus \overline{\mathcal{D}}$. Hence, if the distributions $J(tr(TM))$ and $\overline{\mathcal{D}}$ are parallel distributions on M then by the decomposition theorem of de Rham [9], null hypersurface M of an indefinite Kähler manifold $\overline{M}$ is locally a product manifold of the type $L_{\mathcal{U}} \times M^{\overline{\mathcal{D}}}$, where $L_{\mathcal{U}}$ is a null curve tangent to $J(tr(TM))$ and $M^{\overline{\mathcal{D}}}$ is a leaf of $\overline{\mathcal{D}}$.

Theorem 5.2. *Let M be a null hypersurface of an indefinite Kähler manifold $(\overline{M}, \overline{g}, J)$ with fixed $J-$rigging $\zeta \in \mathcal{R}(J)$. If F is parallel with respect to the induced connection ∇ on M then M is locally a product manifold of the type $M^L \times M^{\overline{\mathcal{D}}}$, where M^L and $M^{\overline{\mathcal{D}}}$ are some leaves of $J(tr(TM))$ and $\overline{\mathcal{D}}$, respectively.*

Proof. The distribution $\overline{\mathcal{D}}$ is parallel on M if and only if $\nabla_U X \in \Gamma(\overline{\mathcal{D}})$, for any $U \in \Gamma(TM)$, $X \in \Gamma(\overline{\mathcal{D}})$ or equivalently $g(\nabla_U \xi, \mathcal{V}) = g(\nabla_U \mathcal{V}, \mathcal{V}) = g(\nabla_U Y, \mathcal{V}) = 0$, for any $U \in \Gamma(TM)$, $X \in \Gamma(\overline{\mathcal{D}})$, $Y \in \Gamma(\mathcal{D})$, $\xi \in \Gamma(Rad(TM))$. Then using (2.1), (2.4), (2.5) and the Theorem 5.1, we get $g(\nabla_U \xi, \mathcal{V}) = -B(U, \mathcal{V}) = 0$, $g(\nabla_U \mathcal{V}, \mathcal{V}) = 0$ and $g(\nabla_U Y, \mathcal{V}) = B(U, JY) = 0$. Hence, the distribution $\overline{\mathcal{D}}$ is a parallel distributions on M.

The distribution $J(tr(TM))$ is a parallel distribution on M if and only if $\nabla_U \mathcal{U} \in \Gamma(J(tr(TM)))$, for any $U \in \Gamma(TM)$ or equivalently,

$$\overline{g}(\nabla_U \mathcal{U}, N) = g(\nabla_U \mathcal{U}, \mathcal{U}) = g(\nabla_U \mathcal{U}, Y) = 0.$$

Then using (2.2), (2.6) and (5.12), we get $\overline{g}(\nabla_U \mathcal{U}, N) = C(U, \mathcal{U}) = 0$, $g(\nabla_U \mathcal{U}, \mathcal{U}) = 0$ and $g(\nabla_U \mathcal{U}, Y) = -C(U, JY) = 0$. Hence the distribution $J(tr(TM))$ is a parallel distributions on M. Consequently, by the decomposition theorem of de Rham [9], null hypersurface M of an indefinite Kähler manifold $\overline{M}$ is locally a product manifold of the type $L_{\mathcal{U}} \times M^{\overline{\mathcal{D}}}$, where $L_{\mathcal{U}}$ is a null curve tangent to $J(tr(TM))$ and $M^{\overline{\mathcal{D}}}$ is a leaf of $\overline{\mathcal{D}}$. □

On replacing U by $\mathcal{U}$ in (5.3), we have

$$\widetilde{u}(\mathcal{U}) = 1 + \alpha(N). \tag{5.13}$$

Now, replace V by $\mathcal{U}$ in (5.10) and using (5.13) with Gauss-Weingarten formulae for lightlike hypersurface, we derive $\widetilde{u}(\nabla_U \mathcal{U}) = (1 + \alpha(N))\tau(U) - \alpha(A_N U)$. On using $(2.6)_2$ and (3.1), it follows that $\alpha(A_N U) = \overline{g}(\zeta, A_N U) = \overline{g}(N, A_N U) = 0$, then from the last expression, we have $\tau(U) = \frac{1}{(1+\alpha(N))}\widetilde{u}(\nabla_U \mathcal{U})$, for any $U \in \Gamma(TM)$. Now, replace $V = \mathcal{V}$ in (5.10) and use the facts that $\widetilde{u}(\mathcal{V}) = 1$, $F\mathcal{V} = J\mathcal{V} = \xi$ and $B(U, \xi) = 0$, for any $U \in \Gamma(TM)$, we derive $\widetilde{u}(\nabla_U \mathcal{V}) = \alpha(i_\star J \nabla_U \mathcal{V})$. Further using (2.1) and (2.4), we obtain $\alpha(i_\star J \nabla_U \mathcal{V}) = -\tau(U)$, hence from the last expression it yields

$$\tau(U) = -\widetilde{u}(\nabla_U \mathcal{V}), \quad \forall\, U \in \Gamma(TM). \tag{5.14}$$

It is known that for a fixed $J-$rigging $\zeta \in \mathcal{R}(J)$, $\alpha(i_\star J\xi) = 0$ then it follows that $\widetilde{u}(\xi) = 0$. Next, on taking $V = \xi$ in (5.9) and using (2.4) and (5.11), we obtain $\nabla_U \mathcal{V} = F(A^\star_\xi U) - \tau(U)\mathcal{V}$, further using (5.3), (5.4) and (5.14), we get

$$\nabla_U \mathcal{V} = J(A^\star_\xi U) - \widetilde{u}(A^\star_\xi U)N + \alpha(i_\star J A^\star_\xi U)N + \widetilde{u}(\nabla_U \mathcal{V})\mathcal{V}. \tag{5.15}$$

Finally, take $V = \mathcal{U}$ in (5.9) and using (5.11), we derive $F\nabla_U \mathcal{U} + A_N U = B(U, \mathcal{U})\mathcal{U}$, apply F on the last expression and further using (5.5), it implies that $\nabla_U \mathcal{U} = \tau(U)\mathcal{U} + FA_N U$, further using (5.3) and (5.4), we derive

$$\nabla_U \mathcal{U} = \tau(U)\mathcal{U} + J(A_N U) - \widetilde{u}(A_N U)N + \alpha(i_\star J A_N U)N. \tag{5.16}$$

Hence from (5.15) and (5.16), the following is proved.

Theorem 5.3. *Let $(\overline{M}, \overline{g}, J)$ be an indefinite Kähler manifold with a fixed $J-$rigging $\zeta \in \mathcal{R}(J)$ for a null hypersurface $M \subset \overline{M}$. Then we have*

(i) The vector field $\mathcal{V}$ is parallel with respect to ∇ if and only if $A^\star_\xi U = \widetilde{u}(A^\star_\xi U)\mathcal{U} - \alpha(i_\star J A^\star_\xi U)\mathcal{U} + \widetilde{u}(\nabla_U \mathcal{V})\xi$.
(ii) The vector field $\mathcal{U}$ is parallel with respect to ∇ if and only if $A_N U = \widetilde{u}(A_N U)\mathcal{U} - \alpha(i_\star J A_N U)\mathcal{U}$, and τ vanishes on M.

Lemma 5.3. *Let $(\overline{M}, \overline{g}, J)$ be an indefinite Kähler manifold with a fixed closed Killing $J-$rigging $\zeta \in \mathcal{R}(J)$ for a null hypersurface $M \subset \overline{M}$. Then*

$$\widetilde{\nabla}_U \mathcal{V} - \nabla_U \mathcal{V} = \widetilde{u}(A^\star_\xi U)\xi - v(A^\star_\xi U)\xi, \tag{5.17}$$

$$\widetilde{\nabla}_U \mathcal{U} - \nabla_U \mathcal{U} = \widetilde{v}(A^\star_\xi U)\xi - v(A_N U)\xi, \tag{5.18}$$

for any vector field U on M.

Proof. Using (2.3), (3.23) and (3.24), we have $\widetilde{\nabla}_U \mathcal{V} - \nabla_U \mathcal{V} = \widetilde{g}(A^\star_\xi U, \mathcal{V})\xi - C(U, \mathcal{V})\xi$. Further, using the definition of $\widetilde{u}$ and (4.18), we get $\widetilde{\nabla}_U \mathcal{V} - \nabla_U \mathcal{V} = \widetilde{u}(A^\star_\xi U)\xi - B(U, \mathcal{U})\xi$, on using (2.5) and then definition of v, we get the equation (5.17). Similarly, we can derive (5.18). □

Above Lemma 5.3 with (5.15) and (5.16), lead to the following.

Corollary 5.1. *Let $(\overline{M}, \overline{g}, J)$ be an indefinite Kähler manifold with a fixed closed Killing $J-$rigging $\zeta \in \mathcal{R}(J)$ for a null hypersurface $M \subset \overline{M}$. Then*

(i) The vector field $\mathcal{V}$ is parallel with respect to $\widetilde{\nabla}$ if and only if

$$A^{\star}_{\xi}U = \{\widetilde{u}(A^{\star}_{\xi}U) - \alpha(i_{\star}JA^{\star}_{\xi}U)\}\mathcal{U} + \{v(A^{\star}_{\xi}U) - \widetilde{u}(A^{\star}_{\xi}U)\}\mathcal{V} + \widetilde{u}(\nabla_U\mathcal{V})\xi.$$

(ii) The vector field $\mathcal{U}$ is parallel with respect to $\widetilde{\nabla}$ if and only if

$$A_N U = \{\widetilde{u}(A_N U) - \alpha(i_{\star}JA_N U)\}\mathcal{U} - \{\widetilde{v}(A^{\star}_{\xi}U) - v(A_N U)\}\mathcal{V}.$$

(iii) if $A^{\star}_{\xi}U \in \Gamma(\mathcal{D})$ then vector field $\mathcal{V}$ is parallel with respect to $\widetilde{\nabla}$ and ∇, simultaneously.

Theorem 5.4. *Let $(\overline{M}, \overline{g}, J)$ be an indefinite Kähler manifold with a fixed $J-$rigging $\zeta \in \mathcal{R}(J)$ for a null hypersurface $M \subset \overline{M}$. Then M is totally geodesic if and only if*

$$(\nabla_U F)V = 0, \quad \forall\, U \in \Gamma(TM), V \in \Gamma(\mathcal{D} \oplus_{orth} J(TM^{\perp})),$$

$$A_N U = -F\nabla_U\mathcal{U}, \quad \forall\, U \in \Gamma(TM).$$

Proof. From (5.3), we have $\widetilde{u}(\xi) = 0$, also $\alpha(i_{\star}J\xi) = 0$. Take $V = \xi$ in (5.9) and by using the above facts, we obtain

$$(\nabla_U F)\xi = 0. \tag{5.19}$$

Take $V \in \Gamma(\mathcal{D})$ in (5.9) and further using the fact that $\widetilde{u}(V) = 0$, we get

$$(\nabla_U F)V = -B(U, V)\mathcal{U}. \tag{5.20}$$

Take $V = J\xi$ in (5.9) and further using $\widetilde{u}(J\xi) = -1$, $\alpha(J^2\xi) = -1$, we derive

$$(\nabla_U F)J\xi = -B(U, J\xi)\mathcal{U}. \tag{5.21}$$

Finally, put $V = \mathcal{U}$ in (5.9) and (5.11). Since $\widetilde{u}(\mathcal{U}) = 1 + \alpha(N)$ and $\alpha(i_{\star}J\mathcal{U}) = \alpha(N)$,

$$A_N U + F\nabla_U\mathcal{U} = B(U, \mathcal{U})\mathcal{U}. \tag{5.22}$$

The assertion follows from (5.19), (5.20), (5.21) and (5.22). □

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] Atindogbe, C., Gutierrez, M., Hounnonkpe, R.: *New properties on normalized null hypersurfaces*. Mediterr. J. Math. **15**(166), 1–19 (2018).
[2] Barros, M., Romero, A.: *Indefinite Kähler manifolds*. Math. Ann. **261**, 55–62 (1982).
[3] Bejancu, A., Duggal, K.L.: *Lightlike submanifolds of semi-Riemannian manifolds*. Acta Appl. Math. **38**, 197–215 (1995).
[4] Duggal, K.L., Bejancu, A.: Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications. Kluwer Academic Publishers. Netherlands (1996).
[5] Duggal, K.L., Sahin, B.: Differential Geometry of Lightlike Submanifolds. Birkhauser Verlag AG. Berlin (2010).
[6] Gutierrez, M., Olea, B.: *Induced Riemannian structures on null hypersurfaces*. Math. Nachr. **289**, 1219–1236 (2016).
[7] Ngakeu, F., Tetsing, H.F.: *$\alpha-$associated metrics on rigged null hypersurfaces*. Turkish J. Math. **43**, 1161–1181 (2019).
[8] O'Neill, B.: Semi-Riemannian Geometry with Applications to Relativity. Academic Press. New York (1983).
[9] de Rham, G.: *Sur la réductibilité d'un espace de Riemannian*. Comm. Math. Helv. **26**, 328–344 (1952).
[10] Singh, A.P., Atindogbe, C., Kumar, R., Jain, V.: *Chen-like inequalities on null hypersurfaces with closed rigging of a Lorentzian manifold*. Int. J. Geom. Methods Mod. Phys. **18**(8), 1–23 (2021).
[11] Tetsing, H.F., Ngakeu, F., Olea, B.: *Rigging technique for 1-lightlike submanifolds and preferred rigged connections*. Mediterr. J. Math. **16**, 1–20 (2019).

Affiliations

AMRINDER PAL SINGH
ADDRESS: Punjabi University, Dept. of Mathematics, 147002, Patiala-India.
E-MAIL: amrinderpalsinghramsingh@yahoo.com
ORCID ID:0000-0001-8896-7539

CYRIAQUE ATINDOGBE
ADDRESS: University of Abomey-Calavi, Dept. of Mathematics, BP 526, Cotonou-Benin.
E-MAIL: atincyr@imsp-uac.org
ORCID ID:0000-0001-8346-4027

RAKESH KUMAR
ADDRESS: Punjabi University, Dept. of Mathematics, 147002, Patiala-India.
E-MAIL: dr_rk37c@yahoo.co.in
ORCID ID:0000-0001-8896-7539

VARUN JAIN
ADDRESS: Multani Mal Modi College, Dept. of Mathematics, 147001, Patiala-India.
E-MAIL: varun82jain@gmail.com
ORCID ID:0000-0003-1846-6805