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Research Article

On the geometry of fixed points and discontinuity

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Abstract

Recently, there has been a considerable effort to obtain new solutions to the Rhoades' open problem on the existence of contractive mappings that admit discontinuity at the fixed point. An extended version of this problem is also stated using a geometric approach. In this paper, we obtain new solutions to this extended version of the Rhoades' open problem. A related problem, the fixed-circle problem (resp. fixed-disc problem) is also studied. Both of these problems are related to the geometric properties of the fixed point set of a self-mapping on a metric space. Furthermore, a new result about metric completeness and a short discussion on the activation functions used in the study of neural networks are given. By providing necessary examples, we show that our obtained results are effective.

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1. Introduction

Let f be a self-mapping of a metric space (X, d) and the set $Fix(f) = \{x \in X : fx = x\}$ be the fixed point set of f. Throughout the paper $\mathbb{R}_+$ and $\mathbb{R}$ stand for the set of non-negative real numbers and real numbers, respectively. In [37], Rhoades presented a detailed study on the continuity of a large number of contractive mappings at their fixed points. Then, he stated the question whether there exists a contractive definition which is strong enough to generate a fixed point but which does not force the mapping to be continuous at the fixed point. In [29], Pant gave the first solution to this open problem.

Theorem 1.1 ([29, Theorem 1]). *Let f be a self-mapping of a complete metric space (X, d) such that for any $x, y \in X$*

(a) $d(fx, fy) \leq \phi(\max\{d(x, fx), d(y, fy)\})$, *where the function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is such that $\phi(t) < t$ for each $t > 0$,*

(b) *Given $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\varepsilon < \max\{d(x, fx), d(y, fy)\} < \varepsilon + \delta \implies d(fx, fy) \leq \varepsilon.$$

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Then f has a unique fixed point, say z. Moreover, f is continuous at z if and only if $\lim_{x\to z}\max\{d(x,fx),d(y,fy)\}=0$.

We now give an apparently counter intuitive example of a contractive mapping which is discontinuous everywhere, yet, possesses a fixed point.

Example 1.2. Let $X=[0,2]$ and d be the Euclidean metric defined by $d(x,y)=|x-y|$. Define $f:X\to X$ by

$$fx=\begin{cases} 0 & if \quad x>1 \text{ and } x \text{ is rational} \\ \frac{2+x}{3} & if \quad x\leq 1 \text{ and } x \text{ is rational} \\ 1 & if \quad x \text{ is irrational}\end{cases}.$$

Then f is discontinuous everywhere, possesses a unique fixed point $x=1$ and satisfies the conditions:

(a) $d(fx,fy)\leq\varphi(\max\{d(x,fx),d(y,fy)\})$, where the function $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ is such that $\varphi(t)=1$ if $t>1$ and $\varphi(t)=\frac{t}{2}$ if $t\leq 1$,

(b) Given $\varepsilon>0$ there exists a number $\delta=\delta(\varepsilon)>0$ such that

$$\varepsilon<\max\{d(x,fx),d(y,fy)\}<\varepsilon+\delta\Rightarrow d(fx,fy)\leq\varepsilon$$

where $\delta(\varepsilon)=\min\{\varepsilon,1-\varepsilon\}$ if $\varepsilon<1$ and $\delta(\varepsilon)=\varepsilon$ when $\varepsilon\geq 1$.

This follows from the following computations:

Case 1. If x,y are irrational numbers, then we have $d(fx,fy)=0$ and

$$0<\max\{d(x,fx),d(y,fy)\}=\max\{|1-x|,|1-y|\}<1.$$

Case 2. If $x<y$ are rational numbers such that ≤ 1, then $d(fx,fy)=\frac{y-x}{3}$ and

$$0<\max\{d(x,fx),d(y,fy)\}=\frac{2(1-x)}{3}\leq\frac{2}{3}.$$

Case 3. If $y>x>1$ are rational numbers, then $d(fx,fy)=0$ and

$$1<\max\{d(x,fx),d(y,fy)\}=y\leq 2.$$

Case 4. If $x\leq 1, y>1$ are rational numbers, then, $d(fx,fy)=\frac{2+x}{3}\leq 1$ and

$$1<\max\{d(x,fx),d(y,fy)\}=y\leq 2.$$

Case 5. If $x\leq 1$ rational and y irrational numbers, then $d(fx,fy)=\frac{1-x}{3}\leq\frac{1}{3}$ and

$$0<\max\{d(x,fx),d(y,fy)\}=\max\left\{\frac{2(1-x)}{3},|1-y|\right\}<1.$$

Case 6. If $x>1$ rational and y irrational numbers, then $d(fx,fy)=1$ and

$$1<\max\{d(x,fx),d(y,fy)\}=x\leq 2.$$

This is the first example of a contractive mapping which is discontinuous everywhere yet ensures the existence of a fixed point.

After this first result, several new solutions were given (see, for example, [1,3–5,7,24,27, 29–34, 44, 50] and the references therein). All these results are important both in theory and practice since the investigation of the neural network systems with discontinuous activation functions have been intensively studied in recent years (see [2, 8–12, 15, 17–21, 47, 48]). In a recent paper, new answers to this question was obtained by proving some fixed point theorems under contractive conditions which admit discontinuity at the fixed point [34]. All these new answers require the uniqueness of the fixed point, that is, for a self-mapping f in question, the set $Fix(f)$ is a singleton. On the other hand, there exist a lot of non-unique fixed point results in the literature (for example see [14] for a recent survey on this topic). Also, discontinuous activation functions, which have more than one fixed point, have appeared in the study of various neural networks. In recent years, geometric properties of non-unique fixed points have been investigated as the *fixed-circle problem* (*resp. fixed-disc problem*). For more details about this problem on metric (resp. generalized metric) spaces, one can see [13, 16, 23, 25–28, 30, 31, 38, 43, 45, 46] and the references therein. The fixed-circle problem (resp. fixed-disc problem) is the investigation of some necessary conditions such that the fixed point set $Fix(f)$ of a self-mapping f contains a circle $C_{x_0,r} = \{x \in X : d(x, x_0) = r\}$ (resp. a disc $D_{x_0,r} = \{x \in X : d(x, x_0) \leq r\}$). In other words, a circle $C_{x_0,r}$ (resp. a disc $D_{x_0,r}$) is called a fixed circle (resp. a fixed disc) of a self-mapping f if $fx = x$ for all $x \in C_{x_0,r}$ (resp. $x \in D_{x_0,r}$).

In [7], considering these types of applications and using a geometric approach, an extended version of the Rhoades' question has been stated as follows:

> *What are the conditions that generate a fixed circle but do not force the self-mapping to be continuous on its fixed circle?*

An answer to this extended version was presented in [7]. Such geometric approaches have an important role for some real-life problems. For example, in [35], an efficient application of the notion of an Apollonius circle was used to obtain a geometric method in data point analysis. In [41], orthogonal families of ellipses and hyperbolas were used to make a better decision making in the study of a wind prediction system for the wind power generation using ensemble of multiple complex extreme learning machines (C-ELM). Similar studies show the importance and effectiveness of the non-unique fixed point results and geometric methods (see [43] for more details). Geometric approaches are also crucial for theoretical studies (see, for instance, [36]).

In summarizing, if the fixed point set $Fix(f)$ is not a singleton, $Fix(f)$ can contain a circle (or a disc) or $Fix(f)$ can be a circle (or a disc). Then the resulting question is the investigation of fixed-circle (resp. fixed-disc) problem which is a recent approach to study new fixed-point results using the geometric properties of the fixed points (see [25] and [23]). Consequently, the fixed-circle problem and the extended version of the Rhoades' open problem are related to each other through the geometric properties of the set $Fix(f)$.

For a self-mapping f of a metric space (X, d), let us denote:

$$m(x,y) = \max\left\{ \begin{array}{c} d(x,y), ad(x,fx) + (1-a)d(y,fy), \\ (1-a)d(x,fx) + ad(y,fy), \frac{b[d(x,fy)+d(y,fx)]}{2} \end{array} \right\}, 0 \leq a, b < 1. \qquad (1.1)$$

In [34], using the number $m(x, y)$, Pant et al. presented new solutions to the Rhoades' original question. In 2019, Pant et al. [34] obtained the following theorem:

Theorem 1.3 ([34, Theorem 3.1]). *Let f be a self-mapping of a complete metric space (X, d) such that for any $x, y \in X$ we have*

(i) *Given $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\varepsilon < m(x,y) < \varepsilon + \delta \implies d(fx, fy) \leq \varepsilon,$$

(ii) $d(fx, fy) \leq \phi(m(x, y)), \phi : \mathbb{R}_+ \to \mathbb{R}_+$ *is such that* $\phi(t) < t$ *for each* $t > 0$.

Then f has a unique fixed point, say z, and for each x in X the sequence of iterates $\{f^n x\}$ converges to the fixed point. If $0 < a < 1$ then f is continuous at z, and if $a = 0$ then f is continuous at z if and only if $\lim_{x\to z} m(x, z) = 0$.

In this paper, our aim is twofold. First, we prove that the fixed point theorems proved in [34] imply completeness of the metric space. Then we focus on the extended version of the Rhoades' question stated in [7]. We present new results to the fixed-circle problem (resp. fixed-disc problem). Finally, we give a short discussion on the activation functions used in the study of neural networks.

2. A new result about metric completeness

Now, we show that Theorem 1.3 ([34, Theorem 3.1]) characterizes completeness of the metric space.

Theorem 2.1. *Let (X, d) be a metric space. If every self-mapping of X satisfying conditions (i) and (ii) of Theorem 1.3 has a fixed point, then X is complete.*

Proof. Assume that every self-mapping of X satisfying conditions (i) and (ii) of Theorem 1.3 has a fixed point. We show that X is complete. On the contrary, suppose X is not complete. There exists a Cauchy sequence $S = \{u_1, u_2, u_3, \ldots\}$ in X which does not converge and consisting of distinct points. For any given $x \in X$, since x is not a limit point of the sequence S, $d(x, S - \{x\}) > 0$ and there exists a least positive integer $n(x)$ such that $x \neq u_{n(x)}$ and for each $m \geq n(x)$ we have

$$d(u_{n(x)}, u_m) < \frac{1}{4} d(x, u_{n(x)}). \tag{2.1}$$

If we define the map $f : X \to X$ by $f(x) = u_{n(x)}$, then, $f(x) \neq x$ for each x and, using (2.1), for any $x, y \in X$ we get

$$d(fx, fy) = d(u_{n(x)}, u_{n(y)}) < \frac{1}{4} d(x, u_{n(x)}) = \frac{1}{4} d(x, fx) \text{ if } n(x) \leq n(y)$$

or

$$d(fx, fy) = d(u_{n(x)}, u_{n(y)}) < \frac{1}{4} d(y, u_{n(y)}) = \frac{1}{4} d(y, fy) \text{ if } n(x) > n(y).$$

This implies that

$$\begin{aligned} d(fx, fy) &< \frac{1}{4} \max\{d(x, fx), d(y, fy)\} \\ &< \frac{1}{2} \max \left\{ \begin{array}{c} d(x, y), ad(x, fx) + (1-a)d(y, fy), \\ (1-a)d(x, fx) + ad(y, fy), \frac{b[d(x,fy)+d(y,fx)]}{2} \end{array} \right\} \\ &= \frac{1}{2} m(x, y). \end{aligned} \tag{2.2}$$

That is, given $\varepsilon > 0$ we can choose $\delta(\varepsilon) = \varepsilon$ such that

$$\varepsilon < m(x, y) < \varepsilon + \delta \Rightarrow d(fx, fy) \leq \varepsilon. \tag{2.3}$$

Similarly, using (2.2), we can define $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ by $\phi(t) = \frac{t}{2}$ such that

$$d(fx, fy) \leq \phi(m(x, y)). \tag{2.4}$$

By the inequalities (2.2), (2.3) and (2.4) we deduce that the map f satisfies conditions (i) and (ii) of Theorem 1.3. Furthermore, f is a fixed point free map whose range is contained in the non-convergent Cauchy sequence $S = \{u_n\}$. Hence, we have a self-mapping f of X which satisfies all the conditions of Theorem 1.3, but does not possess a fixed point. We get a contradiction with the hypotheses of Theorem 1.3. Hence, X is complete. □

Now, we give an example to indicate the difference between Theorem 2.1 and the following theorem of Subrahmanyam [42]:

Theorem 2.2 ([42, Theorem 1])**.** *A metric space (X, d) in which every mapping f of X onto itself, satisfying the conditions*:

(A) $d(fx, fy) \leq \lambda \max\{d(x, fx), d(y, fy)\}$, $x, y \in X$, for a fixed $\lambda > 0$;

(B) $f(X)$ is countable;

has a fixed-point, is complete.

The converse of Theorem 2.2 does not hold unless $0 < \lambda < 1$, that is, a self-mapping f of a complete metric space (X, d) satisfying conditions (A) and (B) of Theorem 2.2 need not have a fixed point. On the other hand, by virtue of Theorem 1.3, the converse of Theorem 2.1 always holds. The following examples illustrate this.

Example 2.3. Let $X=\{1\}\cup\left\{\frac{2^n-1}{2^n}:n=1,2,3,\cdots\right\}$ equipped with the Euclidean metric. Define $f:X\to X$ by

$$fx=\left\{\begin{array}{ccc}\frac{1}{2} & \text{if} & x=1\\ \frac{2^{n+1}-1}{2^{n+1}} & \text{if} & x=\frac{2^n-1}{2^n}\end{array}\right..$$

Then X is a complete metric space, $f(X)$ is countable and f satisfies the condition (A) of Theorem 2.2 with $\lambda=1$ but f does not possess a fixed point. This shows that the converse of Theorem 2.2 does not hold unless $0<\lambda<1$. It may be noted that f does not satisfy condition (ii) of Theorem 1.3.

Example 2.4. Let $X=[0,2]$ equipped with the Euclidean metric. Define $f:X\to X$ by

$$fx=\left\{\begin{array}{ccc}1 & \text{if} & x\leq 1\\ 0 & \text{if} & x>1\end{array}\right..$$

Then X is a complete metric space and $f(X)$ is countable. f satisfies the conditions of Theorem 1.3 with $a=0$ and the functions

$$\delta(\varepsilon)=\left\{\begin{array}{ccc}\varepsilon & \text{if} & \varepsilon\geq 1\\ 1-\varepsilon & \text{if} & \varepsilon<1\end{array}\right.$$

and

$$\phi(t)=\left\{\begin{array}{ccc}1 & \text{if} & t>1\\ \frac{t}{2} & \text{if} & t\leq 1\end{array}\right..$$

Further, f possesses a unique fixed point $z=1$. This shows that the converse of Theorem 2.1 holds. It may be noted that the mapping f in this example satisfies the condition (A) of Theorem 2.2 with $\lambda=1$. This means that the converse of Theorem 2.1 holds and is not limited to $0<\lambda<1$.

3. A geometric viewpoint to the Rhoades' open problem

Consider the number $m(x,y)$ defined in (1.1). By fixing the second variable y as $y=x_0$ in the definition of $m(x,y)$, we get

$$m(x,x_0)=\max\left\{\begin{array}{c}d(x,x_0),ad(x,fx)+(1-a)d(x_0,fx_0),(1-a)d(x,fx)+ad(x_0,fx_0),\\ \frac{b[d(x,fx_0)+d(x_0,fx)]}{2}\end{array}\right\},$$

where $0\leq a,b<1$. We give the following definition modifying the condition (iv) in [34, Theorem 3.1].

Definition 3.1. Let f be a self-mapping on a metric space (X,d). If there exists $x_0\in X$ and a function $\phi:\mathbb{R}_+\to\mathbb{R}_+$ such that $\phi(t)<t$ for each $t>0$ satisfying

$$d(x,fx)\leq\phi(m(x,x_0)),$$

for all $x\in X$, then f is called an m_{x_0}-type contraction.

In the following fixed-circle theorem, using the m_{x_0}-type contractive property of the self-mapping with a geometric condition, we give a solution to the extended version of the Rhoades' question without any assumption on X.

Theorem 3.2. *Let (X,d) be a metric space, f be a self-mapping on X and the number ρ be defined as follows :*

$$\rho=\inf\left\{d(x,fx):x\notin Fix(f),x\in X\right\}. \tag{3.1}$$

If there exists a point $x_0\in X$ such that

(i) *f is an m_{x_0}-type contraction with x_0,*

(ii) *$d(x_0,fx)\leq\rho$ for all $x\in C_{x_0,\rho}$,*

then $x_0\in Fix(f)$ and $C_{x_0,\rho}$ is a fixed circle of f, that is, $C_{x_0,\rho}\subset Fix(f)$. Furthermore, f is continuous at any $u\in C_{x_0,\rho}$ if and only if $\lim_{x\to u}m(x,u)=0$.

Proof. First, we show that $x_0 \in Fix(f)$. If $d(x_0, fx_0) > 0$, using the m_{x_0}-type contractive property and definition of the function ϕ, we get

$$\begin{aligned} d(fx_0, x_0) &\leq \phi\left(m(x_0, x_0)\right) \\ &= \phi\left(\max\left\{\begin{array}{c} d(x_0, x_0), ad(x_0, fx_0) + (1-a)d(x_0, fx_0), \\ (1-a)d(x_0, fx_0) + ad(x_0, fx_0), \frac{b[d(x_0, fx_0)+d(x_0, fx_0)]}{2} \end{array}\right\}\right) \\ &= \phi\left(d(x_0, fx_0)\right) \\ &< d(x_0, fx_0) = d(fx_0, x_0), \end{aligned}$$

a contradiction. This implies $d(fx_0, x_0) = 0$, that is, $fx_0 = x_0$.

If $\rho = 0$, clearly we have $C_{x_0,\rho} = \{x_0\}$ and $C_{x_0,\rho} \subset Fix(f)$.

Let $\rho \neq 0$. For any $x \in C_{x_0,\rho}$, assume that $fx \neq x$ and so $d(fx, x) > 0$. Considering definition of the number ρ, we have

$$\rho \leq d(fx, x). \tag{3.2}$$

Using the m_{x_0}-type contractive property, definition of the function ϕ, (3.2) and the geometric condition (ii) we obtain

$$\begin{aligned} d(fx, x) &\leq \phi\left(m(x, x_0)\right) \\ &= \phi\left(\max\left\{\begin{array}{c} d(x, x_0), ad(x, fx) + (1-a)d(x_0, fx_0), \\ (1-a)d(x, fx) + ad(x_0, fx_0), \frac{b[d(x, fx_0)+d(x_0, fx)]}{2} \end{array}\right\}\right) \\ &= \phi\left(\max\left\{\begin{array}{c} d(x, x_0), ad(x, fx), (1-a)d(x, fx), \\ \frac{b[d(x, x_0)+d(x_0, fx)]}{2} \end{array}\right\}\right) \qquad (3.3) \\ &\leq \phi\left(\max\left\{\rho, ad(x, fx), (1-a)d(x, fx), b\rho\right\}\right) \\ &< \max\left\{\rho, ad(x, fx), (1-a)d(x, fx), b\rho\right\}. \end{aligned}$$

Let

$$\alpha = \max\left\{\rho, ad(x, fx), (1-a)d(x, fx), b\rho\right\}.$$

Then, α can be equal to ρ, $ad(x, fx)$ or $(1-a)d(x, fx)$ and using (3.3) we get a contradiction with the definitions of the numbers ρ and a. Hence, it should be $fx = x$. Consequently, f fixes the circle $C_{x_0,r}$.

For the last part of the proof, assume that f is continuous at $u \in C_{x_0,\rho}$ and $x_n \to u$. Hence $fx_n \to fu = u$ and using the triangle inequality we obtain $d(x_n, fx_n) \to 0$. Then we have

$$\begin{aligned} \lim m(x_n, u) &= \lim\left(\max\left\{\begin{array}{c} d\left(x_n, u\right), ad\left(x_n, fx_n\right) + (1-a)d(u, fu), \\ (1-a)d\left(x_n, fx_n\right) + ad(u, fu), \frac{b[d(x_n, fu)+d(u, fx_n)]}{2} \end{array}\right\}\right) \\ &= \lim\left(\max\left\{\begin{array}{c} d\left(x_n, u\right), ad\left(x_n, fx_n\right), (1-a)d\left(x_n, fx_n\right), \\ \frac{b[d(x_n, u)+d(u, fx_n)]}{2} \end{array}\right\}\right) = 0. \end{aligned}$$

Conversely, if $\lim_{x_n \to u} m(x_n, u) = 0$, then clearly, we get $d(x_n, fx_n) \to 0$ as $x_n \to u$. So we deduce $fx_n \to u = fu$, that is, f is continuous at u. $\square$

Remark 3.3. We have seen that x_0 is a fixed point of any m_{x_0}-type contraction. Hence, we can simplify the definition of an m_{x_0}-type contraction as follows:

Let f be a self-mapping on a metric space (X, d). For a point $x_0 \in Fix(f)$, if there exists a function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\phi(t) < t$ for each $t > 0$ satisfying

$$\begin{aligned} &d(x, fx) \\ &\leq \phi\left(\max\left\{d(x, x_0), cd(x, fx), \frac{b\left[d(x, x_0) + d(x_0, fx)\right]}{2}\right\}\right), \quad \left(c \in \left[\frac{1}{2}, 1\right] \text{ and } b \in [0, 1)\right) \end{aligned}$$

for all $x \in X$, then f is called an m_{x_0}-type contraction. Observe that if $a \in [0,1)$, then $\max\{a, 1-a\} \in \left[\frac{1}{2}, 1\right]$.

Using the new version of an m_{x_0}-type contraction, we can give the following immediate consequences of Theorem 3.2.

Corollary 3.4. *Let (X,d) be a metric space, f be a self-mapping on X and the number ρ be defined as in (3.1). If there exists a point $x_0 \in Fix(f)$ such that*

(i) *f is an m_{x_0}-type contraction with x_0,*
(ii) *$d(x_0, fx) \leq \rho$ for all $x \in D_{x_0,\rho}$,*

then $D_{x_0,\rho}$ is a fixed disc of f, that is, the set $Fix(f)$ contains the disc $D_{x_0,\rho}$.

Proof. For any $r \in [0,\rho]$, it is sufficient to show that the circle $C_{x_0,r}$ is contained in $Fix(f)$. Assume that $d(x,fx) > 0$ for any $x \in C_{x_0,r}$. This implies

$$r \leq \rho \leq d(fx, x). \tag{3.4}$$

Considering this last inequality and using the m_{x_0}-type contractive property, we have

$$\begin{aligned} d(x,fx) &\leq \phi\left(\max\left\{d(x,x_0), cd(x,fx), \frac{b\left[d(x,x_0)+d(x_0,fx)\right]}{2}\right\}\right) \\ &< \max\left\{d(x,x_0), cd(x,fx), \frac{b\left[d(x,x_0)+d(x_0,fx)\right]}{2}\right\} \\ &\leq \max\left\{d(x,x_0), cd(x,fx), \frac{b\left[r+\rho\right]}{2}\right\}. \end{aligned}$$

From this last inequality and considering (3.4), the proof follows by similar arguments used in the proof of Theorem 3.2. □

Corollary 3.5. *Let (X,d) be a metric space, f be a self-mapping on X and the number ρ be defined as in (3.1). If there exists a point $x_0 \in Fix(f)$ such that*

(i) *$d(x,fx) > 0$ implies $d(x,fx) < m(x,x_0)$ for all $x \in X$ and*
(ii) *$d(x_0, fx) \leq \rho$ for all $x \in D_{x_0,\rho}$,*

then $D_{x_0,\rho} \subset Fix(f)$ and so any circle $C_{x_0,r}$ with $r \leq \rho$ is contained in $Fix(f)$.

Observe that we have not used the m_{x_0}-type contractive property in the proof of the last part of Theorem 3.2. In fact, the last part of this theorem is also true for any self map f of X and any fixed point $x \in Fix(f)$. Therefore, we can restate this part of the theorem as a separate proposition.

Proposition 3.6. *Let (X,d) be a metric space and f be a self-mapping on X. Then f is continuous at any $u \in Fix(f)$ if and only if $\lim_{x\to u} m(x,u) = 0$.*

Remark 3.7. We note that the converse statement of Theorem 3.2 (resp. Corollary 3.4 and Corollary 3.5) is not true even if the geometric condition (*ii*) is satisfied by the self-mapping f. For example, consider the self-mapping f_r defined by

$$f_r x = \begin{cases} x & \text{if} \quad x \in D_{x_0,\rho} \\ x_0 & , \quad \text{otherwise} \end{cases},$$

where $r \in (0,\infty)$ (see Example 2 in [45]). We obtain

$$\begin{aligned} \rho &= \inf\{d(x,fx) : x \notin Fix(f), x \in X\} \\ &= \inf\{d(x,x_0) : x \in X \text{ such that } d(x,x_0) > r\} \\ &= r \end{aligned}$$

and the condition (ii) of Theorem 3.2 is satisfied by f. For any $x \in X$ with $d(x, fx) > 0$ we have $fx = x_0$ and hence

$$\begin{aligned} d(x, fx) &= d(x, x_0) \leq \phi(m(x, x_0)) \\ &= \phi\left(\max\left\{d(x, x_0), ad(x, x_0), (1-a)d(x, x_0), \frac{bd(x, x_0)}{2}\right\}\right) \\ &= \phi(d(x, x_0)) < d(x, x_0). \end{aligned}$$

This is a contradiction. Then f is not an m_{x_0}-type contraction with $x_0 \in X$. But, we have $Fix(f) = D_{x_0,\rho}$ and so, $C_{x_0,\rho} \subset Fix(f)$.

Now, we give an illustrative example.

Example 3.8. Let $(\mathbb{R}, d)$ be the usual metric space and consider the self-mapping $f : \mathbb{R} \to \mathbb{R}$ defined by

$$fx = \begin{cases} 2x & if \quad x > 2 \\ x & if \quad x \leq 2 \end{cases},$$

for all $x \in \mathbb{R}$. We have

$$\begin{aligned} \rho &= \inf\{d(x, fx) : x \notin Fix(f), x \in X\} \\ &= \inf\{|2x - x| : x > 2\} \\ &= \inf\{|x| : x > 2\} = 2 \end{aligned}$$

and the self-mapping f satisfies the conditions of Theorem 3.2 with $x_0 = 0$, $a = \frac{1}{2}$, $b = \frac{4}{5}$, $\phi(t) = \frac{7}{8}t$. First, since $0 \in Fix(f)$, for each $x > 2$ we obtain

$$\begin{aligned} m(x, 0) &= \max\left\{d(x, 0), \frac{1}{2}d(x, fx), (1-\frac{1}{2})d(x, fx), \frac{\frac{4}{5}[d(x, 0) + d(0, fx)]}{2}\right\} \\ &= \max\left\{|x|, \frac{1}{2}|x|, \frac{2}{5}(|x| + |2x|)\right\} = \max\left\{|x|, \frac{1}{2}|x|, \frac{6}{5}|x|\right\} = \frac{6}{5}|x| \end{aligned}$$

and so

$$d(x, fx) = |x| \leq \phi(m(x, 0)) = \phi\left(\frac{6}{5}|x|\right) = \frac{7}{8}.\frac{6}{5}|x| = \frac{21}{20}|x|.$$

Also, for all $x \leq 2$, we have $fx = x$ and

$$m(x, 0) = \max\left\{|x|, 0, 0, \frac{2}{5}(|x| + |x|)\right\} = \max\left\{|x|, 0, 0, \frac{4}{5}|x|\right\} = |x|,$$

$$d(x, fx) = 0 \leq \phi(m(x, 0)) = \phi(|x|) = \frac{7}{8}|x|.$$

This shows that f is an m_{x_0}-type contraction with $x_0 = 0$. Clearly, for the circle $C_{0,2} = \{-2, 2\}$, the condition (ii) of Theorem 3.2 is satisfied by f, and $C_{0,2}$ is a fixed circle of f (particularly, the set $Fix(f) = (-\infty, 2]$ contains the disc $D_{0,2} = [-2, 2]$). Furthermore, we have $\lim_{x\to-2} m(x, -2) = 0$ and hence f is continuous at the fixed point -2. f is discontinuous at the fixed point 2 since the limit $\lim_{x\to2} m(x, 2)$ does not exist (we have $\lim_{x\to2^-} m(x, 2) = 0$ while $\lim_{x\to2^+} m(x, 2) \neq 0$).

f is also an m_{x_0}-type contraction with $x_0 = -2$, $a = \frac{1}{4}$, $b = \frac{6}{7}$, $\phi(t) = \frac{7}{9}t$. Indeed, for each $x > 2$, we obtain

$$\begin{aligned} m(x, -2) &= \max\left\{d(x, -2), \frac{1}{4}d(x, fx), (1-\frac{1}{4})d(x, fx), \frac{\frac{6}{7}[d(x, -2) + d(-2, fx)]}{2}\right\} \\ &= \max\left\{|x+2|, \frac{1}{4}|x|, \frac{3}{4}|x|, \frac{3}{7}(|x+2| + |2x+2|)\right\} \\ &= \frac{3}{7}(|x+2| + |2x+2|) \end{aligned}$$

and

$$d(x,fx) = |x| \leq \phi(m(x,-2)) = \phi\left(\frac{3}{7}(|x+2|+|2x+2|)\right) = \frac{1}{3}(|x+2|+|2x+2|).$$

For all $x \leq 2$, since $x \in Fix(f)$, we find

$$m(x,-2) = \max\left\{|x+2|, 0, 0, \frac{\frac{6}{7}[|x+2|+|x+2|]}{2}\right\} = \max\left\{|x+2|, 0, 0, \frac{6}{7}|x+2|\right\} = |x+2|$$

and

$$d(x,fx) = 0 \leq \phi(m(x,0)) = \phi(|x+2|) = \frac{7}{9}|x+2|.$$

The circle $C_{-2,2} = \{-4, 0\}$ is another fixed circle of f and also we have $D_{-2,2} = [-4, 0] \subset Fix(f)$. It is easy to see that $\lim_{x\to -4} m(x,-4) = 0$ and $\lim_{x\to 0} m(x,0) = 0$ and so, f is continuous on the fixed circle $C_{-2,2}$.

This example shows that the radius ρ of the fixed circle (resp. fixed disc) is independent from the center x_0 in Theorem 3.2 (resp. Corollary 3.4 and Corollary 3.5). We note that the fixed circle (resp. fixed-disc) produced by Theorem 3.2 (resp. Corollary 3.4 and Corollary 3.5) is not maximal. That is, if $C_{x_0,\mu}$ is another fixed circle (resp. fixed disc) of f then it is possible to be $\rho < \mu$. For example, in the above example, the circle $C_{-2,3}$ (resp. the disc $D_{-2,3}$) is another fixed circle (resp. fixed disc) of the self-mapping f with the center $x_0 = -2$. Notice that the maximal circle (resp. fixed disc) with the center $x_0 = -2$ and contained in the set $Fix(f)$, is $C_{-2,4}$ (resp. $D_{-2,4}$).

Theorem 3.9. *Let (X,d) be a metric space, f be a self-mapping on X and $x_0 \in X$ be any point. Assume*

$$\mu = \inf\{\lambda d(x,fx) : x \notin Fix(f), x \in X\} > 0, \tag{3.5}$$

where $\lambda = \max\{a, 1-a\}$. If $\varphi : (0,\infty) \to \mathbb{R}$ be a strictly increasing function such that

$$\lambda + \varphi(\lambda d(x,fx)) \leq \varphi(\lambda m(x,x_0))$$

whenever $d(x,fx) > 0$ for all $x \in X$ and $d(x_0,fx) \leq \mu$ for all $x \in C_{x_0,\mu}$ then we have $x_0 \in Fix(f)$ and $C_{x_0,\mu} \subset Fix(f)$, that is, $C_{x_0,\mu}$ is a fixed circle of f.

Proof. First, assume that $fx_0 \neq x_0$ and so $d(fx_0,x_0) > 0$. Then $x_0 \notin Fix(f)$ and by the hypothesis, we have

$$0 < \mu \leq \lambda d(x_0,fx_0)$$

and hence

$$\begin{aligned} \varphi(\mu) &\leq \varphi(\lambda d(x_0,fx_0)) \leq \varphi(\lambda m(x_0,x_0)) - \lambda \\ &< \varphi(\lambda m(x_0,x_0)) = \varphi(\lambda d(x_0,fx_0)), \end{aligned}$$

a contradiction. Hence we find $fx_0 = x_0$.

Let $x \in C_{x_0,\mu}$ be any point such that $d(x,fx) > 0$. Then by the definition of μ and the hypothesis, we have

$$0 < \mu \leq \lambda d(x,fx)$$

and

$$\begin{aligned} \varphi(\mu) &\leq \varphi(\lambda d(x,fx)) \leq \varphi(\lambda m(x,x_0)) - \lambda \\ &< \varphi(\lambda m(x,x_0)) \\ &= \varphi\left(\lambda \max\left\{d(x,x_0), ad(x,fx), (1-a)d(x,fx), \frac{b[d(x,x_0)+d(x_0,fx)]}{2}\right\}\right) \\ &\leq \varphi(\lambda \max\{\mu, ad(x,fx), (1-a)d(x,fx), b\mu\}) = \varphi\left(\lambda^2 d(x,fx)\right), \end{aligned}$$

a contradiction.

Consequently, we obtain $fx = x$ for all $x \in C_{x_0,\mu}$, that is, $C_{x_0,\mu} \subset Fix(f)$. □

Corollary 3.10. *Let (X,d) be a metric space, f be a self-mapping on X and $x_0 \in X$ be any point. Assume*

$$\mu = \inf\left\{\lambda d\left(x, fx\right) : x \notin Fix\left(f\right), x \in X\right\} > 0$$

where $\lambda = \max\{a, 1-a\}$. If $\varphi : (0,\infty) \to \mathbb{R}$ be a strictly increasing function such that

$$\lambda + \varphi\left(\lambda d\left(x, fx\right)\right) \leq \varphi\left(\lambda m\left(x, x_0\right)\right)$$

whenever $d(x, fx) > 0$ for all $x \in X$ and $d(x_0, fx) \leq \mu$ for all $x \in D_{x_0,\mu}$ then we have $D_{x_0,\mu} \subset Fix(f)$, that is, $D_{x_0,\mu}$ is a fixed disc of f.

Example 3.11. Let $(\mathbb{R}, d)$ be the usual metric space. Consider the self-mapping $f : \mathbb{R} \to \mathbb{R}$ defined by

$$fx = \begin{cases} \frac{7}{3}x & if \quad |x| > 1 \\ x & if \quad |x| \leq 1 \end{cases}.$$

If we choose the auxiliary function $\varphi(t) = 14t$ and $a = \frac{3}{4}$ then we have $\lambda_1 = \max\{\frac{3}{4}, \frac{1}{4}\} = \frac{3}{4}$ and

$$\begin{aligned} \mu_1 &= \inf\left\{\lambda_1 d\left(x, fx\right) : x \notin Fix\left(f\right), x \in X\right\} \\ &= \inf\left\{\frac{3}{4}\left|\frac{7}{3}x - x\right| : |x| > 1\right\} = \inf\left\{|x| : |x| > 1\right\} = 1. \end{aligned}$$

The self-mapping f satisfies the conditions of Theorem 3.9 and Corollary 3.10 with $x_0 = 0$ and $b = \frac{6}{7}$. Indeed, for each x with $|x| > 1$ we get

$$\begin{aligned} m\left(x, 0\right) &= \max\left\{d(x,0), \frac{3}{4}d(x,fx), (1-\frac{3}{4})d(x,fx), \frac{\frac{6}{7}\left[d(x,0) + d(0,fx)\right]}{2}\right\} \\ &= \max\left\{|x|, \frac{1}{3}|x|, \frac{\frac{6}{7}\left[|x| + \frac{7}{3}|x|\right]}{2}\right\} = \max\left\{|x|, \frac{1}{3}|x|, \frac{10}{7}|x|\right\} \\ &= \frac{10}{7}|x| \end{aligned}$$

and

$$\begin{aligned} \lambda_1 + \varphi\left(\lambda_1 d\left(x, fx\right)\right) &= \frac{3}{4} + \varphi\left(\frac{3}{4} \cdot \frac{4}{3}|x|\right) = \frac{3}{4} + \varphi\left(|x|\right) = \frac{3}{4} + 14|x| \\ &\leq \varphi\left(\lambda_1 m\left(x, x_0\right)\right) = \varphi\left(\frac{3}{4} \cdot \frac{10}{7}|x|\right) = \varphi\left(\frac{15}{14}|x|\right) = 15|x|. \end{aligned}$$

For all $x \in D_{0,1}$, the condition $d(0, fx) = |x| \leq \mu_1 = 1$ is also satisfied. Clearly, the circle $C_{0,1} = \{-1, 1\}$ is a fixed circle of f. Particularly, we have $Fix(f) = D_{0,1} = [-1, 1]$. In other words, the disc $D_{0,1}$ is the maximal disc contained in $Fix(f)$. Notice that f is not continuous on the circle $C_{0,1}$ since the limits $\lim_{x\to -1} m\left(x, -1\right)$ and $\lim_{x\to 1} m\left(x, 1\right)$ do not exist.

On the other hand, if we use the auxiliary function $\varphi(t) = 48t$ and choose $a = \frac{1}{2}$, we find

$$\lambda_2 = \frac{1}{2}$$

and

$$\begin{aligned} \mu_2 &= \inf\left\{\lambda_2 d\left(x, fx\right) : fx \neq x, x \in X\right\} \\ &= \inf\left\{\frac{2}{3}|x| : |x| > 1\right\} = \frac{2}{3}. \end{aligned}$$

f satisfies the conditions of Theorem 3.9 and Corollary 3.10 with $x_0 = 0$ and $b = \frac{7}{8}$. Since we have

$$\begin{aligned} m(x,0) &= \max\left\{d(x,0), \frac{1}{2}d(x,fx), (1-\frac{1}{2})d(x,fx), \frac{\frac{7}{8}[d(x,0)+d(0,fx)]}{2}\right\} \\ &= \max\left\{|x|, \frac{2}{3}|x|, \frac{\frac{7}{8}\left[|x|+\frac{7}{3}|x|\right]}{2}\right\} = \max\left\{|x|, \frac{2}{3}|x|, \frac{35}{24}|x|\right\} \\ &= \frac{35}{24}|x| \end{aligned}$$

and

$$\begin{aligned} \lambda_2 + \varphi(\lambda_2 d(x,fx)) &= \frac{1}{2} + \varphi\left(\frac{1}{2}\cdot\frac{4}{3}|x|\right) = \frac{1}{2} + \varphi\left(\frac{2}{3}|x|\right) = \frac{1}{2} + 24|x| \\ &\leq \varphi(\lambda_2 m(x,x_0)) = \varphi\left(\frac{1}{2}\cdot\frac{35}{24}|x|\right) = \varphi\left(\frac{35}{48}|x|\right) = 35|x|. \end{aligned}$$

The circle $C_{0,\frac{2}{3}} = \left\{-\frac{2}{3}, \frac{2}{3}\right\}$ is another fixed circle and the disc $D_{0,\frac{2}{3}} = \left[-\frac{2}{3}, \frac{2}{3}\right]$ is another fixed disc of f.

In the above example, we have seen that it is possible to obtain a maximal fixed disc or the case $Fix(f) = D_{x_0,\mu}$ can occur.

Now we give another fixed-disc theorem.

Theorem 3.12. *Let (X,d) be a metric space, f be a self-mapping on X, $x_0 \in X$ be any point and the number μ be defined as follows*:

$$\mu = \inf\{\lambda(x)\, d(x,fx) : x \notin Fix(f), x \in X\},$$

where $\lambda : X \to (0,\infty)$ is a function. If there exists $x_0 \in X$ such that

(i) $\lambda(x)\, d(x,fx) < m(x,x_0)$ *whenever* $d(x,fx) > 0$ *for all* $x \in X$,
(ii) $d(x_0,fx) \leq \mu$ *and* $\lambda(x) \geq \max\{a, 1-a\}$ *for all* $x \in D_{x_0,\mu}$,
(iii) $\lambda(x_0) \geq 1$,

then $D_{x_0,\mu} \subset Fix(f)$, that is, $D_{x_0,\mu}$ is a fixed disc of f.

Proof. If $fx_0 \neq x_0$ then by the condition (i) we have

$$\lambda(x_0)\, d(x_0,fx_0) < m(x_0,x_0) = d(x_0,fx_0),$$

a contradiction with the hypothesis $\lambda(x_0) \geq 1$. Hence, we get $x_0 \in Fix(f)$.

Let $x \in D_{x_0,\mu}$ be any point such that $d(x,fx) > 0$. Then by the definition of μ and the hypothesis, we have

$$\mu \leq \lambda(x)\, d(x,fx)$$

and

$$\begin{aligned} \lambda(x)\, d(x,fx) &< m(x,x_0) = \max\left\{\begin{array}{c} d(x,x_0), ad(x,fx) + (1-a)d(x_0,fx_0), \\ (1-a)d(x,fx) + ad(x_0,fx_0), \frac{b[d(x,fx_0)+d(x_0,fx)]}{2} \end{array}\right\} \\ &\leq \max\{\mu, ad(x,fx), (1-a)d(x,fx), b\mu\}. \end{aligned}$$

Let $\alpha = \max\{\mu, ad(x,fx), (1-a)d(x,fx)\}$. If $\alpha = \mu$ then we have $\lambda(x)\, d(x,fx) < \mu$ and this is a contradiction with the definition of μ.

If $\alpha = ad(x,fx)$ or $(1-a)d(x,fx)$, we have $\lambda(x)\, d(x,fx) < ad(x,fx)$ or $\lambda(x)\, d(x,fx) < (1-a)\, d(x,fx)$. We get a contradiction since $\lambda(x) \geq \max\{a, 1-a\}$ for all $x \in D_{x_0,\mu}$.

Consequently, we obtain $fx = x$ for all $x \in D_{x_0,\mu}$, that is, $D_{x_0,\mu}$ is a fixed disc of f. □

Corollary 3.13. *Let (X,d) be a metric space, f be a self-mapping on X, $x_0 \in X$ be any point and the number μ be defined as follows :*

$$\mu = \inf\{\lambda(x)\, d(x, fx) : x \notin Fix(f), x \in X\},$$

where $\lambda : X \to (0,\infty)$ is a function. If there exists $x_0 \in X$ such that

(i) $\lambda(x)\, d(x, fx) < m(x, x_0)$ *whenever* $d(x, fx) > 0$ *for all* $x \in X$,

(ii) $d(x_0, fx) \leq \mu$ *and* $\lambda(x) \geq \max\{a, 1-a\}$ *for all* $x \in C_{x_0,\mu}$,

(iii) $\lambda(x_0) \geq 1$,

then $C_{x_0,\mu} \subset Fix(f)$, that is, $C_{x_0,\mu}$ is a fixed circle of f.

Remark 3.14. 1) We note that it is possible to obtain a maximal fixed disc by choosing an appropriate auxiliary function $\lambda : X \to (0,\infty)$ for a point $x_0 \in Fix(f)$.

2) In the fixed-circle problem, uniqueness of the fixed circle is possible. Various uniqueness conditions have been given (for example see [25] and [27]). In the following, we give a new uniqueness theorem using the number $m(x,y)$.

Theorem 3.15. *Let (X,d) be a metric space and f be a self-mapping on X having a fixed circle $C_{x_0,\mu}$. If the inequality*

$$d(fx, fy) < m(x,y) \tag{3.6}$$

is satisfied for all $x \in C_{x_0,\mu}$ and $y \in X \setminus C_{x_0,\mu}$, then $C_{x_0,\mu}$ is the unique fixed-circle of f, that is, $Fix(f) = C_{x_0,\mu}$.

Proof. Since $0 \leq b < 1$, if there exists another fixed circle $C_{u_0,\gamma}$ of f then by (3.6) we get

$$d(fx, fy) = d(x,y) < m(x,y) = \max\{d(x,y), bd(x,y)\} = d(x,y)$$

for arbitrary distinct points $x \in C_{x_0,\mu}$, $y \in C_{u_0,\gamma}$. This is a contradiction. Hence, f has a unique fixed circle. □

As a corollary, we have the following condition for a maximal fixed disc of a self-mapping.

Corollary 3.16. *Let (X,d) be a metric space and f be a self-mapping on X having a fixed disc $D_{x_0,\mu}$. If the inequality (3.6) is satisfied for all $x \in D_{x_0,\mu}$ and $y \in X \setminus D_{x_0,\mu}$, then the fixed disc $D_{x_0,\mu}$ of f is maximal, that is, $Fix(f) = D_{x_0,\mu}$.*

Example 3.17. Let $\mathbb{C}$ be the set of all complex numbers and $(\mathbb{C}, d)$ be the usual metric space with $d(z,w) = |z-w|$ for all $z, w \in \mathbb{C}$. Let the self-mapping $f : \mathbb{C} \to \mathbb{C}$ be defined by

$$fz = \left\{ \begin{array}{ccc} \frac{z}{|z|^2} & if & z \neq 0 \\ 1 & if & z = 0 \end{array} \right\}.$$

Clearly, we have $Fix(f) = C_{0,1}$ and the fixed circle is unique for this self-mapping. But the inequality (3.6) is not satisfied for all $z \in C_{0,1}$ and $w \in \mathbb{C} \setminus C_{0,1}$. This example shows that the converse statement of Theorem 3.15 does not hold in general.

4. Activation functions having fixed discs

It is known that activation functions are the primary neural networks decision-making units. Hence, the determination of the activation function is essential in the design of a neural network and the choice of each specific activation function defines different types of neural networks (for more details see [6], [39] and the references therein). Furthermore, the existence and stability of stationary patterns for neural networks rely on the characteristics of activation functions. In recent years, new classes of generalized activation functions were proposed [49].

As stated in [23], most of the frequently used activation functions have a fixed disc. For example, let us consider the bipolar ramp activation function f_b, which is one of the most widely used activation functions, defined by

$$f_b(x) = \begin{cases} 1 & if \quad x > 1 \\ x & if \quad -1 \leq x \leq 1 \\ -1 & if \quad x < -1 \end{cases}.$$

For more details, for example, see [48]. For this activation function, we have

$$\rho = \inf\{d(x, fx) : fx \neq x, x \in X\} = 0.$$

For any $x_0 \in Fix(f)$, Theorem 3.2 (resp. Corollary 3.4 and Corollary 3.5) produces a fixed circle (resp. fixed disc) with one element.

If we define

$$\lambda(x) = \begin{cases} \frac{1}{|x-1|} & if \quad x > 1 \\ 1 & if \quad -1 \leq x \leq 1 \\ \frac{1}{|x+1|} & if \quad x < -1 \end{cases},$$

then we have

$$\begin{aligned} \mu &= \inf\{\lambda(x) d(x, fx) : fx \neq x, x \in X\} = \inf \begin{cases} \frac{1}{|x-1|}|x-1| = 1 & ; \quad x > 1 \\ \frac{1}{|x+1|}|x+1| = 1 & ; \quad x < -1 \end{cases} \\ &= 1. \end{aligned}$$

For the point $0 \in Fix(f)$, the function f_b satisfies the conditions of Theorem 3.12 with $a = \frac{1}{3}$, $b = 0$. Consequently, $D_{0,1}$ is a fixed-disc of f_b. Moreover, we have $Fix(f_b) = D_{0,1}$. It is easy to check that the inequality (3.6) is satisfied for all $x \in [-1, 1]$ and $y \in (-\infty, -1) \cup (1, \infty)$.

Another example of the most effective activation functions is the ReLU function defined by

$$f(x) = \max(0, x) = \begin{cases} x & if \quad x \geq 0 \\ 0 & if \quad x < 0 \end{cases}. \tag{4.1}$$

In [40], an automated method for diagnosis of COVID-19 from X-ray images was proposed. This model is based on XceptionNet that uses depth wise separable convolutions. It was noted that the convergence of the neural net largely depends on the choice of activation function and that the most effective activation function used in hidden layers of deep neural network is ReLU function (see [9], [22] and [40] for more details). Let $x_0 \in \left[\frac{1}{2}, \infty\right)$ be a fixed real number and consider the function $\lambda_{x_0}(x)$ defined by

$$\lambda_{x_0}(x) = \begin{cases} \frac{x_0}{|x|} & if \quad x \neq 0 \\ x_0 & if \quad x = 0 \end{cases},$$

for all $x \in \mathbb{R}$. Then we have

$$\begin{aligned} \mu_{x_0} &= \inf\{\lambda_{x_0}(x) d(x, fx) : fx \neq x, x \in \mathbb{R}\} \\ &= \inf\left\{\frac{x_0}{|x|}|x - 0| : x \in (-\infty, 0)\right\} = x_0. \end{aligned}$$

Clearly, we have $x_0 \in Fix(f) = [0, \infty)$ and it is easy to check that the function $f(x)$ defined in (4.1) satisfies the conditions of Theorem 3.12 with $a = \frac{1}{2}$ and $b = 0$. Notice that the disc $D_{x_0,x_0} = [0, 2x_0]$ is contained in $Fix(f)$.

The above examples show the effectiveness of our fixed-disc results. As a final remark, we note that some fixed point results (e.g. Banach fixed point theorem, Brouwer's fixed point theorem) have been intensively used in the theoretical studies of neural networks. In this context, obtained theoretical results of this paper can be used to design a new neural network with a more generalized activation function and a geometric viewpoint.

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