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On the Characterisations of Curves with Modified Orthogonal Frame in $\mathbb{E}^3$

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Abstract − This study analyses (k, m)-type slant helices in compliance with the modified orthogonal frame in 3-dimensional Euclidean space ($\mathbb{E}^3$). Furthermore, we perform some characterisations of curves with modified orthogonal frames in $\mathbb{E}^3$.

Keywords *– Helices, (k,m)-type slant helices, modified orthogonal frame.*

Mathematics Subject Classification (2020) – 53A04, 53A25

1. Introduction

Certain particular curves and surfaces have notable importance in differential geometry. Frenet equality and curvatures of the curve are important in works related to curves. Frenet equations are employed in the structure of plenty of curve theories. One of the primary importance of these curves is the helix curve. The characterisations of curvature and torsion of helix curves perform a remarkable role in describing particular curve types. A helix is a curve whose tangent makes a constant angle with a fixed direction [1]. There is an essential categorisation of a helix. In such a way, a curve is called a helix if $\frac{\kappa}{\tau} = constant$ [2]. The curve is a (k, m)-type slant helix. Then, there is a non-zero fixed vector field, and with this constant vector field, the vector fields that have the identical index of a parallel transport frame do a constant angle. Lately, many search works related to this concept. The slant helix notion in 3-dimensional Euclidean space ($\mathbb{E}^3$) is described by Takeuchi and Izumiya [3]. Soliman et al. [4] investigated the progression of space curves using the type 3-Bishop frame. Bektaş and Yılmaz [5,6] described (k, m)-type slant helices in $\mathbb{E}^4$ and null curves in 4-dimensional Minkowski space.

Additionally, Bulut and Bektaş [7] acquired particular helices upon isomorphic differential geometry of spacelike curves in Minkowski spacetime. Furthermore, Bükcü et al. [8] investigated the spherical curves according to two types of modified orthogonal frames in $\mathbb{E}^3$. Besides, Eren and Köksal [9] investigated the development of space curves with modified orthogonal frames. Azak [10] described the notion of involute-evolute curve according to modified orthogonal frames in $\mathbb{E}^3$. This paper studies (k, m)-type slant helices in compliance with the aforesaid frame in $\mathbb{E}^3$.

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2. Preliminaries

This section presents some basic definitions to be employed in the following sections.

Definition 2.1. The curve formed by making a fixed angle in a fixed direction is called a helix. If ratio $\frac{\tau}{\kappa}$ is constant, it means that the curves are helices [13].

Throughout this paper, let C^∞ denote the curves with continuous partial derivatives of every order ∞. Suppose that $\varphi(s) \in C^3$ is parameterised by arc length s in $\mathbb{E}^3$ and its curvature $\kappa(s)$ is different from zero. Thus, an orthonormal frame $\{t, n, b\}$ provides the Serret-Frenet equations

$$\begin{bmatrix} t'(s) \\ n'(s) \\ b'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} t(s) \\ n(s) \\ b(s) \end{bmatrix} \tag{2.1}$$

Here, t is the unit tangent vector, n is the unit principal normal, b is the unit binormal. $\tau(s)$ is also the torsion of the curve. Besides a provided C^1 function $\kappa(s)$ as well as a continuous function $\tau(s)$, there is a curve C^3 with an orthonormal frame $\{t, n, b\}$ that provides the Serret-Frenet frame given in Equation (2.1) [11].

Suppose that $\varphi(t)$ is a common analytic curve that may be reparameterised by its arc length s. Herein, $s \in I$ and I is a non-empty open interval. Suggesting that the curvature function has discrete zero points or $\kappa(s)$ is not identically zero, we have an orthogonal frame $\{T, N, B\}$ named as noted below:

$$T = \frac{d\varphi}{ds}, \qquad N = \frac{dT}{ds}, \qquad \text{and} \qquad B = T \times N$$

Here, $T \times N$ is the vector multiplying T and N. The relation between $\{T, N, B\}$ and former Frenet frame vectors at non-zero spots of κ are as follows:

$$T = t, \qquad N = \kappa n, \qquad \text{and} \qquad B = \kappa b \tag{2.2}$$

Hence, we find out that $N(s_0) = B(s_0) = 0$ while $\kappa(s_0) = 0$ and squares of the length of N and B change analytically according to s. By way of Equation (2.2), it is simply to be calculated

$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\kappa^2 & \frac{\kappa'}{\kappa} & \tau \\ 0 & -\tau & \frac{\kappa'}{\kappa} \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix} \tag{2.3}$$

where the whole of the differentiations are executed according to the arc length (s) and

$$\tau = \tau(s) = \frac{det(\varphi', \varphi'', \varphi''')}{\kappa^2}$$

is the torsion of φ. By way of the Serret-Frenet Equation, we acknowledge that whichever spot in which $\kappa^2 = 0$ is a removable singularity of τ. Suggesting that $\langle , \rangle$ be the standard inner multiplying of $\mathbb{E}^3$, in that case $\{T, N, B\}$ provide for

$$\langle T, T\rangle = 1, \qquad \langle N, N\rangle = \langle B, B\rangle = \kappa^2, \qquad \text{and} \qquad \langle T, N\rangle = \langle T, B\rangle = \langle N, B\rangle = 0 \tag{2.4}$$

The orthogonal frame described in Equation (2.3) providing Equation (2.4) is called modified orthogonal frames [8,11]. Then, we acknowledge that for $\kappa = 1$, the Serret-Frenet frame matches up with the modified orthogonal frames.

Suppose that I be an open interval of real line $\mathbb{R}$ and M be a n-dimensional Riemannian manifold and $T_p(M)$ be a tangent space of M at a point $p \in M$. A curve on M is a smooth mapping $\psi: I \to M$. I, a submanifold of $\mathbb{R}$, has a coordinate system consisting of the identity map u of I. The velocity vector of ψ at $s \in I$ is noted by

$$\psi'(s) = \left.\frac{d\psi(u)}{du}\right|_s \in T_{\psi(s)}(M)$$

If $\psi'(s) \neq 0$ for whichever s, then a curve $\psi(s)$ is the regular curve. Suppose that $\psi(s)$ is a space curve on M and $\{t, n, b\}$ the moving Frenet frame along ψ, thus we have the below-mentioned features

$$\begin{aligned} \psi'(s) &= t \\ D_t t &= \kappa n \\ D_t n &= -\kappa t + \tau b \\ D_t b &= -\tau n \end{aligned} \tag{2.5}$$

where D indicates the covariant differentiation on M [12].

In the following, let vectors $V_K, K \in \{1,2,3\}$ be given as $V_1 = T, V_2 = N, V_3 = B$ and let a fixed direction vector u define the axis for the helix.

Definition 2.2. A curve $\alpha(s)$ is said to be a K-type slant helix if there exists a non-zero fixed direction vector u such that $\langle V_K, u\rangle = c$, where c is a real constant and $K \in \{1,2,3\}$ [14].

Definition 2.3. Let α be a regular unit speed curve $\mathbb{E}^3$ with Frenet frame $\{V_1, V_2, V_3\}$. We call α is a (k, m)-type slant helices if there exists a non-zero constant vector field $u \in \mathbb{E}^3$ satisfies $\langle V_k, u\rangle = a$ and $\langle V_m, u\rangle = b$. Here, a and b are constant concerning Frenet frame $\{V_1, V_2, V_3\}$ [5,6].

3. On the Characterisations of Curves with Modified Orthogonal Frame in $\mathbb{E}^3$

This section presents (k, m)-type slant helices in compliance with the modified orthogonal frame in $\mathbb{E}^3$. Furthermore, we get some characterisations of curves with modified orthogonal frames in $\mathbb{E}^3$.

Theorem 3.1. A unit speed curve ψ is a helix in compliance with orthonormal frame $\{t, n, b\}$ necessary and sufficient condition ψ is a helix in compliance with the modified orthogonal framework $\{T, N, B\}$.

PROOF. $\Rightarrow$ Let ψ be a helix in compliance with the orthonormal framework $\{t, n, b\}$, we know that

$$\langle t, u\rangle = cos\theta = constant \tag{3.1}$$

On the other hand, we write from Equations (2.2) and (3.1),

$$\langle T, u\rangle = cos\theta = constant \tag{3.2}$$

Thus, ψ is a helix in compliance with a modified orthogonal framework $\{T, N, B\}$.

$\Leftarrow$ Suppose that ψ is a helix in compliance with the modified orthogonal framework $\{T, N, B\}$. Thus, we can write

$$\langle T, u\rangle = c = constant. \tag{3.3}$$

Thus, from Equations (3.3) and (2.2), we obtain $\langle t, u\rangle = c = constant$. In this case, ψ is a helix in compliance with the orthonormal framework $\{t, n, b\}$.

Theorem 3.2. There are no (1,2)-type slant helix in compliance with the modified orthogonal framework $\{T, N, B\}$.

PROOF. Suppose that ψ is a (1,2)-type slant helix. Then, we may write

$$\langle T, u\rangle = c_1 \tag{3.4}$$

and

$$\langle N, u\rangle = c_2 \tag{3.5}$$

where c_1 and c_2 are constants. If we take the derivative of these equations, we obtain $\langle N, u\rangle = 0$. That is to say, u is orthogonal to N. Thus, there are no (1,2)-type slant helix in compliance with the modified orthogonal framework $\{T, N, B\}$.

Theorem 3.3. If ψ is a (1,3)-type slant helix in compliance with the modified orthogonal framework $\{T, N, B\}$. Then, κ and τ are constants.

PROOF. Suppose that ψ is a (1,3)-type slant helix in compliance with the modified orthogonal framework $\{T, N, B\}$. Then, we may write

$$\langle T, u\rangle = c_1 \tag{3.6}$$

and

$$\langle B, u\rangle = c_3 \tag{3.7}$$

where c_1 and c_3 are constants.

If we take the derivative of Equations (3.6) and (3.7) and use Equation (2.3), we acquire that κ and τ are constants.

Theorem 3.4. If ψ is a (2,3)-type slant helix in compliance with the modified orthogonal framework $\{T, N, B\}$, then

$$\langle T, u\rangle = \frac{\kappa'}{\kappa^3}c_2 + \frac{\tau}{\kappa^2}c_3 \tag{3.8}$$

$$\frac{1}{\tau}(\ln\kappa)' = constant \tag{3.9}$$

where c_2 and c_3 are constants.

PROOF. Suppose that ψ is a (2,3)-type slant helix in compliance with the modified orthogonal framework $\{T, N, B\}$. Then, we are written as follows:

$$\langle N, u\rangle = c_2 \tag{3.10}$$

and

$$\langle B, u\rangle = c_3 \tag{3.11}$$

Now, if we take the derivative of Equation (3.10) and we use Equation (2.3), we may write

$$\langle T, u\rangle = \frac{\kappa'}{\kappa^3}c_2 + \frac{\tau}{\kappa^2}c_3$$

Similarly, from Equations (3.11) and (2.3), we get

$$\frac{1}{\tau}(\ln\kappa)' = constant$$

Theorem 3.5. If ψ is a 3-type slant helix according to the modified orthogonal frame. Then, the following ordinary differential equation holds

$$\left(\int \frac{\kappa\kappa'}{\tau} ds\right)^2 + \left(\frac{\kappa'}{\kappa\tau}\right)^2 = constant \tag{3.12}$$

PROOF. Supposed that ψ is a 3-type slant helix according to the modified orthogonal framework $\{T, N, B\}$. Thus, we have

$$\langle B, u\rangle = c \tag{3.13}$$

and if we take the derivative of Equation (3.13) and use Equation (2.3), we obtain

$$\langle N, u\rangle = \frac{\kappa'}{\kappa\tau} c \tag{3.14}$$

Moreover, we decompose u as follows

$$u = a_1 T + a_2 N + cB$$

If we take the derivative of constant vector u and use the Equation (2.3), we get

$$a_1' T + a_2' N + a_1 N + a_2 \left(-\kappa^2 T + \frac{\kappa'}{\kappa} N + \tau B\right) + c\left(-\tau N + \frac{\kappa'}{\kappa} B\right) = 0 \tag{3.15}$$

and

$$a_1' - a_2 \kappa^2 = 0 \tag{3.16}$$

$$a_2' + a_1 + a_2 \frac{\kappa'}{\kappa} - c\tau = 0 \tag{3.17}$$

$$a_2 \tau + c\frac{\kappa'}{\kappa} = 0 \tag{3.18}$$

From Equations (3.16) and (3.18), we get

$$a_1 = -c \int \frac{\kappa\kappa'}{\tau} ds$$

$$a_2 = \frac{-c\kappa'}{\kappa\tau}$$

We know that u is constant. Moreover, we may select ${a_1}^2 + {a_2}^2 = \|u\|^2 = 1 = constant$. As a result, we are written by the following equations:

$$\left(\int \frac{\kappa\kappa'}{\tau} ds\right)^2 + \left(\frac{\kappa'}{\kappa\tau}\right)^2 = \frac{1}{c^2} = constant$$

4. Conclusion

We analyse (k, m)-form slant helices in compliance with the modified orthogonal frame in $\mathbb{E}^3$. Furthermore, we get some characterisations of curves with modified orthogonal frames in $\mathbb{E}^3$. In the subsequent studies, we will investigate the conformable curves according to a modified orthogonal frame in $\mathbb{E}^3$.

Author Contributions

All authors contributed equally to this work. They all read and approved the last version of the paper.

Conflict of Interest

The authors declare no conflict of interest.

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