e-ISSN: 2564-7954 CUJSE 20(01): 001-008 (2023) Research Article

Çankaya University

Journal of Science and Engineering

https://dergipark.org.tr/cankujse

CUJSE

Pseudo-spectrum and numerical range of matrices Walker of DimensionThree

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Keywords

Bounded operator,
Spectrum,
Pseudo-spectrum,
Numerical range,
Matrix Walker.

Abstract

The study of numerical range, spectrum and pseudo spectrum appears in diffrent scientific fields, for example the domain of spectral theory, the stability of dynamics electricity, physics, the quantum mechanics. In this paper, we find the spectrum, pseudo-spectrum and numerical range on Walker manifolds of dimension three. Two examples are given for metric g_f.

1. Introduction

The domain of function analysis presents an important part of applied mathematics, such as the results of operational equations, spectrum of operators, the field of values. For that we choose to speak about this latter in our study. The eigenvalue was one of the most important in understanding and solving linear equations and appearing of spectral theory with the investigation of localized vibrations of variety of different objects, made so much mathematics and physicals problems solved.

Hilbert was the first who coined the term of "eigenvalue" and the set of eigenvalues"the spectrum". His research laids to the foundation of spectral notion and function analysis. But the spectral objects have some changes in the case of small perturbations. In addition the study of behavior of non normal operator using the spectrum wasn't enough and evident. That what lead Trefethen in 1990 to the concept of "pseudospectrum", and he applied it to plenty of highly interesting problems.

In 1918, Toeplitz introduced the field of values (the numerical range) of matrix and it is generalized with time to the numerical range of operators. This latter plays a main role in studying matrices, polynomial, norm inequality, perturbation theory, numerical analysis. The notion of pseudospectrum was first introduced and studied by numerical analysts such as Trefethen (1997, 1999) who noted significant differences between the theoretical results and the predictions suggested by spectral analysis on the one hand, and on the other hand, the results obtained by numerical simulation for certain mathematical engineering problems where non-self-adjoint operators naturally intervene.

The numerical range of an operator, like the spectrum, is a subset of the complex plane, whose geometric properties make it possible to know some characteristics of the operator, it also has a very important relationship with the spectrum. The notion of the numerical range was introduced by Otto Toeplitz [1] in 1918 for complex matrices, in 1919 F. Hausdorff [2] proved that the numerical range of a complex matrix is convex, in years 1929 and 1932 A. Winter [3] and M. H. Stone [4] studied the relations between the numerical range and the convex hull of the spectrum of a bounded linear operator in a Hilbert space.

Around each eigenvalue in a normal matrix A pseudospectrum are circles with the radius ε. The pseudospectrum for non-normal matrices appears in several ways on the complex plane. The pseudospectrum of thirteen

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Received: July 27, 2022, Accepted: September 26, 2022

severely non-normal matrices is shown in citation [5].

Let T be an operator in $B(H)$ (i.e bounded linear operator on a Hilbert space H), the numerical range of T is the set $W(T)$ of complex numbers defined by

$$W(T) = \{\langle Tu, u\rangle \ : u \in H, \ \|u\| = 1\}.$$

The numerical range has a wide history and there is a lot of new and exclusive researchs on this concept and its generalizations.

The authors of "On the Numerical Range of Some Bounded Operators," M. M. Khorami, F. Ershad, and B. Yousefi, published their work. In this paper, they examined the extreme points of the numerical range of an operator acting on any arbitrary Banach space and provided criteria under which the numerical range of a weighted composition operator operating on a Hilbert space contains zero as an interoir point. Additionally, they provided necessary and sufficient criteria for the closure of the numerical range of an operator on a few Banach spaces. The numerical range of an operator working on Banach weighted Hardy spaces was finally described, see [6].

The present work is devoted to the study of spectra, pseudospectrum and numerical range on manifolds admitting a parallel isotropic vector field, also known as Walker manifolds.

In [7], Chaichi, García-Río and Vázquez-Abal studied three-dimensional Lorentzian manifolds which admit a light-like parallel vector field. This class of varieties, which will be denoted (M, g_f), has a local coordinate system (t, x, y), such that the field $\frac{\partial}{\partial t}$ is parallel of light type, and there is a differentiable function $f = f(x, y)$, such that the Lorentzian metric g_f is defined by :

$$g_f = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \alpha & 0 \\ 1 & 0 & f \end{pmatrix},$$

where $\alpha = \pm 1$. In all that follows, let us denote by (M, g_f) this Lorentzian manifolds, i.e, the Walker manifolds. We will find the spectrum, pseudo-spectrum and numerical range of Lorentzian metric.

The current paper is organized as follows, In section 2 somme definitions, theorems and propositions related to our study are recalled. In section 3, we calculate the spectrum, pseudo-spectrum and numerical range of metric g_f to obtain our result. Finally, we give two numerical examples to illustrate our resut and we icludes the conclusions of the paper.

2. Preliminaries

Pseudospectra of matrices and numerical range

Note that, to definie the eigenvalue z we need the condition of matrix singularity that is, $zI - A$ singular i.e., $zI - A$ is not robust. Then is better to ask robust "is $\left\|(zI - A)^{-1}\right\|$ large". As in the following definition.

Definition 1 (the norm of resolvent) *[8] Let $A \in \mathbb{C}^{n\times n}$ and $\varepsilon > 0$, then the $\varepsilon-$pseudospectrum σ_ε of A is:*

$$\sigma_\varepsilon = \left\{z \in \mathbb{C} : \left\|(zI - A)^{-1}\right\| > \varepsilon^{-1}\right\}.$$

In words, the set $\varepsilon-$pseudospectrum of the complex plane is open and bounded by ε^{-1} .

Definition 2 (the perturbation theory) *[8] Let $A \in \mathbb{C}^{n\times n}$ and $\varepsilon > 0$, then the $\varepsilon-$pseudospectrum σ_ε of A is:*

$$\sigma_\varepsilon = \left\{z \in \mathbb{C} : z \in \sigma(A + E) \ \textit{for some} \ E \in \mathbb{C}^{n\times n} \ \textit{with} \ \|E\| < \varepsilon\right\}.$$

Proposition 1 *[8] For a normal matrix, the ε-pseudospectrum is simply the union of open ε-balls with center eigenvalues and radius ε.*

Definition 3 *[9] Let A be an operator in $B(H)$ (i.e bounded linear operator on a Hilbert space H), the numerical range of A is the set $W(A)$ of complex numbers defined by*

$$W(A) = \{\langle Au, u\rangle \ : u \in H, \ \|u\| = 1\}.$$

Definition 4 *[8] The (2-norm) numerical range or field of values of a matrix $M \in \mathbb{C}^{n\times n}$ is the set*

$$W(M) = \{z^*Mz,\ \ z \in \mathbb{C}^n, \|z\| = 1\},$$

is defined to be where z^* denotes the conjugate transpose of the vector z.

Proposition 2 *[8] Let $A, B \in \mathbb{C}^{n\times n}$, $\alpha, \beta \in \mathbb{C}$ and $\gamma \in \mathbb{C}$ then:*

1)

$$W(A+B) \subset W(A) + W(B).$$

2)

$$W(\gamma A) = \gamma W(A).$$

3)

$$W(\alpha A + \beta I) = \alpha W(A) + \beta.$$

4)

$$W(A^*) = \{\bar{z}, z \in W(A)\}.$$

Definition 5 *[8] Let $A \in \mathbb{C}^{n\times n}$. The numerical radius of matrix A is defined by*

$$\mu(A) = \sup_{z\in W(A)} |z|.$$

Theorem 1 *[8] The numerical range of matrix A is nonempty bounded and convex set.*

Proposition 3 *[8] Let $A \in \mathbb{C}^{n\times n}$. Then*

$$\sigma(A) \subset W(A).$$

Proof 1 *Let $\lambda \in \sigma(A)$ and $x \in H$ such that $\|x\| = 1$ then, $Ax = \lambda x$ and then $\langle (A-\lambda)x, x\rangle = 0$ then, $\langle Ax, x\rangle = \lambda$ then $\lambda \in W(A)$.*

Theorem 2 *[8] Let $A \in \mathbb{C}^{n\times n}$. Then,*

$$\sigma_\varepsilon(A) \subseteq W(A) + \Delta_\varepsilon,$$

where Δ_ε is the closed disk of center 0 and radius ε.

3. The main result

In this paper, we will find spectrum, pseudo spectrum and the numerical range of Walker manifolds of dimension three.

Spectrum of metric g_f

The eigenvalues of matrix g_f are by

$$\sigma(g_f) = \left\{m_1 = \alpha,\ m_2 = \frac{1}{2}f - \frac{1}{2}\sqrt{f^2+4},\ m_3 = \frac{1}{2}f + \frac{1}{2}\sqrt{f^2+4}\right\}.$$

Pseudo-spectrum of metric g_f

As well as we know, if g_f is symmetrical, then matrix g_f is normal. Further, pseudo-spectrum is given by:

$$\Lambda_\varepsilon(g_f) = \{z \in \mathbb{C} : |z - m_i| \leq \varepsilon\}, \text{ with } i \in \{1,2,3\}.$$

Numerical range of metric g_f

The following theorem give the numerical range of the matrix g_f.

Theorem 3 *The numerical range of matrix g_f are given by:*

1.

$$W(g_f) \subseteq [-1, 2+f(x,y)], \text{ if } \alpha = 1 \text{ and } f(x,y) \in \mathbb{R}^+, \quad (1)$$

$$W(g_f) \subseteq [-1+f(x,y), 2], \text{ if } \alpha = 1 \text{ and } f(x,y) \in \mathbb{R}^-, \quad (2)$$

2.

$$W(g_f) \subseteq [-2, 1+f(x,y)], \text{ if } \alpha = -1 \text{ and } f(x,y) \in \mathbb{R}^+, \quad (3)$$

$$W(g_f) \subseteq [-2+f(x,y), 1], \text{ if } \alpha = -1 \text{ and } f(x,y) \in \mathbb{R}^-. \quad (4)$$

Proof 2 *Let $z \in \mathbb{C}^3$ such that $z \neq 0$, we put $z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$, with $z_i = r_i e^{i\theta_i}, i \in \{1,2,3\}$. We have*

$$\begin{aligned} z^* g_f z &= (\overline{z_1}, \overline{z_2}, \overline{z_3}) \begin{pmatrix} 0 & 0 & 1 \\ 0 & \alpha & 0 \\ 1 & 0 & f \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}, \\ &= \alpha |z_2|^2 + |z_3|^2 f(x,y) + z_1\overline{z_3} + z_3\overline{z_1}, \end{aligned}$$

so,

$$\frac{z^* g_f z}{z^* z} = \frac{\alpha |z_2|^2 + |z_3|^2 f(x,y) + z_1\overline{z_3} + z_3\overline{z_1},}{\sum_{i=1}^{3} |z_i|^2}.$$

For making the calcul, of abone equality, we split in several steps.
*Steps*1. *We know that*

$$\frac{|z_j|^2}{\sum_{i=1}^{3} |z_i|^2} \leq 1, \ \forall j \in \{1,2,3\}, \quad (5)$$

because if we have the opposite, i.e, we put $\exists j \in \{1,2,3\}$, such as

$$\frac{|z_j|^2}{\sum_{i=1}^{3} |z_i|^2} > 1$$

so, for example, ($j = 1$) we find

$$|z_2|^2 + |z_3|^2 < 0,$$

and this is a contradiction.
*Steps*2. *We have*

$$z_i\overline{z_j} + z_j\overline{z_i} = 2r_i r_j \cos(\theta_i - \theta_j) \quad (6)$$

so

$$-1 \leq \frac{z_i\overline{z_j} + z_j\overline{z_i}}{\sum_{i=1}^{3} |z_i|^2} \leq 1, \quad \forall i,j \in \{1,2,3\}, \quad (7)$$

1. (i) if $\alpha = 1$ and $f(x,y) \geq 0$. From (5) and (7), we have

$$-1 \leq \frac{z^* g_f z}{z^* z} \leq 2 + f(x,y),$$

(ii) if $\alpha = 1$ and $f(x,y) \leq 0$. From (5) and (7), we have

$$-1 + f(x,y) \leq \frac{z^* g_f z}{z^* z} \leq 2.$$

From (i) and (ii), we deduce (1) *and* (2) *recpectively.*

2.(iii) If $\alpha=-1$ *and* $f(x,y)\geq 0$. *From* (5) *and* (7), *we have*

$$-2\leq \frac{z^*g_fz}{z^*z}=\frac{-|z_2|^2+|z_3|^2f(x,y)+z_1\overline{z_3}+z_3\overline{z_1}}{\sum_{i=1}^{3}|z_i|^2}\leq 1+f(x,y),$$

(iv) if $\alpha=-1$ *and* $f(x,y)\leq 0$. *From* (5) *and* (7), *we have*

$$-2+f(x,y)\leq \frac{z^*g_fz}{z^*z}=\frac{-|z_2|^2+|z_3|^2f(x,y)+z_1\overline{z_3}+z_3\overline{z_1}}{\sum_{i=1}^{3}|z_i|^2}\leq 1.$$

From (iii) and (iv), we deduce (3) *and* (4) *recpectively.*

Example 3.1 *For* $\alpha=1$ *and* $f(x,y)=x+y,\ x,y\in$

If $x=1$ *and* $y=\frac{1}{2}$ *then*

$$g_{\frac{3}{2}}=\begin{pmatrix}0&0&1\\0&1&0\\1&0&\frac{3}{2}\end{pmatrix},$$

so

$$\frac{z^*g_{\frac{3}{2}}z}{z^*z}=\frac{r_2^2+\frac{3}{2}r_3^2+2r_1r_3\cos(\theta_1-\theta_3)}{r_1^2+r_2^2+r_3^2}\leq 2.$$

We chek the above inequality. By contraduction, $\exists\, z\in\mathbb{C}^3$, *such that*

$$2<\frac{r_2^2+\frac{3}{2}r_3^2+2r_1r_3\cos(\theta_1-\theta_3)}{r_1^2+r_2^2+r_3^2},$$

we have

$$\frac{r_2^2+\frac{3}{2}r_3^2+2r_1r_3\cos(\theta_1-\theta_3)}{r_1^2+r_2^2+r_3^2}\leq\frac{r_2^2+\frac{3}{2}r_3^2+2r_1r_3}{r_1^2+r_2^2+r_3^2},$$

so

$$\frac{r_2^2+\frac{3}{2}r_3^2+2r_1r_3}{r_1^2+r_2^2+r_3^2}>2,$$

this is a contradiction.

If $r_2=0$, $r_3=2r_1$ *and* $(\theta_1-\theta_3)=2k\pi, k\in\mathbb{Z}$, *we find* $\frac{z^*g_{\frac{3}{2}}z}{z^*z}=2$. *We deduce that,* $2\in W(g_{\frac{3}{2}})$.

On the other hand, we have

$$\frac{z^*g_{\frac{3}{2}}z}{z^*z}=-\frac{1}{2}+\frac{\frac{1}{2}r_1^2+\frac{3}{2}r_2^2+2r_3^2+2r_1r_3\cos(\theta_1-\theta_3)}{r_1^2+r_2^2+r_3^2}.$$

Since

$$\frac{\frac{1}{2}r_1^2+\frac{3}{2}r_2^2+2r_3^2+2r_1r_3\cos(\theta_1-\theta_3)}{r_1^2+r_2^2+r_3^2}\geq 0,$$

then

$$\frac{z^*g_{\frac{3}{2}}z}{z^*z}=-\frac{1}{2}+\frac{\frac{1}{2}r_1^2+\frac{3}{2}r_2^2+2r_3^2+2r_1r_3\cos(\theta_1-\theta_3)}{r_1^2+r_2^2+r_3^2}\geq-\frac{1}{2}.$$

If $r_2=0$, $r_1=2r_3$ *and* $(\theta_1-\theta_3)=(2k+1)\pi$, $k\in\mathbb{Z}$, *we find* $\frac{z^*g_{\frac{3}{2}}z}{z^*z}=-\frac{1}{2}$. *We deduce that,* $-\frac{1}{2}\in W(g_{\frac{3}{2}})$.

Finally, we find

$$W(g_{\frac{3}{2}})=\left[-\frac{1}{2},2\right].$$

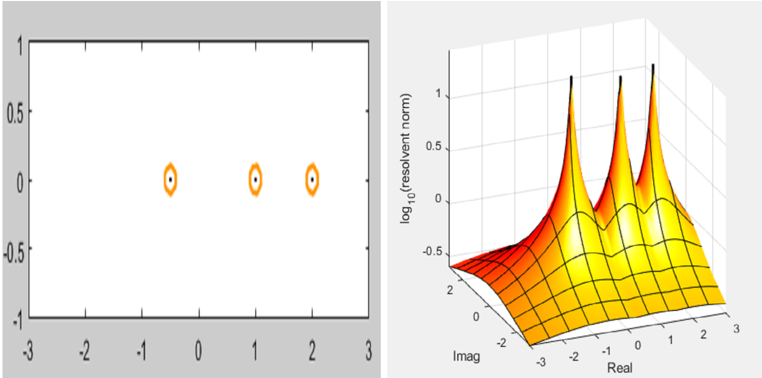

Figure 1. Spectrum and pseudospectrum of the matrix g_f for $\alpha = 1$ and $f(x,y) = \frac{3}{2}$.

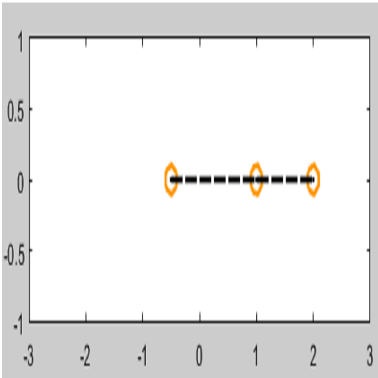

Figure 2. Numerical range of the matrix g_f for $\alpha = 1$ and $f(x,y) = \frac{3}{2}$.

Example 3.2 *For $\alpha = -1$ and $f(x,y) = x+y,\ x,y \in \mathbb{R}^-$.*

If $x = -1$ and $y = -1$ then

$$g_{-2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -2 \end{pmatrix},$$

so

$$\frac{z^* g_{-2} z}{z^* z} = \frac{-r_2^2 - 2r_3^2 + 2r_1 r_3 \cos(\theta_1 - \theta_3)}{r_1^2 + r_2^2 + r_3^2} \leq \frac{1}{2}.$$

We chek the above inequality. By contraduction, $\exists\, z \in \mathbb{C}^3$, such that

$$\frac{1}{2} < \frac{-r_2^2 - 2r_3^2 + 2r_1 r_3 \cos(\theta_1 - \theta_3)}{r_1^2 + r_2^2 + r_3^2},$$

we have

$$\frac{-r_2^2 - 2r_3^2 + 2r_1 r_3 \cos(\theta_1 - \theta_3)}{r_1^2 + r_2^2 + r_3^2} \leq \frac{-r_2^2 - r_3^2 + 2r_1 r_3}{r_1^2 + r_2^2 + r_3^2}.$$

So

$$\frac{-r_2^2 - r_3^2 + 2r_1 r_3}{r_1^2 + r_2^2 + r_3^2} > \frac{1}{2},$$

this is a contradiction.

If $\frac{1}{2} \in W(g_{-2})$, then

$$\frac{z^* g_{-2} z}{z^* z} = \frac{-r_2^2 - 2r_3^2 + 2r_1 r_3 \cos(\theta_1 - \theta_3)}{r_1^2 + r_2^2 + r_3^2} = \frac{1}{2},$$

so, we get

$$3r_2^2+r_3^2+(r_1\cos(\theta_1)-2r_3\cos(\theta_3))^2+(r_1\sin(\theta_1)-2r_3\sin(\theta_3))^2=0,$$

the only condition which verify the above equation is just when $r_1=r_2=r_3=0$, *and that is not possible because* $z\neq 0$. *Then* $\frac{1}{2}\notin W(g_{-2})$.

On the other hand, we have

$$\frac{z^*g_{-2}z}{z^*z}=-\frac{5}{2}+\frac{\frac{5}{2}r_1^2+\frac{3}{2}r_2^2+\frac{1}{2}r_3^2+2r_1r_3\cos(\theta_1-\theta_3)}{r_1^2+r_2^2+r_3^2}.$$

Since

$$\frac{\frac{5}{2}r_1^2+\frac{3}{2}r_2^2+\frac{1}{2}r_3^2+2r_1r_3\cos(\theta_1-\theta_3)}{r_1^2+r_2^2+r_3^2}\geq 0,$$

then

$$\frac{z^*g_{-2}z}{z^*z}=-\frac{5}{2}+\frac{\frac{5}{2}r_1^2+\frac{3}{2}r_2^2+\frac{1}{2}r_3^2+2r_1r_3\cos(\theta_1-\theta_3)}{r_1^2+r_2^2+r_3^2}\geq -\frac{5}{2}.$$

If $-\frac{5}{2}\in W(g_{-2})$, *then*

$$\frac{z^*g_{-2}z}{z^*z}=\frac{-r_2^2-2r_3^2+2r_1r_3\cos(\theta_1-\theta_3)}{r_1^2+r_2^2+r_3^2}=-\frac{5}{2},$$

so we get

$$r_1^2+3r_2^2+(2r_1\cos(\theta_1)+r_3\cos(\theta_3))^2+(2r_1\sin(\theta_1)+r_3\sin(\theta_3))^2=0,$$

the only condition which verify the equation is just when $r_1=r_2=r_3=0$, *and that is not possible because* $z\neq 0$. *Then,* $-\frac{5}{2}\notin W(g_{-2})$.

Finally, we find

$$W(g_{-2})\subset\left]-\frac{5}{2},\frac{1}{2}\right[.$$

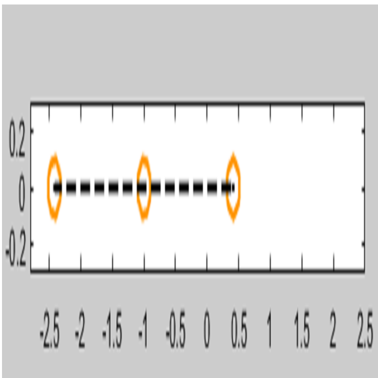

Figure 3. Numerical range of the matrix g_f for $\alpha=-1$ and $f(x,y)=-2$.

4. Conclusion

Our work deal with one of the most important and the newest topics in functional analysis, and it focused on several aspects such as the theory of spectrum, which contribute to the solution of linear equations addition to many mathematics and physicals problems. Also, we made the point in pseudo spectrum which had great role in understanding the perturbations of spectral objects. This latter contributed to many mathemathical discplines such as the theory of operators, matrix polynomials, applications to various areas including C-algebras. And the subject of numerical range of matrices is still open for scientific research in all aspects algebric, geometric. We touched in this paper for a generalization spectrum, pseudo-spectrum and numerical range on Walker manifolds of dimension three.

Declaration of Competing Interest

The author(s), declares that there is no competing financial interests or personal relationships that influence the work in this paper.

Authorship Contribution Statement

Khalef Bouich: Data creation, Draft preparation, Writing, Reviewing.
Rafik Derkaoui: Methodology, Writing, Editing, Investigation.
Abderrahmane Smail: Methodology, Revieing, Supervision, Investigation.

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