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Petrie Paths in Triangular Normalizer Maps

Nazlı Yazıcı Gözütok[1]

[1] *Department of Mathematics, Faculty of Science and Arts, Marmara University, Istanbul, Turkey*

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Abstract

This study is devoted to investigate the Petrie paths in the normalizer maps and regular triangular maps corresponding to the subgroups $\Gamma_0(N)$ of the modular group Γ. We show that each regular triangular map admits a closed Petrie path. Thus, for each regular map, we find the Petrie length of the corresponding map.

1. Introduction

Let $PSL(2,\mathbb{R})$ denote the group of all linear fractional transformations

$$T : z \to \frac{az+b}{cz+d},$$

where $a,b,c,d \in \mathbb{R}$ and $ad-bc=1$. In terms of matrix representation, the elements of $PSL(2,\mathbb{R})$ correspond to the matrices $\pm\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a,b,c,d \in \mathbb{R}$ and the determinant is 1. This is the automorphism group of the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\}$. The modular group Γ is the subgroup of $PSL(2,\mathbb{R})$ such that $a,b,c,d \in \mathbb{Z}$ [1]. $\Gamma_0(N)$, a well-known congruence subgroup of the modular group, consists of the transformations of Γ such that $N \mid c$.

Let $\mathscr{U}$ denote the space consisting of the points of upper half plane and the extended rational numbers. The normalizer of the congruence subgroups $\Gamma_0(N)$ of the modular group Γ in $PSL(2,\mathbb{R})$ is $\Gamma_B(N)$ which is simply called "the normalizer". We refer reader to [2–7] and references therein for results concerning the normalizer. It is known that the normalizer is a triangle group, and it acts transitively on the set of extended rational numbers $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ for $N = 2^\alpha 3^\beta$ with $\alpha \in \{0,2,4,6\}$ and $\beta \in \{0,2\}$ [8,9]. We denote the values of $N = 2^\alpha 3^\beta$ by N_1 for $\alpha \in \{0,2,4,6\}$ and $\beta \in \{0,2\}$. The normalizer maps are universal maps on $\mathscr{U}$ which are arose from the action of $\Gamma_B(N_1)$ on $\hat{\mathbb{Q}}$. Using the normalizer maps, regular triangular maps can be obtained by the quotients $\mathscr{M}_3^h/\Gamma_0(N_1)$. These regular maps are introduced in [8]. However, combinatorial properties of these maps has not been addressed. In this manner, to reveal the complete structure of the normalizer and the maps corresponding to the normalizer, we consider the Petrie paths in these maps. This paper investigates the Petrie paths in the normalizer maps which are the universal maps arising from the action of the normalizer on the extended rationals, and the regular triangular maps corresponding to the subgroups $\Gamma_0(N)$ which are the quotient maps of the normalizer maps.

This paper is organized as follows. Section 2 describes the normalizer $\Gamma_B(N)$ and its associated group structure which is important to construct regular maps. Section 3 introduces a brief information about the basic concept of the normalizer maps which is already addressed. Section 4 provides the results concerning the Petrie paths in the normalizer maps, where we concluded that the Petrie paths in the normalizer maps are related to well-known Fibonacci sequence. Section 5 investigates the Petrie paths in the regular triangular maps. Finally, Section 6 presents our conclusions.

2. The Normalizer

As described in [10], the normalizer $\Gamma_B(N)$ of $\Gamma_0(N)$ consists of the transformations corresponding to the matrices

$$\begin{pmatrix} ae & b/h \\ cN/h & de \end{pmatrix}$$

Email address and ORCID number: **yazici.gozutok@marmara.edu.tr, 0000-0002-3645-0623 (N. Yazıcı Gözütok)**

where all symbols represent integers, h is the largest divisor of 24 for which $h^2 \mid N$, $e > 0$ is an exact divisor of N/h^2, and the determinant is e. (We say that r is an exact divisor of s if $r \mid s$ and $(r, s/r) = 1$).
The following results are taken from [11] which characterizes the structure of the normalizer.

Theorem 2.1 ([11]). *Let $N = 2^{\alpha}3^{\beta}$ and $\beta = 0$ or 2. Then, $\Gamma_B(N)$ is a triangle group if and only if $\alpha \leq 8$. In these cases*

$$\Gamma_B(N) \text{ has signature } \begin{cases} (0;2,3,\infty) & \text{if } \alpha = 0,2,4,6 \\ (0;2,4,\infty) & \text{if } \alpha = 1,3,5,7 \\ (0;2,\infty,\infty) & \text{if } \alpha = 8. \end{cases}$$

Theorem 2.2 ([11]). *Let $N = 2^{\alpha}3^{\beta}$ and $\beta = 1$ or 3. Then $\Gamma_B(N)$ is a triangle group if and only if $\alpha = 0,2,4,6$. In these cases $\Gamma_B(N)$ has signature $(0;2,6,\infty)$.*

In this paper we only consider the case $\beta = 0,2$ and $\alpha = 0,2,4,6$ so that the normalizer maps and the regular maps corresponding to the quotients of the normalizer maps will be all triangular. Thus if $\beta = 0,2$ and $\alpha = 0,2,4,6$, then the normalizer $\Gamma_B(N_1)$ is the set of all transformations corresponding to the matrices

$$\begin{pmatrix} a & b/h \\ ch & d \end{pmatrix}, \; ad - bc = 1,$$

where h is the largest divisor of 24 for which $h^2 \mid N_1$.

3. The Normalizer Maps

The information in this section is given in [8,9]. We denote the normalizer maps corresponding to $\Gamma_B(N_1)$ by $\mathcal{M}_3^h$. Vertices of $\mathcal{M}_3^h$ are the fractions $\frac{a}{ch}$ with $(a,c) = 1$ and two vertices, $\frac{a}{ch}$ and $\frac{b}{dh}$, are joined by an edge if and only if $ad - bc = \pm 1$. $\mathcal{M}_3^h$ has the following properties:

(1) There is a triangle with vertices $\frac{1}{0}, \frac{1}{h}, \frac{0}{h}$.
(2) $\Gamma_B(N_1)$ acts as a group of homomorphisms of $\mathcal{M}_3^h$.
(3) There is a triangle with vertices $\frac{a}{ch}, \frac{a+b}{(c+d)h}, \frac{b}{dh}$.

One can easily see that when $h = 1$, $\mathcal{M}_3^1$ is just the Farey map. When $h > 1$, the normalizer map $\mathcal{M}_3^h$ is the Farey map scaled by a factor of $1/h$ (see Fig. 3.1).

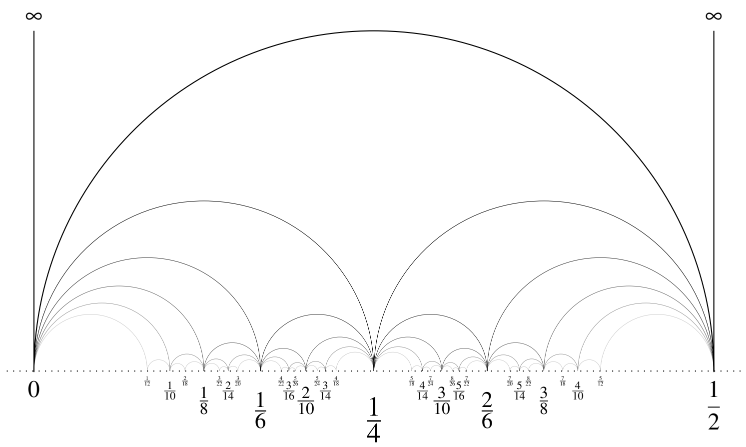


Figure 3.1: Part of the normalizer map $\mathcal{M}_3^2$.

It is easily seen that normalizer maps are all triangular maps each of which corresponds to a value of h with triangular faces given by (3). We refer reader to [8,9] for further details about normalizer maps and regular maps corresponding to quotients of normalizer maps.

4. Petrie Paths in the Normalizer Maps

A Petrie polygon in a map is a zig-zag path in the corresponding map. In other words, we start by choosing a vertex on the map. This vertex is the starting point of the polygon. On that vertex, we go along an edge to an adjacent vertex. On the last vertex we turn left and then go to another adjacent vertex. Then we turn right, and so on, but interchanging left and right consecutively. In a finite map, this procedure ends on the first vertex. Thus a path or polygon is obtained by this procedure. This path or polygon is called a Petrie path or Petrie polygon. The number of edges of a Petrie polygon in a map is called Petrie length of the map.

In this section, we investigate the Petrie paths in $\mathcal{M}_3^h$. Now consider a Petrie path in $\mathcal{M}_3^h$. By the transitivity (see [8]), first edge of the Petrie path can be chosen the edge from $E_1 = \frac{1}{0}$ to $E_2 = \frac{0}{h}$. Keeping in mind that left turns correspond to $ad-bc=-1$ and right turns correspond to $ad-bc=1$, the first left turn goes to the vertex $E_3 = \frac{1}{h}$. Then a right turn goes to the vertex $E_4 = \frac{1}{2h}$. As this procedure goes on, the following vertices of the Petrie path can be found as $\frac{a}{ch}, \frac{b}{dh}, \frac{a+b}{(c+d)h}$. Now, if we denote the elements of the well-known Fibonacci sequence by f_k, where $f_0=0, f_1=1$ and $f_k = f_{k-1}+f_k$, $k\geq 1$, then we can express the $(k+1)$th vertex of the Petrie path as $E_{k+1} = \frac{f_{k-1}}{hf_k}$. Thus the Petrie path in normalizer maps can be found as

$$\frac{1}{0}, \frac{0}{h}, \frac{1}{h}, \frac{1}{2h}, \frac{2}{3h}, \frac{3}{5h}, \dots, \frac{f_{k-1}}{hf_k}, \dots.$$

Note that if $h=1$, then the above Petrie path is exactly the Petrie path in the Farey map (see [12]).

Example 4.1. *The Petrie path in the normalizer map $\mathcal{M}_3^2$ is*

$$\frac{1}{0}, \frac{0}{2}, \frac{1}{2}, \frac{1}{4}, \frac{2}{6}, \frac{3}{10}, \dots, \frac{f_{k-1}}{2f_k}, \dots.$$

We close this section by the following results concerning the relation between the vertices of the Petri paths in normalizer maps.

Lemma 4.2. *The transformation corresponding to the matrix $T = \begin{pmatrix} 0 & 1/h \\ h & 1 \end{pmatrix}$ maps each vertex of the Petrie polygon in the normalizer maps $\mathcal{M}_3^h$ to the next vertex.*

Proof. The first vertex of the Petrie path is $E_1 = \frac{1}{0}$ and $(k+1)$th vertex is $\frac{f_{k-1}}{hf_k}$ such that $k \geq 1$. We use induction on k. So the second vertex can be found by taking $k=1$ as $E_2 = \frac{f_0}{hf_1} = \frac{0}{h}$. Now one can easily see that $T(E_1) = E_2$. Let us assume that $T(E_k) = E_{k+1}$ for each $k>1$. We will show that $T(E_{k+1}) = E_{k+2}$.Thus

$$T(E_k) = \frac{f_{k-1}}{h(f_{k-2}+f_{k-1})}.$$

Since f_k is the Fibonacci sequence, we have $f_{k-2}+f_{k-1}) = f_k$. Using this equality, we have

$$T(E_k) = \frac{f_{k-1}}{hf_k}.$$

By the definition of $E_{k+1} = \frac{f_{k-1}}{hf_k}$, we have

$$T(E_{k+1}) = \begin{pmatrix} 0 & 1/h \\ h & 1 \end{pmatrix} \begin{pmatrix} f_{k-1} \\ hf_k \end{pmatrix} = \begin{pmatrix} f_k \\ h(f_{k-1}+f_k) \end{pmatrix} = \begin{pmatrix} f_k \\ hf_{k+1} \end{pmatrix} = E_{k+2}.$$

This completes the proof. □

Proposition 4.3. *The transformations corresponding to the matrices $T^k = \begin{pmatrix} f_{k-1} & f_k/h \\ hf_k & f_{k+1} \end{pmatrix}$ maps each vertex of the Petrie polygon in the normalizer maps $\mathcal{M}_3^h$ to the next vertex.*

Proof. Applying induction on k, using Lemma 4.2 and the properties of the Fibonacci sequence f_k, the proof follows. □

We denote the determinant $f_{k-1}f_{k+1} - f_k^2$ of the matrix T^k by $D(k) := f_{k-1}f_{k+1} - f_k^2$.

Proposition 4.4. *For $k \geq 1$, $D(k) = (-1)^k$.*

Proof. We use induction on k. So let $k=1$, then we have $D(1) = -1$. Now let $D(k) = (-1)^k$. We will show that $D(k+1) = (-1)^{k+1}$ holds. By the definition of $D(k)$, we have

$$D(k+1) = f_k f_{k+2} - f_{k+1}^2.$$

By the definition of the Fibonacci sequence, substituting $f_{k+2} = f_k + f_{k-1}$ in the above equality, we get

$$D(k+1) = f_k(f_k+f_{k-1}) - f_{k+1}^2 = f_k^2 + f_k f_{k-1} - f_{k+1}^2.$$

Again substituting $f_k = f_{k+1} - f_{k-1}$ in the above equality, it is obtained that

$$D(k+1) = f_k^2 + f_{k+1}^2 - f_{k-1}f_{k+1} - f_{k+1}^2 = -D(k) = -(-1)^k = (-1)^{k+1}.$$

The above equation leads to $D(k+1) = (-1)^{k+1}$. By the induction, the proof is completed. □

Corollary 4.5. *If $k \geq 1$ is even, then T^k is an element of the normalizer.*

Proof. By Proposition 4.4, if k is even, then $D(k) = 1$. Thus T^k has determinant 1 if k is even. By the definition of the normalizer T^k is an element of the normalizer. $\square$

5. Petrie Paths in the Regular Triangular Maps

The regular triangular maps corresponding to the subgroups $\Gamma_0(N_1)$ of the modular group Γ are defined by the quotients $\mathscr{M}_3^h(N_1) = \mathscr{M}_3^h/\Gamma_0(N_1)$. The complete table of these regular triangular maps can be found in [8]. In this section we determine the Petrie polygons in $\mathscr{M}_3^h(N_1)$.

In [8], the set of vertices of the maps $\mathscr{M}_3^h(N_1)$ is $\{(a,ch) \mid a,c \in \mathbb{Z}_h, (a,c,h) = 1\}/\sim$, where $(a,ch) \sim (h-a,(h-c)h)$. We denote any vertex of this kind by $\left[\frac{a}{ch}\right]$. Also, the vertices $\left[\frac{a}{ch}\right]$ and $\left[\frac{b}{dh}\right]$ is combined by an edge in $\mathscr{M}_3^h(N_1)$ if and only if $ad - bc \equiv \pm 1 \mod h$.

As in the previous section, we can define the Petrie path of the regular triangular maps $\mathscr{M}_3^h(N_1)$. The Petrie path has vertices

$$\left[\frac{1}{0}\right], \left[\frac{0}{h}\right], \left[\frac{1}{h}\right], \left[\frac{1}{2h}\right], \ldots, \left[\frac{f_{r-1}}{hf_r}\right], \ldots.$$

Now we are going to find Petrie polygons in $\mathscr{M}_3^h(N_1)$ explicitly for some low values of N_1. Further details about regular triangular maps $\mathscr{M}_3^h(N_1)$ can be found in [8].

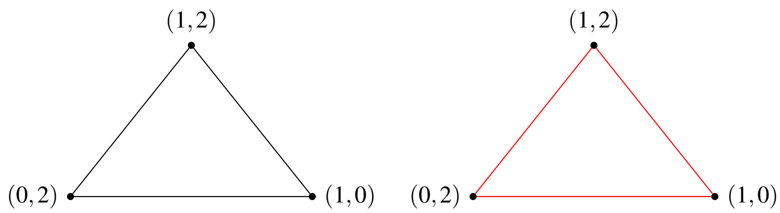
1) $N_1 = 4$. In this case $h = 2$, then $\mathscr{M}_3^2(4)$ is the triangle with vertices

$$(1,0), (0,2), (1,2).$$

Then the vertices of the Petrie polygon in $\mathscr{M}_3^2(4)$ are

$$\left[\frac{1}{0}\right], \left[\frac{0}{2}\right], \left[\frac{1}{2}\right].$$

Below we illustrate the regular triangular map $\mathscr{M}_3^2(4)$ (see Fig. 5.1a) and the Petrie polygon, edges of the polygon are highlighted in red, in the map (see Fig. 5.1b)



(a) $\mathscr{M}_3^2(4)$ **(b)** Petrie polygon in $\mathscr{M}_3^2(4)$

Figure 5.1: The map $\mathscr{M}_3^2(4)$ and Petrie polygon

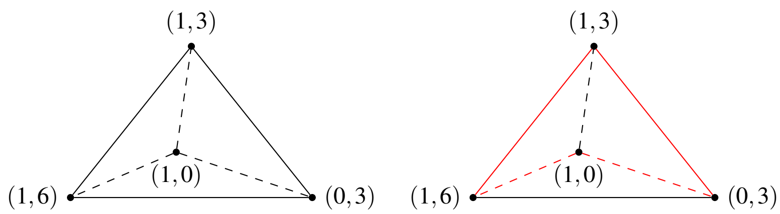
2) $N_1 = 9$. In this case $h = 3$, then $\mathscr{M}_3^3(9)$ is the tetrahedron with vertices

$$(1,0), (0,3), (1,6), (1,3).$$

Then the vertices of the Petrie polygon in $\mathscr{M}_3^3(9)$ are

$$\left[\frac{1}{0}\right], \left[\frac{0}{3}\right], \left[\frac{1}{3}\right], \left[\frac{1}{6}\right].$$

Below we illustrate the regular triangular map $\mathscr{M}_3^3(9)$ (see Fig. 5.2a) and the Petrie polygon in the map (see Fig. 5.2b)



(a) $\mathscr{M}_3^3(9)$ **(b)** Petrie polygon in $\mathscr{M}_3^3(9)$

Figure 5.2: The map $\mathscr{M}_3^3(9)$ and Petrie polygon

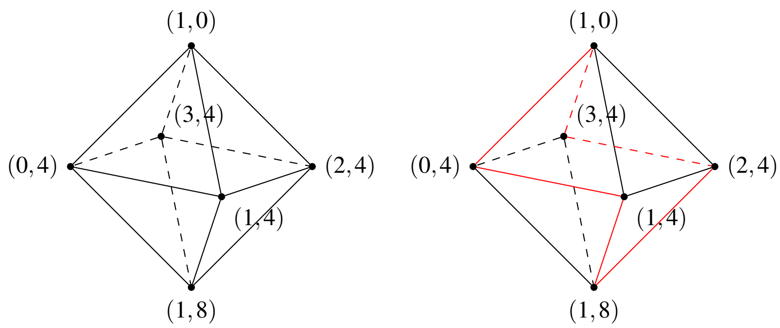
3) $N_1 = 16$. In this case $h = 4$, then $\mathscr{M}_3^4(16)$ is the octahedron with vertices

$$(1,0),(0,4),(1,8),(1,8),(2,4),(3,4).$$

Then the vertices of the Petrie polygon in $\mathscr{M}_3^4(16)$ are

$$\left[\frac{1}{0}\right],\left[\frac{0}{4}\right],\left[\frac{1}{4}\right],\left[\frac{1}{8}\right],\left[\frac{2}{4}\right],\left[\frac{3}{4}\right].$$

Below we illustrate the regular triangular map $\mathscr{M}_3^4(16)$ (see Fig. 5.3a) and the Petrie polygon in the map (see Fig. 5.3b)



(a) $\mathscr{M}_3^4(16)$ **(b)** Petrie polygon in $\mathscr{M}_3^4(16)$

Figure 5.3: The map $\mathscr{M}_3^4(16)$ and Petrie polygon

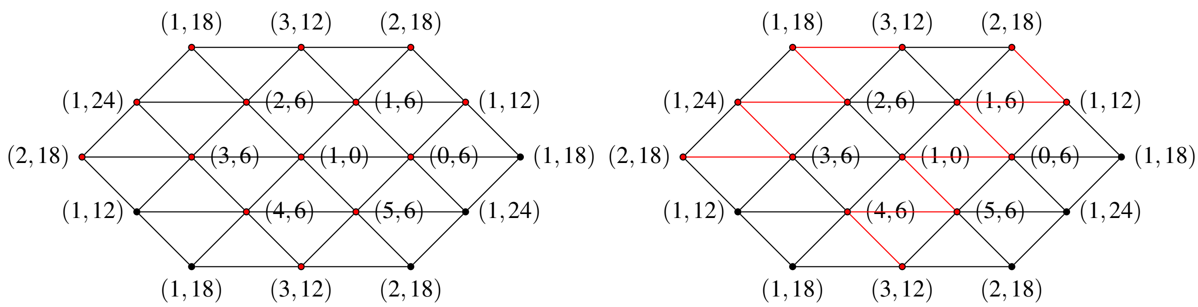
4) $N_1 = 36$. In this case $h = 6$, then $\mathscr{M}_3^6(36)$ is the map $\{3,6\}_{12}$ with vertices

$$(1,0),(0,6),(1,6),(1,12),(1,18),(1,24),(2,6),(2,18),(3,6),(3,12),(4,6),(5,6).$$

Then the vertices of the Petrie polygon in $\mathscr{M}_3^6(36)$ are

$$\left[\frac{1}{0}\right],\left[\frac{0}{6}\right],\left[\frac{1}{6}\right],\left[\frac{1}{12}\right],\left[\frac{2}{18}\right],\left[\frac{3}{6}\right],\left[\frac{1}{24}\right],\left[\frac{2}{6}\right],\left[\frac{1}{18}\right],\left[\frac{3}{12}\right],\left[\frac{4}{6}\right],\left[\frac{5}{6}\right].$$

Below we illustrate the regular triangular map $\mathscr{M}_3^6(36)$ (see Fig. 5.4a) and the Petrie polygon in the map (see Fig. 5.4b).



(a) $\mathscr{M}_3^6(36)$ **(b)** Petrie polygon in $\mathscr{M}_3^6(36)$

Figure 5.4: The map $\mathscr{M}_3^6(36)$ and Petrie polygon

Now we are going to formulate Petrie lenghts of the Petrie polygons in the regular triangular maps $\mathscr{M}_3^h(N_1)$. In order to do that let us recall a well-known notion called Pisano period [13], and the notion of semi period of a Fibonacci sequence.

Definition 5.1 ([12]). *Let f_k be the Fibonacci sequence. The Pisano period $\pi(n)$ of f_k is defined to be the least positive integer m such that $f_{m-1} \equiv 1 \mod n$ and $f_m \equiv 0 \mod n$. The semi-period $\sigma(n)$ of f_k is defined to be the least positive integer m such that $f_{m-1} \equiv \pm 1 \mod n$ and $f_m \equiv 0 \mod n$.*

Example 5.2. *Let f_k be the Fibonacci sequence and $n = 5$. The elements of the Fibonacci sequence modulo 5 are*

$$0,1,1,2,3,0,3,3,1,-1,0,-1,-1,3,2,0,2,2,-1,1,0,1,1,2,3,\ldots.$$

So the Pisano period of f_k is $\pi(5) = 20$ and the semi period is $\sigma(5) = 10$.

Now we denote the Petrie polygon in the regular triangular maps $\mathscr{M}_3^h(N_1)$ by $\mathscr{P}(N_1)$, and we denote the Petrie length of $\mathscr{P}(N_1)$ by $\mathscr{L}(\mathscr{P})(N_1)$.

Theorem 5.3. $\mathscr{L}(\mathscr{P})(N_1) = \sigma(h)$.

Proof. By the definition, $\mathscr{P}(N_1)$ in $\mathscr{M}_3^h(N_1)$ has vertices

$$\left[\frac{1}{0}\right], \left[\frac{0}{h}\right], \left[\frac{1}{h}\right], \left[\frac{1}{2h}\right], \ldots, \left[\frac{f_{r-1}}{hf_r}\right], \ldots.$$

Also, by the definition, $f_{\sigma(h)-1} \equiv \pm 1 \mod h$ and $f_{\sigma(h)} \equiv 0 \mod h$. So the Petrie polygon $\mathscr{P}(N_1)$ is exactly the following polygon

$$\left[\frac{1}{0}\right] \to \left[\frac{0}{h}\right] \to \left[\frac{1}{h}\right] \to \left[\frac{1}{2h}\right] \to \ldots \to \left[\frac{f_{\sigma(h)-2}}{hf_{\sigma(h)-1}}\right] \to \left[\frac{f_{\sigma(h)-1}}{hf_{\sigma(h)}}\right] = \left[\frac{1}{0}\right].$$

Number of edges in this polygon is $\sigma(h)$. This completes the proof. □

Please see Table 1 for the complete table of the Petrie lengths of the regular triangular maps $\mathscr{M}_3^h(N_1)$.

N_1	h	Map	Petrie Length ($\sigma(h)$)
4	2	Triangle	3
9	3	Tetrahedron	4
16	4	Octahedron	6
36	6	$\{3,6\}$	12
64	8	Dual Dyck	12
144	12	$\{3,12\}$	24
576	24	$\{3,24\}$	24

Table 1: The complete table of the Petrie lengths of the regular triangular maps $\mathscr{M}_3^h(N_1)$

6. Conclusions

In this paper we addressed the problem of determining the Petrie paths in certain maps. The maps that are considered in this paper are the normalizer maps corresponding to the universal maps arising from the action of the normalizer of $\Gamma_0(N)$ in $PSL(2,\mathbb{R})$ on $\hat{\mathbb{Q}}$, and the regular triangular maps corresponding to the subgroups $\Gamma_0(N)$ of the modular group which are the quotient maps of the normalizer maps. We completely determined the Petrie paths in these maps. We showed that the Petrie paths in the normalizer maps are strongly related to Fibonacci sequence. Using this result we managed to determine the Petrie paths in the regular triangular maps. Finally we showed that the Petrie paths in the regular triangular maps are related to Fibonacci sequence modulo an integer, h. As the regular triangular maps are finite maps, we found the Petrie lengths of these maps. Other than that, it would be interesting to determine the Petrie paths in the regular quadrilateral and hexagonal maps corresponding to the subgroups $\Gamma_0(N)$. However it needs further investigation to determine these maps, since currently it is not known whether Fibonacci sequence can be used in that case. Thus we keep this approach as a future research.

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Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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