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ANOTHER APPLICATION OF SMARANDACHE BASED RULED SURFACES WITH THE DARBOUX VECTOR ACCORDING TO FRENET FRAME IN E^3

Süleyman ŞENYURT[1], Davut CANLI[2], Elif ÇAN[3] and Sümeyye GÜR MAZLUM[4]

[1,2,3]Department of Mathematics, Ordu University, Ordu, TÜRKİYE
[4]Department of Computer Technology, Gümüşhane University, Gümüşhane, TÜRKİYE

Abstract. In this study, we first define the Smarandache curves derived from the Frenet vectors and the Darboux vector of any curve. Then, we construct new ruled surfaces along these Smarandache curves with the direction vectors obtained from the Frenet vectors and the Darboux vector, and give the equations of these surfaces. In addition, we calculate the Gaussian and mean curvatures of these surfaces separately and present the conditions to be minimal and developable for these surfaces. Finally, as an example, we obtain ruled surfaces whose base curves are Viviani's curves and plot the graphics of these surfaces.

1. Introduction

A surface is defined to be the image of a function with two real variables in three-dimensional space. Surfaces can be characterized by their curvatures and engaged accordingly to many fields, especially in architecture and engineering. Research on the surface curvature went through various stages starting from Ancient Greece. After the studies of Descartes, Kepler, Fermat and Huygens, it gained momentum with the calculations developed by Newton and Leibniz in the 17^{th} century. The curvature of curves and surfaces is an important topic in differential geometry, today. The method of calculating the curvature of a surface was defined by Gauss in the 19^{th} century and therefore named Gaussian curvature. Gaussian curvature is related to the dimension of the surface. The developability of a surface depends

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[1] ssenyurt@odu.edu.tr; 0000-0003-1097-5541
[2] davutcanli@odu.edu.tr; 0000-0003-0405-9969
[3] eliff.cann@hotmail.com; 0000-0001-5870-114X
[4] sumeyyegur@gumushane.edu.tr-Corresponding author; 0000-0003-2471-1627.

on its Gaussian curvature. A surface with zero Gaussian curvature at every point is known to be a developable surface. Since the average curvature of the surface is a ratio, it is independent of the size of the surface. Surfaces with a mean curvature of zero at every point are minimal surfaces. Therefore, it is one of the most used surfaces in architectural designs. There are numerous studies on surfaces, [5, 16, 17, 22, 24]. In surface theory, there are special surfaces of which one is named the ruled surface. A ruled surface is formed by infinitely many lines that move along a given curve. The basics related to this type of surfaces are given in [7, 23, 25]. And there are various other studies on ruled surfaces, [2, 3, 10, 12, 13, 27, 33, 36]. On the other hand, the theory of curves also occupies an important place in differential geometry. There are many studies on various special curves, [7–9, 11, 14, 15, 21]. In addition, studies with Smarandache curves are available in [1, 4, 6, 24, 26–31, 34, 35]. Recently, in [18–20], generating the way of new ruled surfaces have been given by exploiting the idea of Smarandache geometry and using the Frenet, the Darboux or the alternative frame. In the light of all these informations in this study, we obtain the ruled surfaces from the direction vector obtained from the Frenet and Darboux vectors of any curve and from the Smarandache base curve obtained in the same way. And we study some properties of these surfaces. Finally, we show all these results on an example and plot the graphs of the surfaces. This study is another application of our previous paper: Smarandache based ruled surfaces with the Darboux vector according to Frenet frame in E^3, [32].

2. Preliminaries

$\alpha : I \to E^3$ be a unit speed regular curve. The Frenet frame $\{T, N, B\}$, the curvature κ, the torsion τ and the Frenet derivative formulae of the curve α are given by

$$T = \alpha',\ N = \frac{\alpha''}{\|\alpha''\|},\ B = T \wedge N,\ \kappa = \|\alpha''\|,\ \tau = \langle N', B\rangle,$$

and

$$T' = \kappa N,\ N' = -\kappa T + \tau B,\ B' = -\tau N.$$

here T, N and B are the tangent, normal and binormal vectors of α, respectively, [7]. Also $\langle , \rangle$ is the inner product, $\|\ \|$ is the norm and $\wedge$ is the vectorial product functions in E^3, [7]. The Darboux vector corresponding to the Frenet frame $\{T, N, B\}$ is defined by $W = \tau T + \kappa B$. Thus, we write the unit form of Darboux vector as

$$C = \sin\omega\, T + \cos\omega\, B,$$

where $\angle (B, W) = \omega$ and

$$\begin{aligned}\cos\omega &= \frac{\kappa}{\sqrt{\kappa^2+\tau^2}},\ \sin\omega = \frac{\tau}{\sqrt{\kappa^2+\tau^2}},\\ \omega' &= \left(\frac{\tau}{\kappa}\right)'\left(1+\frac{\tau^2}{\kappa^2}\right),\end{aligned} \tag{1}$$

[6]. On the other hand, a unit vector based on the Frenet frame elements can be defined by

$$\gamma = \frac{aT+bN+cB}{\sqrt{a^2+b^2+c^2}}, \tag{2}$$

where a, b, c are some real valued functions. For $\forall s \in I$, the locus of the endpoints of the vector γ defines a differentiable curve. If γ is taken to be the position vector then the generated curve is called as Smarandache curve, [25]. A ruled surface is defined as a one parameter family of lines and it has the form

$$X(s,v) = \alpha(s) + vr(s), \tag{3}$$

here $\alpha(s)$ is the base curve, $r(s)$ is the direction vector of the ruled surface $X(s,v)$ and v is any real number, [9]. The normal vector field, the Gaussian and the mean curvatures of $X(s,v)$ are given by the relations [7]

$$N_X = \frac{X_s \wedge X_v}{\|X_s \wedge X_v\|}, \tag{4}$$

$$K = \frac{eg-f^2}{EG-F^2},\ H = \frac{Eg-2fF+eG}{2(EG-F^2)}, \tag{5}$$

respectively. Here, the coefficients of the first and the second fundamental forms are defined by [7]

$$E = \langle X_s, X_s\rangle,\ F = \langle X_s, X_v\rangle,\ G = \langle X_v, X_v\rangle, \tag{6}$$

$$e = \langle X_{ss}, N_X\rangle,\ f = \langle X_{sv}, N_X\rangle,\ g = \langle X_{vv}, N_X\rangle. \tag{7}$$

3. Another Application of Smarandache Based Ruled Surfaces with the Darboux Vector According to Frenet Frame in E^3

Let us remind the given expression (2). We consider some special cases to generate new kind of Smarandache curves by choosing appropriate a, b, c functions:

- For $a = 1+\sin\omega,\ b = 0,\ c = \cos\omega$, we define $TC-$ Smarandache curve $\gamma_1 = \dfrac{T+C}{\sqrt{2+2\sin\omega}}$, whose position vector is $\gamma = \dfrac{T+C}{\sqrt{2+2\sin\omega}}$,

- For $a = sin\omega,\ b = 1,\ c = \cos\omega$, we define $NC-$ Smarandache curve $\gamma_2 = \dfrac{N+C}{\sqrt{2}}$, whose position vector is $\gamma = \dfrac{N+C}{\sqrt{2}}$,

- For $a = \sin\omega,\ b = 0,\ c = 1+\cos\omega$, we define $BC-$ Smarandache curve $\gamma_3 = \dfrac{B+C}{\sqrt{2+2\cos\omega}}$, whose position vector is $\gamma = \dfrac{B+C}{\sqrt{2+2\cos\omega}}$.

By this study, we define and consider the ruled surfaces where the base curve is one of these Smarandache curves and the generator line is one of the given position vectors. For each surface, we calculate the corresponding the Gaussian and mean curvatures.

Definition 1. *Let's define a ruled surface generated by continuously moving the vector $T+C$ along the $TC-$ Smarandache curve. Thus, we provide its parametric form as*

$$\begin{aligned}\Sigma(s,v) &= \frac{T+C}{\sqrt{2+2\sin\omega}} + v\frac{T+C}{\sqrt{2+2\sin\omega}},\\ \Sigma(s,v) &= \frac{1+v}{\sqrt{2}}\left(\sqrt{1+\sin\omega}T+\sqrt{1-\sin\omega}B\right).\end{aligned}$$

The first and the second partial differentials of $\Sigma(s,v)$ are

$$\Sigma_s = \frac{1+v}{2\sqrt{2}}\left(\begin{array}{l}\omega'\sqrt{1-\sin\omega}T+2\left(\kappa\sqrt{1+\sin\omega}-\tau\sqrt{1-\sin\omega}\right)N\\ -\omega'\sqrt{1+\sin\omega}B\end{array}\right),$$

$$\Sigma_v = \frac{1}{\sqrt{2}}\left(\sqrt{1+\sin\omega}T+\sqrt{1-\sin\omega}B\right),$$

$$\Sigma_{vv} = 0,$$

$$\Sigma_{sv} = \frac{1}{2\sqrt{2}}\left(\begin{array}{l}\omega'\sqrt{1-\sin\omega}T+2\left(\kappa\sqrt{1+\sin\omega}-\tau\sqrt{1-\sin\omega}\right)N\\ -\omega'\sqrt{1+\sin\omega}B\end{array}\right),$$

$$\Sigma_{ss} = \frac{1+v}{4\sqrt{2}}\left[\begin{array}{l}\left((2\omega''+4\tau\kappa)\sqrt{1-\sin\omega}-\left({\omega'}^2+4\kappa^2\right)\sqrt{1+\sin\omega}\right)T\\ +\left((2\kappa\omega'-2\tau')\sqrt{1-\sin\omega}+(2\kappa'+2\tau\omega')\sqrt{1+\sin\omega}\right)N\\ +\left((-2\omega''+4\tau\kappa)\sqrt{1-\sin\omega}-\left({\omega'}^2+4\tau^2\right)\sqrt{1+\sin\omega}\right)B\end{array}\right].$$

And the vectorial product of the vectors Σ_s , Σ_v and its norm are

$$\Sigma_s \wedge \Sigma_v = \frac{1+v}{2}\left(\left(\sqrt{\kappa^2+\tau^2}-\tau\right)T-\omega' N-\kappa B\right),$$

$$\|\Sigma_s \wedge \Sigma_v\| = \frac{1+v}{2}\sqrt{2\kappa^2+2\tau^2+\omega'^2-2\tau\sqrt{\kappa^2+\tau^2}}.$$

If we denote the normal vector field of the surface by N_Σ, then from the expression (4), we have

$$N_\Sigma = \frac{\left(\sqrt{\kappa^2+\tau^2}-\tau\right)T-\omega' N-\kappa B}{\left(\omega'^2+2\kappa^2+2\tau^2-2\tau\sqrt{\kappa^2+\tau^2}\right)^{\frac{1}{2}}}.$$

From the expressions (6) and (7), we compute the coefficients of the first and the second fundamental forms as

$$E_\Sigma = \frac{(1+v)^2}{4}\left(\omega'^2+2\kappa^2+2\tau^2-2\tau\sqrt{\kappa^2+\tau^2}\right),$$
$$F_\Sigma = 0,\ G_\Sigma = 1$$

and

$$e_\Sigma = \frac{\left[\begin{array}{l}\left((2\omega''+4\tau\kappa)\left(\sqrt{\kappa^2+\tau^2}-\tau\right)-\omega'\left(2\kappa\omega'-2\tau'\right)-\kappa\left(-2\omega''+4\tau\kappa\right)\right)\\ .\sqrt{1-\sin\omega}\\ -\left(\left(\omega'^2+4\kappa^2\right)\left(\sqrt{\kappa^2+\tau^2}-\tau\right)+\omega'\left(2\kappa'+2\tau\omega'\right)-\kappa\left(\omega'^2+4\tau^2\right)\right)\\ .\sqrt{1+\sin\omega}\end{array}\right]}{4(1+v)^{-1}\left(\omega'^2+2\kappa^2+2\tau^2-2\tau\sqrt{\kappa^2+\tau^2}\right)^{\frac{1}{2}}},$$

$$f_\Sigma = \frac{\omega'\left(\sqrt{\kappa^2+\tau^2}+\tau\right)\sqrt{1-\sin\omega}-\kappa\omega'\sqrt{1+\sin\omega}}{4(1+v)^{-1}\left(\omega'^2+2\kappa^2+2\tau^2-2\tau\sqrt{\kappa^2+\tau^2}\right)^{\frac{1}{2}}},$$

$$g_\Sigma = 0,$$

respectively. Finally, by using the expression (5), we get the Gaussian and the mean curvatures

$$K_{\Sigma}=-\left[\frac{\omega'\left(\sqrt{\kappa^2+\tau^2}+\tau\right)\sqrt{1-\sin\omega}-\kappa\omega'\sqrt{1+\sin\omega}}{2\left(\omega'^2+2\kappa^2+2\tau^2-2\tau\sqrt{\kappa^2+\tau^2}\right)}\right]^2,$$

$$H_{\Sigma}=\frac{\left[\begin{array}{l}\left((2\omega''+4\tau\kappa)\left(\sqrt{\kappa^2+\tau^2}-\tau\right)-\omega'\left(2\kappa\omega'-2\tau'\right)-\kappa\left(-2\omega''+4\tau\kappa\right)\right)\\ .\sqrt{1-\sin\omega}\\ -\left(\left(\omega'^2+4\kappa^2\right)\left(\sqrt{\kappa^2+\tau^2}-\tau\right)+\omega'\left(2\kappa'+2\tau\omega'\right)-\kappa\left(\omega'^2+4\tau^2\right)\right)\\ .\sqrt{1+\sin\omega}\end{array}\right]}{2\left(\omega'^2+2\kappa^2+2\tau^2-2\tau\sqrt{\kappa^2+\tau^2}\right)^{\frac{3}{2}}},$$

respectively.

Corollary 1. *If $\alpha(s)$ is a general helix, then Σ is a developable surface.*

Definition 2. *Let's define a ruled surface generated by continuously moving the vector $N+C$ along the $TC-$ Smarandache curve. Thus, we provide its parametric form as*

$$\begin{aligned}\Delta(s,v)&=\frac{1}{\sqrt{2+2\sin\omega}}(T+C)+\frac{v}{\sqrt{2}}(N+C),\\ \Delta(s,v)&=\frac{1}{\sqrt{2}}\left[\left(\sqrt{1+\sin\omega}+v\sin\omega\right)T+vN+\left(\sqrt{1-\sin\omega}+v\cos\omega\right)B\right].\end{aligned}$$

The first and the second partial differentials of $\Delta(s,v)$ are

$$\Delta_s=\frac{1}{2\sqrt{2}}\left(xT+yN+zB\right),$$

$$\Delta_v=\frac{1}{\sqrt{2}}\left(\sin\omega T+N+\cos\omega B\right),$$

$$\Delta_{vv}=0,$$

$$\Delta_{sv}=\frac{1}{\sqrt{2}}\left(\left(-\kappa+\cos\omega\right)T+\left(\kappa\sqrt{1+\sin\omega}-\tau\sqrt{1-\sin\omega}\right)N+\left(\tau-\sin\omega\right)B\right),$$

$$\Delta_{ss}=\frac{1}{2\sqrt{2}}\left(\left(x'-y\kappa\right)T+\left(x\kappa-z\tau+y'\right)N+\left(y\tau+z'\right)B\right).$$

And the vectorial product of the vectors Δ_s , Δ_v and its norm are

$$\Delta_s \wedge \Delta_v = \frac{1}{4}\left(\left(-z + y\cos\omega\right)T - \left(x\cos\omega - z\sin\omega\right)N + \left(x - y\sin\omega\right)B\right),$$

$$\|\Delta_s \wedge \Delta_v\| = \frac{1}{4}\Big(\left(-z + y\cos\omega\right)^2 + \left(x\cos\omega - z\sin\omega\right)^2 + \left(x - y\sin\omega\right)^2\Big)^{\frac{1}{2}},$$

where the coefficients x, y, z are

$$x = \omega'\sqrt{1-\sin\omega} + 2v\left(-\kappa + \cos\omega\right),$$

$$y = 2\kappa\sqrt{1+\sin\omega} - 2\tau\sqrt{1-\sin\omega},$$

$$z = -\omega'\sqrt{1+\sin\omega} + 2v\left(\tau - \sin\omega\right).$$

Thus, from the expression (4), the normal of the surface N_Δ is given as

$$N_\Delta = \frac{\left(-z + y\cos\omega\right)T - \left(x\cos\omega - z\sin\omega\right)N + \left(x - y\sin\omega\right)B}{\Big(\left(-z + y\cos\omega\right)^2 + \left(x\cos\omega - z\sin\omega\right)^2 + \left(x - y\sin\omega\right)^2\Big)^{\frac{1}{2}}}.$$

By following the expressions (6) and (7), the coefficients of the first and the second fundamental forms are

$$E_\Delta = \frac{1}{8}\left(x^2 + y^2 + z^2\right),$$

$$F_\Delta = \frac{1}{4}\left(x\sin\omega + y + z\cos\omega\right),$$

$$G_\Delta = 1$$

and

$$e_\Delta = \frac{\left(\begin{array}{l}\left(x' - y\kappa\right)\left(-z + y\cos\omega\right) - \left(x\kappa - z\tau + y'\right)\left(x\cos\omega - z\sin\omega\right)\\ + \left(y\tau + z'\right)\left(x - y\sin\omega\right)\end{array}\right)}{2\sqrt{2}\Big(\left(-z + y\cos\omega\right)^2 + \left(x\cos\omega - z\sin\omega\right)^2 + \left(x - y\sin\omega\right)^2\Big)^{\frac{1}{2}}},$$

$$f_\Delta = \frac{\left(\begin{array}{l}z\kappa + x\tau - z\cos\omega - x\sin\omega + y\left(1 - \sqrt{\kappa^2+\tau^2}\right)\\ - \left(x\cos\omega - z\sin\omega\right)\left(\kappa\sqrt{1+\sin\omega} - \tau\sqrt{1-\sin\omega}\right)\end{array}\right)}{\sqrt{2}\Big(\left(-z + y\cos\omega\right)^2 + \left(x\cos\omega - z\sin\omega\right)^2 + \left(x - y\sin\omega\right)^2\Big)^{\frac{1}{2}}},$$

$$g_\Delta = 0,$$

respectively. Finally, from the expression (5), the Gaussian and mean curvatures are obtained as

$$K_{\Delta}=-\frac{8\begin{bmatrix} z\kappa+x\tau+y\left(1-\sqrt{\kappa^2+\tau^2}\right)-(z\cos\omega+x\sin\omega) \\ -(x\cos\omega-z\sin\omega)\left(\kappa\sqrt{1+\sin\omega}-\tau\sqrt{1-\sin\omega}\right)\end{bmatrix}^2}{\left((-z+y\cos\omega)^2+(x\cos\omega-z\sin\omega)^2+(x-y\sin\omega)^2\right)^2},$$

$$H_{\Delta}=\frac{\begin{bmatrix} (x'-y\kappa)(-z+y\cos\omega)-(x\kappa-z\tau+y')(x\cos\omega-z\sin\omega) \\ +(y\tau+z')(x-y\sin\omega)-z\kappa+x\tau-z\cos\omega-x\sin\omega \\ +y\left(1-\sqrt{\kappa^2+\tau^2}\right)(x\sin\omega+y+z\cos\omega) \\ +(x\cos\omega-z\sin\omega)(x\sin\omega+y+z\cos\omega)\left(\kappa\sqrt{1+\sin\omega}-\tau\sqrt{1-\sin\omega}\right)\end{bmatrix}}{\left((-z+y\cos\omega)^2+(x\cos\omega-z\sin\omega)^2+(x-y\sin\omega)^2\right)^{\frac{3}{2}}},$$

respectively.

Definition 3. *Let's define a ruled surface generated by continuously moving the vector $B+C$ along the TC–Smarandache curve. Thus, we provide its parametric form as*

$$\begin{aligned}\Upsilon(s,v) &= \frac{1}{\sqrt{2+2\sin\omega}}(T+C)+\frac{v}{\sqrt{2+2\cos\omega}}(B+C),\\ \Upsilon(s,v) &= \frac{1}{\sqrt{2}}\left[\left(\sqrt{1+\sin\omega}+v\sqrt{1-\cos\omega}\right)T+\left(\sqrt{1-\sin\omega}+v\sqrt{1+\cos\omega}\right)B\right].\end{aligned}$$

If we assign $p(s,v)=\sqrt{1+\sin\omega}+v\sqrt{1-\cos\omega}$ and $q(s,v)=\sqrt{1-\sin\omega}+v\sqrt{1+\cos\omega}$, then we can rewrite the surface in a simple form as

$$\Upsilon(s,v)=\frac{1}{\sqrt{2}}(pT+qB).$$

Next, the first and the second partial differentials of $\Upsilon(s,v)$ are

$$
\begin{aligned}
&\Upsilon_s = \frac{1}{\sqrt{2}}\ \left(p_s T + (\kappa p - \tau q)\, N + q_s B\right),\\
&\Upsilon_v = \frac{1}{\sqrt{2}} \left(p_v T + q_v B\right),\\
&\Upsilon_{vv} = 0,\\
&\Upsilon_{ss} = \frac{1}{\sqrt{2}} \left[\begin{array}{l} \left(p_{ss} - \kappa^2 p + \tau\kappa q\right) T + \left(p_s\kappa - q_s\tau + (\kappa p - \tau q)_s\right) N \\ + \left(q_{ss} + \kappa\tau p - \tau^2 q\right) B \end{array}\right],\\
&\Upsilon_{sv} = \frac{1}{\sqrt{2}} \left(p_{sv} T + (\kappa p_v - \tau q_v)\, N + q_{sv} B\right).
\end{aligned}
$$

And the vectorial product of the vectors Υ_s , Υ_v and its norm are

$$
\begin{aligned}
&\Upsilon_s \wedge \Upsilon_v = \frac{1}{2}\left((\kappa p q_v - \tau q q_v)\, T + (q_s p_v - p_s q_v)\, N - (\kappa p p_v - \tau q p_v)\, B\right),\\
&\|\Upsilon_s \wedge \Upsilon_v\| = \frac{1}{2}\sqrt{(\kappa p - \tau q)^2 \left({p_v}^2 + {q_v}^2\right) + (p_v q_s - p_s q_v)^2}.
\end{aligned}
$$

From the expression (4), the normal of the surface N_Υ is

$$
N_\Upsilon = \frac{q_v\,(\kappa p - \tau q)\, T + (p_v q_s - p_s q_v)\, N - p_v\,(\kappa p - \tau q)\, B}{\sqrt{(\kappa p - \tau q)^2 \left({p_v}^2 + {q_v}^2\right) + (p_v q_s - p_s q_v)^2}}
$$

From the expressions (6) and (7) to compute the coefficients of fundamental forms, we get

$$
\begin{aligned}
&E_\Upsilon = \frac{1}{2}\ \left((p_s)^2 + (\kappa p - \tau q)^2 + (q_s)^2\right),\\
&F_\Upsilon = \frac{1}{2}\left(p_s p_v + q_s q_v\right),\\
&G_\Upsilon = 1
\end{aligned}
$$

and

$$e_{\Upsilon}=\frac{\left[\begin{array}{l}\left(\kappa p-\tau q\right)\left(q_v\left(p_{ss}-\kappa^2 p+\tau\kappa q\right)-p_v\left(q_{ss}+\kappa\tau p-\tau^2 q\right)\right)\\ \\ +\left(p_v q_s-p_s q_v\right)\left(p_s\kappa-q_s\tau+\left(\kappa p-\tau q\right)_s\right)\end{array}\right]}{\sqrt{\left(\kappa p-\tau q\right)^2\left({p_v}^2+{q_v}^2\right)+\left(p_v q_s-p_s q_v\right)^2}},$$

$$f_{\Upsilon}=\frac{\left(\kappa p-\tau q\right)\left(q_v p_{sv}-p_v q_{sv}\right)+\left(p_v q_s-p_s q_v\right)\left(\kappa p_v-\tau q_v\right)}{\sqrt{2}\sqrt{\left(\kappa p-\tau q\right)^2\left({p_v}^2+{q_v}^2\right)+\left(p_v q_s-p_s q_v\right)^2}},$$

$$g_{\Upsilon}=0,$$

respectively. Finally, from the expression (5), we obtain the Gaussian and mean curvatures as

$$K_{\Upsilon}=-\left[\frac{\left(\kappa p-\tau q\right)\left(q_v p_{sv}-p_v q_{sv}\right)+\left(p_v q_s-p_s q_v\right)\left(\kappa p_v-\tau q_v\right)}{\left(\kappa p-\tau q\right)^2\left({p_v}^2+{q_v}^2\right)+\left(p_v q_s-p_s q_v\right)^2}\right]^2,$$

$$H_{\Upsilon}=\frac{\left[\begin{array}{l}\left(\kappa p-\tau q\right)\left(q_v p_{ss}-p_v q_{ss}-\left(\kappa p-\tau q\right)\left(\kappa q_v+\tau p_v\right)\right)\\ \\ +\left(p_v q_s-p_s q_v\right)\left(2p_s\kappa-2q_s\tau+\kappa' p-\tau' q\right)\\ \\ -\left(\kappa p-\tau q\right)\left(p_s p_v+q_s q_v\right)\left(p_{sv}q_v-q_{sv}p_v+p_v q_s-p_s q_v\right)\end{array}\right]}{\sqrt{2}\left(\left(\kappa p-\tau q\right)^2\left({p_v}^2+{q_v}^2\right)+\left(p_v q_s-p_s q_v\right)^2\right)^{\frac{3}{2}}}.$$

Definition 4. *Let's define a ruled surface generated by continuously moving the vector $T+C$ along the $NC-$ Smarandache curve. Thus, we provide its parametric form as*

$$\begin{aligned}\chi\left(s,v\right) &= \frac{1}{\sqrt{2}}\left(N+C\right)+\frac{v}{\sqrt{2+2\sin\omega}}(T+C),\\ \chi\left(s,v\right) &= \frac{1}{\sqrt{2}}\left[\left(\sin\omega+v\sqrt{1+\sin\omega}\right)T+N+\left(\cos\omega+v\sqrt{1-\sin\omega}\right)B\right].\end{aligned}$$

If we assign $m\left(s,v\right)=\sin\omega+v\sqrt{1+\sin\omega}$ and $r\left(s,v\right)=\cos\omega+v\sqrt{1-\sin\omega}$, then we can rewrite the surface in a simple form as

$$\chi\left(s,v\right)=\frac{1}{\sqrt{2}}\left(mT+N+rB\right).$$

Next, the first and second partial differentials of $\chi(s,v)$ are

$$\chi_s = \frac{1}{\sqrt{2}}\left((m_s-\kappa)T + (m\kappa - r\tau)N + (r_s+\tau)B\right),$$

$$\chi_v = \frac{1}{\sqrt{2}}\left(m_v T + r_v B\right),$$

$$\chi_{vv} = 0,$$

$$\chi_{sv} = \frac{1}{\sqrt{2}}\left(m_{sv}T + (m_v\kappa - r_v\tau)N + r_{sv}B\right),$$

$$\chi_{ss} = \frac{1}{\sqrt{2}}\begin{bmatrix} \left(m_{ss} - m\kappa^2 - \kappa' + r\kappa\tau\right)T \\ + \left(2\kappa m_s - 2\tau r_s - \kappa^2 - \tau^2 + m\kappa' - r\tau'\right)N \\ + \left(r_{ss} + m\tau\kappa + \tau' - r\tau^2\right)B \end{bmatrix}.$$

And the vectorial product of the vectors χ_s , χ_v and its norm are

$$\chi_s \wedge \chi_v = \frac{1}{2}\left(r_v(m\kappa - r\tau)T + (m_v(r_s+\tau) - r_v(m_s-\kappa))N - m_v(m\kappa - r\tau)B\right),$$

$$\|\chi_s \wedge \chi_v\| = \frac{1}{2}\left((m_v(r_s+\tau) - r_v(m_s-\kappa))^2 + 2(m\kappa - r\tau)^2\right)^{\frac{1}{2}}.$$

From the expression (4), the normal of the ruled surface $\chi(s,v)$ is

$$N_\chi = \frac{r_v(m\kappa - r\tau)T + (m_v(r_s+\tau) - r_v(m_s-\kappa))N - m_v(m\kappa - r\tau)B}{\left((m_v(r_s+\tau) - r_v(m_s-\kappa))^2 + 2(m\kappa - r\tau)^2\right)^{\frac{1}{2}}}.$$

From the expression (6) and (7, the coefficients of the first and the second fundamental forms are

$$E_\chi = \frac{1}{2}\left((m_s-\kappa)^2 + (m\kappa - r\tau)^2 + (r_s+\tau)^2\right),$$

$$F_\chi = \frac{1}{2}\left(m_v(m_s-\kappa) + r_v(r_s+\tau)\right),$$

$$G_\chi = 1$$

and

$$e_{\chi}=\frac{\left[\begin{array}{l}(m\kappa-r\tau)\left(r_v\left(m_{ss}-m\kappa^2-\kappa'+r\kappa\tau\right)-m_v\left(r_{ss}+m\tau\kappa+\tau'-r\tau^2\right)\right)\\ +\left(m_v\left(r_s+\tau\right)-r_v\left(m_s-\kappa\right)\right)\left(2\kappa m_s-2\tau r_s-\kappa^2-\tau^2+m\kappa'-r\tau'\right)\end{array}\right]}{\sqrt{2}\Big(\left(m_v\left(r_s+\tau\right)-r_v\left(m_s-\kappa\right)\right)^2+2(m\kappa-r\tau)^2\Big)^{\frac{1}{2}}},$$

$$f_{\chi}=\frac{(m\kappa-r\tau)\left(r_v m_{sv}-m_v r_{sv}\right)+\left(m_v\left(r_s+\tau\right)-r_v\left(m_s-\kappa\right)\right)\left(m_v\kappa-r_v\tau\right)}{\sqrt{2}\Big(\left(m_v\left(r_s+\tau\right)-r_v\left(m_s-\kappa\right)\right)^2+2(m\kappa-r\tau)^2\Big)^{\frac{1}{2}}},$$

$$g_{\chi}=0,$$

respectively. Finally, from the expression (5), we compute the Gaussian and mean curvatures as

$$K_{\chi}=-\left[\frac{(m\kappa-r\tau)\left(r_v m_{sv}-m_v r_{sv}\right)+\left(m_v\left(r_s+\tau\right)-r_v\left(m_s-\kappa\right)\right)\left(m_v\kappa-r_v\tau\right)}{\left(m_v\left(r_s+\tau\right)-r_v\left(m_s-\kappa\right)\right)^2+2(m\kappa-r\tau)^2}\right]^2,$$

$$H_{\chi}=\frac{\left[\begin{array}{l}(m\kappa-r\tau)\left(\begin{array}{l}r_v m_{ss}-r_v m\kappa^2-r_v\kappa'+r_v r\kappa\tau-m_v r_{ss}\\ -m_v m\tau\kappa-m_v\tau'+m_v r\tau^2\end{array}\right)\\ -(m\kappa-r\tau)\left(r_v m_{sv}-m_v r_{sv}\right)\left(m_v m_s-\kappa m_v+r_v r_s+r_v\tau\right)\\ +\left(m_v r_s+m_v\tau-r_v m_s+r_v\kappa\right)\left(2\kappa m_s-2\tau r_s-\kappa^2-\tau^2+m\kappa'-r\tau'\right)\\ -\left(m_v r_s+m_v\tau-r_v m_s+r_v\kappa\right)\left(m_v m_s-m_v\kappa+r_v r_s+r_v\tau\right)\\ .\left(m_v\kappa-r_v\tau\right)\end{array}\right]}{\sqrt{2}\Big(\left(m_v\left(r_s+\tau\right)-r_v\left(m_s-\kappa\right)\right)^2+2(m\kappa-r\tau)^2\Big)^{\frac{3}{2}}}.$$

Definition 5. *Let's define a ruled surface generated by continuously moving the vector $N+C$ along the NC– Smarandache curve. Thus, we provide its parametric form as*

$$\begin{aligned}\mathrm{P}(s,v) &= \frac{N+C}{\sqrt{2}}+v\frac{N+C}{\sqrt{2}},\\ \mathrm{P}(s,v) &= \frac{1+v}{\sqrt{2}}\left(\sin\omega T+N+\cos\omega B\right).\end{aligned}$$

Next, the first and the second partial differentials of $\mathrm{P}(s,v)$ are

$$\mathrm{P}_s = \frac{1+v}{\sqrt{2}}\left(\left(-\kappa+\omega'\cos\omega\right)T+\left(\tau-\omega'\sin\omega\right)B\right),$$

$$\mathrm{P}_v = \frac{1}{\sqrt{2}}\left(\sin\omega T+N+\cos\omega B\right),$$

$$\mathrm{P}_{ss} = \frac{1}{\sqrt{2}}\left(1+v\right)\begin{bmatrix} (-\kappa+\omega'\cos\omega)'T-\left(\kappa^2+\tau^2-\omega'\left(\kappa\cos\omega+\tau\sin\omega\right)\right)N \\ +\left(\tau-\omega'\sin\omega\right)'B \end{bmatrix},$$

$$\mathrm{P}_{sv} = \frac{1}{\sqrt{2}}\left(\left(-\kappa+\omega'\cos\omega\right)T+\left(\tau-\omega'\sin\omega\right)B\right),$$

$$\mathrm{P}_{vv} = 0.$$

And the vectorial product of the vectors P_s , P_v and its norm are

$$\mathrm{P}_s\wedge\mathrm{P}_v = \frac{(1+v)}{2}\left[\left(\omega'\sin\omega-\tau\right)T+\left(\kappa\cos\omega+\tau\sin\omega-\omega'\right)N+\left(\omega'\cos\omega-\kappa\right)B\right],$$

$$\left\|\mathrm{P}_s\wedge\mathrm{P}_v\right\| = \frac{(1+v)}{2}\sqrt{\left(\omega'\sin\omega-\tau\right)^2+\left(\omega'\cos\omega-\kappa\right)^2+\left(\kappa\cos\omega+\tau\sin\omega-\omega'\right)^2}.$$

If we denote the normal vector of the surface by N_{P}, then from the expression (4), we get

$$N_{\mathrm{P}} = \frac{\left(\omega'\sin\omega-\tau\right)T+\left(\kappa\cos\omega+\tau\sin\omega-\omega'\right)N+\left(\omega'\cos\omega-\kappa\right)B}{\sqrt{\left(\omega'\sin\omega-\tau\right)^2+\left(\omega'\cos\omega-\kappa\right)^2+\left(\kappa\cos\omega+\tau\sin\omega-\omega'\right)^2}}.$$

By using the expressions (6) and (7), the coefficients of the first and the second fundamental forms are given as

$$E_{\mathrm{P}} = \frac{1}{2}\left((1+v)^2\left(\left(-\kappa+\omega'\cos\omega\right)^2+\left(\tau-\omega'\sin\omega\right)^2\right)\right),$$

$$F_{\mathrm{P}} = 0,$$

$$G_{\mathrm{P}} = 1$$

and

$$e_{\mathrm{P}} = \frac{(1+v)\begin{bmatrix} (\omega'\cos\omega-\kappa)'(\omega'\sin\omega-\tau)+(\tau-\omega'\sin\omega)'(\omega'\cos\omega-\kappa) \\ +(\kappa\cos\omega+\tau\sin\omega-\omega')^2 \end{bmatrix}}{\sqrt{2}\Big((\omega'\sin\omega-\tau)^2+(\omega'\cos\omega-\kappa)^2+(\kappa\cos\omega+\tau\sin\omega-\omega')^2\Big)^{\frac{1}{2}}},$$

$$f_{\mathrm{P}} = 0,$$

$$g_{\mathrm{P}} = 0,$$

respectively. Finally, from the expression (5), the Gaussian and mean curvatures are obtained as:

$$K_{\mathrm{P}} = 0,$$

$$H_{\mathrm{P}} = \frac{\begin{bmatrix} \left(\sqrt{2}\,(1+v)\right)^{-1}(-\kappa+\omega'\cos\omega)'(\omega'\sin\omega-\tau) \\ +(\tau-\omega'\sin\omega)'(-\kappa+\omega'\cos\omega)+(\kappa\cos\omega+\tau\sin\omega-\omega')^2 \end{bmatrix}}{\begin{bmatrix} \left(\kappa^2+\tau^2+\omega'^2-2\omega'(\kappa\cos\omega+\tau\sin\omega)\right) \\ .\sqrt{(\omega'\sin\omega-\tau)^2+(\omega'\cos\omega-\kappa)^2+(\kappa\cos\omega+\tau\sin\omega-\omega')^2} \end{bmatrix}}.$$

Corollary 2. *The ruled surface* $\mathrm{P}(s,v)$ *is always developable.*

Definition 6. *Let's define a ruled surface generated by continuously moving the vector* $B+C$ *along the* $NC-$ *Smarandache curve. Thus, we provide its parametric form as*

$$\begin{aligned} \delta(s,v) &= \frac{1}{\sqrt{2}}(N+C)+\frac{v}{\sqrt{2+2\cos\omega}}(B+C), \\ \delta(s,v) &= \frac{1}{\sqrt{2}}\left[\left(\sin\omega+v\sqrt{1+\cos\omega}\right)T+N+\left(\cos\omega+v\sqrt{1-\cos\omega}\right)B\right]. \end{aligned}$$

If we assign $p^*(s,v)=\sin\omega+v\sqrt{1+\cos\omega}$ and $q^*(s,v)=\cos\omega+v\sqrt{1-\cos\omega}$, then we can rewrite the surface in a simple form as

$$\delta(s,v)=\frac{1}{\sqrt{2}}\left(p^*T+N+q^*B\right).$$

Next, the first and second partial differentials of $\delta(s,v)$ are

$$\delta_s = \frac{1}{\sqrt{2}} \left(\left(-\kappa + p_s^*\right) T + \left(\kappa p^* - \tau q^*\right) N + \left(\tau + q_s^*\right) B \right),$$

$$\delta_v = \frac{1}{\sqrt{2}} \left(p_v^* T + q_v^* B \right),$$

$$\delta_{vv} = 0,$$

$$\delta_{sv} = \frac{1}{\sqrt{2}} \left(p_{sv}^* T + \left(\kappa p_v^* - \tau q_v^*\right) N + q_{sv}^* B \right),$$

$$\delta_{ss} = \frac{1}{\sqrt{2}} \left[\begin{array}{l} \left(-\kappa' + p_{ss}^* - \kappa^2 p^* + \tau\kappa q^*\right) T + \left(\tau' + q_{ss}^* + \kappa\tau p^* - \tau^2 q^*\right) B \\ + \left(2\kappa p_s^* - 2\tau q_s^* - \tau^2 - \kappa^2 + \kappa' p^* - \tau' q^*\right) N \end{array} \right].$$

And the vectorial product of the vectors δ_s , δ_v and its norm are

$$\delta_s \wedge \delta_v = \frac{1}{2} \left[\begin{array}{l} q_v^* \left(\kappa p^* - \tau q^*\right) T + \left(p_v^* \left(\tau + q_s^*\right) - q_v^* \left(-\kappa + p^* s\right)\right) N \\ -p_v^* \left(\kappa p^* - \tau q^*\right) B \end{array} \right],$$

$$\|\delta_s \wedge \delta_v\| = \frac{1}{2} \left[\left(q_v^{*2} + p_v^{*2}\right) \left(\kappa p^* - \tau q^*\right)^2 + \left(p_v^* \left(\tau + q_s^*\right) - q_v^* \left(-\kappa + p_s^*\right)\right)^2 \right]^{\frac{1}{2}}.$$

From the expression (4), we compute the normal of the surface denoted by N_δ as

$$N_\delta = \frac{q_v^* \left(\kappa p^* - \tau q^*\right) T + \left(p_v^* \left(\tau + q_s^*\right) - q_v^* \left(-\kappa + p_s^*\right)\right) N - p_v^* \left(\kappa p^* - \tau q^*\right) B}{\left[\left(q_v^{*2} + p_v^{*2}\right) \left(\kappa p^* - \tau q^*\right)^2 + \left(p_v^* \left(\tau + q_s^*\right) - q_v^* \left(-\kappa + p_s^*\right)\right)^2 \right]^{\frac{1}{2}}}.$$

By the expressions (6) and (7), the coefficients of fundamental forms are given as

$$E_\delta = \frac{1}{2} \left[\left(-\kappa + p_s^*\right)^2 + \left(\kappa p^* - \tau q^*\right)^2 + \left(\tau + q_s^*\right)^2 \right],$$

$$F_\delta = \frac{1}{2} \left[\left(-\kappa + p_s^*\right) p_v^* + \left(\tau + q_s^*\right) q_v^* \right],$$

$$G_\delta = 1$$

and

$$e_\delta = \frac{\left[\begin{array}{l} (\kappa p^* - \tau q^*)\left[q_v^*\left(-\kappa' + p_{ss}^* - \kappa^2 p^* + \tau\kappa q^*\right) - p_v^*\left(\tau' + q_{ss}^* + \kappa\tau p^* - \tau^2 q^*\right)\right] \\ + \left(p_v^*\left(\tau + q_s^*\right) - q_v^*\left(-\kappa + p_s^*\right)\right)\left(2\kappa p_s^* - 2\tau q_s^* - \tau^2 - \kappa^2 + \kappa' p^* - \tau' q^*\right) \end{array}\right]}{\sqrt{2}\Big(\left(q_v^{*2} + p_v^{*2}\right)(\kappa p^* - \tau q^*)^2 + \left(p_v^*\left(\tau + q_s^*\right) - q_v^*\left(-\kappa + p_s^*\right)\right)^2\Big)^{\frac{1}{2}}},$$

$$f_\delta = \frac{\left(q_v^* p_{sv}^* - p_v^* q_{sv}^*\right)(\kappa p^* - \tau q^*) + \left(p_v^*\left(\tau + q_s^*\right) - q_v^*\left(-\kappa + p_s^*\right)\right)\left(\kappa p_v^* - \tau q_v^*\right)}{\sqrt{2}\Big(\left(q_v^{*2} + p_v^{*2}\right)(\kappa p^* - \tau q^*)^2 + \left(p_v^*\left(\tau + q_s^*\right) - q_v^*\left(-\kappa + p_s^*\right)\right)^2\Big)^{\frac{1}{2}}},$$

$$g_\delta = 0,$$

respectively. Finally, from the expression (5), we compute the Gaussian and mean curvatures as

$$K_\delta = -2\left[\frac{\left(q_v^* p_{sv}^* - p_v^* q_{sv}^*\right)(\kappa p^* - \tau q^*) + \left(p_v^*\left(\tau + q_s^*\right) - q_v^*\left(-\kappa + p_s^*\right)\right)\left(\kappa p_v^* - \tau q_v^*\right)}{\left(q_v^{*2} + p_v^{*2}\right)(\kappa p^* - \tau q^*)^2 + \left(p_v^*\left(\tau + q_s^*\right) - q_v^*\left(-\kappa + p_s^*\right)\right)^2}\right]^2,$$

$$H_\delta = \frac{\left[\begin{array}{l} (\kappa p^* - \tau q^*)\left[q_v^*\left(-\kappa' + p_{ss}^* - \kappa^2 p^* + \tau\kappa q^*\right) - p_v^*\left(\tau' + q_{ss}^* + \kappa\tau p^* - \tau^2 q^*\right)\right] \\ + \left(p_v^*\left(\tau + q_s^*\right) - q_v^*\left(-\kappa + p_s^*\right)\right)\left(2\kappa p_s^* - 2\tau q_s^* - \tau^2 - \kappa^2 + \kappa' p^* - \tau' q^*\right) \\ - \left(q_v^* p_{sv}^* - p_v^* q_{sv}^*\right)(\kappa p^* - \tau q^*)\left(\left(-\kappa + p_s^*\right) p_v^* + \left(\tau + q_s^*\right) q_v^*\right) \\ - \left(p_v^*\left(\tau + q_s^*\right) - q_v^*\left(-\kappa + p_s^*\right)\right)\left(p_v^*\left(-\kappa + p_s^*\right) + q_v^*\left(\tau + q_s^*\right)\right)\left(\kappa p_v^* - \tau q_v^*\right) \end{array}\right]}{2^{-\frac{1}{2}}\left[\left(q_v^{*2} + p_v^{*2}\right)(\kappa p^* - \tau q^*)^2 + \left(p_v^*\left(\tau + q_s^*\right) - q_v^*\left(-\kappa + p_s^*\right)\right)^2\right]^{\frac{3}{2}}}.$$

Definition 7. *Let's define a ruled surface generated by continuously moving the vector $T + C$ along the $BC-$ Smarandache curve. Thus, we provide its parametric form as*

$$\begin{aligned} \eta(s, v) &= \frac{1}{\sqrt{2 + 2\cos\omega}}(B + C) + \frac{v}{\sqrt{2 + 2\sin\omega}}(T + C), \\ \eta(s, v) &= \frac{1}{\sqrt{2}}\Big[\Big(\left(\sqrt{1 - \cos\omega}\right) + v\left(\sqrt{1 + \sin\omega}\right)\Big) T \\ &\qquad + \Big(\left(\sqrt{1 + \cos\omega}\right) + v\left(\sqrt{1 - \sin\omega}\right)\Big) B\Big]. \end{aligned}$$

If we assign $m^*(s, v) = \left(\sqrt{1 - \cos\omega} + v\sqrt{1 + \sin\omega}\right)$ and $n^*(s, v) = \left(\sqrt{1 + \cos\omega} + v\sqrt{1 - \sin\omega}\right)$, then we can rewrite the surface in a simple form as

$$\eta(s,v)=\frac{1}{\sqrt{2}}\left(m^*(s,v)\,T(s)+n^*(s,v)\,B(s)\right).$$

Next, the first and the second partial differentials of $\eta(s,v)$ are

$$\eta_s=\frac{1}{\sqrt{2}}\left(m_s^*T+(\kappa m^*-\tau n^*)\,N+n_s^*B\right),$$

$$\eta_v=\frac{1}{\sqrt{2}}\left(m_v^*T+n_v^*B\right),$$

$$\eta_{vv}=0,$$

$$\eta_{ss}=\frac{1}{\sqrt{2}}\left[\begin{array}{l}\left(m_{ss}^*-\kappa^2m^*+\tau\kappa n_s^*\right)T+\left(\kappa' m^*-\tau' n_s^*+2\kappa m_s^*-2\tau n_s^*\right)N\\ +\left(n_{ss}^*+\kappa\tau m^*-\tau^2 n_s^*\right)B\end{array}\right],$$

$$\eta_{sv}=\frac{1}{\sqrt{2}}\left(m_{sv}^*T+(\kappa m_v^*-\tau n_v^*)\,N+n_{sv}^*B\right).$$

And the vectorial product of the vectors η_s , η_v and its norm are

$$\eta_s\wedge\eta_v=\frac{1}{2}\left(n_v^*(\kappa m^*-\tau n^*)\,T+(n_s^*m_v^*-m_s^*n_v^*)\,N-m_v^*(\kappa m^*-\tau n^*)\,B\right),$$

$$\|\eta_s\wedge\eta_v\|=\frac{1}{2}\Big((\kappa m^*-\tau n^*)^2\left(m_v^{*2}+n_v^{*2}\right)+(n_s^*m_v^*-m_s^*n_v^*)^2\Big)^{\frac{1}{2}}.$$

By using the expression (4), the normal vector field denoted by N_η can be computed as:

$$N_\eta=\frac{n_v^*(\kappa m^*-\tau n^*)\,T+(n_s^*m_v^*-m_s^*n_v^*)\,N-m_v^*(\kappa m^*-\tau n^*)\,B}{\Big((\kappa m^*-\tau n^*)^2\left(m_v^{*2}+n_v^{*2}\right)+(n_s^*m_v^*-m_s^*n_v^*)^2\Big)^{\frac{1}{2}}}.$$

From the expressions (6) and (7), the coefficients of fundamental forms can be given as:

$$E_\eta=\frac{1}{2}\left({m_s}^{*2}+{n_s}^{*2}+(\kappa m^*-\tau n^*)^2\right),$$

$$F_\eta=\frac{1}{2}\left({m_s}^*{m_v}^*+{n_s}^*{n_v}^*\right),$$

$$G_\eta=1$$

and

$$e_\eta = \frac{\begin{bmatrix} (\kappa m^* - \tau n^*)\left(n_v^*\left(m_{ss}^* - \kappa^2 m^* + \tau\kappa n_s^*\right) - m_v^*\left(n_{ss}^* + \kappa\tau m^* - \tau^2 n_s^*\right)\right) \\ + \left(n_s^* m_v^* - m_s^* n_v^*\right)\left(\kappa' m^* - \tau' n_s^* + 2\kappa m_s^* - 2\tau n_s^*\right) \end{bmatrix}}{\sqrt{2}\left((\kappa m^* - \tau n^*)^2\left(m_v^{*2} + n_v^{*2}\right) + \left(n_s^* m_v^* - m_s^* n_v^*\right)^2\right)^{\frac{1}{2}}},$$

$$f_\eta = \frac{(\kappa m^* - \tau n^*)\left(n_v^* m_v^* - m_v^* n_v^*\right)}{\sqrt{2}\left((\kappa m^* - \tau n^*)^2\left(m_v^{*2} + n_v^{*2}\right) + \left(n_s^* m_v^* - m_s^* n_v^*\right)^2\right)^{\frac{1}{2}}},$$

$$g_\eta = 0,$$

respectively. Finally, from the expression (5), the Gaussian and mean curvatures are obtained as

$$K_\eta = -\frac{2(\kappa m^* - \tau n^*)^2 (n_v^* m_v^* - m_v^* n_v^*)^2}{\left((\kappa m^* - \tau n^*)^2\left(m_v^{*2} + n_v^{*2}\right) + \left(n_s^* m_v^* - m_s^* n_v^*\right)^2\right)^2},$$

$$H_\eta = \frac{\begin{bmatrix} (\kappa m^* - \tau n^*)\left[n_v^*\left(m_{ss}^* - \kappa^2 m^* + \tau\kappa n_s^*\right) - m_v^*\left(n_{ss}^* + \kappa\tau m^* - \tau^2 n_s^*\right)\right] \\ + \left(n_s^* m_v^* - m_s^* n_v^*\right)\left(\kappa' m^* - \tau' n_s^* + 2\kappa m_s^* - 2\tau n_s^*\right) \\ - (\kappa m^* - \tau n^*)\left(n_v^* m_v^* - m_v^* n_v^*\right)\left(m_s^* m_v^* + n_s^* n_v^*\right) \end{bmatrix}}{2^{-\frac{1}{2}}\left((\kappa m^* - \tau n^*)^2\left(m_v^{*2} + n_v^{*2}\right) + \left(n_s^* m_v^* - m_s^* n_v^*\right)^2\right)^{\frac{3}{2}}}.$$

Definition 8. *Let's define a ruled surface generated by continuously moving the vector $N + C$ along the $BC-$ Smarandache curve. Thus, we provide its parametric form as*

$$\begin{aligned} \varphi(s,v) &= \frac{1}{\sqrt{2 + 2\cos\omega}}(B + C) + \frac{v}{\sqrt{2}}(N + C), \\ \varphi(s,v) &= \frac{1}{\sqrt{2}}\left(\left(\sqrt{1 - \cos\omega} + v\sin\omega\right)T + vN + \left(\sqrt{1 + \cos\omega} + v\cos\omega\right)B\right). \end{aligned}$$

If we assign $\mu(s,v) = \left(\sqrt{1 - \cos\omega} + v\sin\omega\right)$ and $\rho(s,v) = \left(\sqrt{1 + \cos\omega} + v\cos\omega\right)$, then we can rewrite the surface in a simple form as

$$\varphi(s,v) = \frac{1}{\sqrt{2}}(\mu T + vN + \rho B).$$

Next, the first and second partial differentials of $\varphi(s,v)$ are

$$\varphi_s = \frac{1}{\sqrt{2}}\left(\left(\mu_s - v\kappa\right)T + \left(\kappa\mu - \tau\rho\right)N + \left(\rho_s + v\tau\right)B\right),$$

$$\varphi_v = \frac{1}{\sqrt{2}}\left(\mu_v T + N + \rho_v B\right),$$

$$\varphi_{vv} = 0,$$

$$\varphi_{sv} = \frac{1}{\sqrt{2}}\left(\left(\mu_{sv} - \kappa\right)T + \left(\kappa\mu_v - \tau\rho_v\right)N + \left(\rho_{sv} + \tau\right)B\right),$$

$$\varphi_{ss} = \frac{1}{\sqrt{2}}\begin{bmatrix} \left(\mu_{ss} - v\kappa' - \kappa^2\mu + \tau\kappa\rho\right)T \\ +\left(2\kappa\mu_s - 2\tau\rho_s + \kappa'\mu - \tau'\rho - v\kappa^2 - v\tau^2\right)N \\ +\left(\rho_{ss} + v\tau' + \tau\kappa\mu - \tau^2\rho\right)B \end{bmatrix}.$$

And the vectorial product of the vectors φ_s , φ_v and its norm are

$$\varphi_s \wedge \varphi_v = \frac{1}{2}\begin{bmatrix} \left(\rho_v\left(\kappa\mu - \tau\rho\right) - \left(\rho_s + v\tau\right)\right)T + \left(\mu_v\left(\rho_s + v\tau\right) - \rho_v\left(\mu_s - v\kappa\right)\right)N \\ +\left(\left(\mu_s - v\kappa\right) - \mu_v\left(\kappa\mu - \tau\rho\right)\right)B \end{bmatrix},$$

$$\left\|\varphi_s \wedge \varphi_v\right\| = \frac{1}{2}\begin{bmatrix} \left(\rho_v\left(\kappa\mu - \tau\rho\right) - \left(\rho_s + v\tau\right)\right)^2 + \left(\left(\mu_s - v\kappa\right) - \mu_v\left(\kappa\mu - \tau\rho\right)\right)^2 \\ +\left(\mu_v\left(\rho_s + v\tau\right) - \rho_v\left(\mu_s - v\kappa\right)\right)^2 \end{bmatrix}^{\frac{1}{2}}.$$

From the expression (4), the normal of the surface $\varphi(s,v)$ denoted by N_φ can then be given as:

$$N_\varphi = \frac{\begin{bmatrix} \left(\rho_v\left(\kappa\mu - \tau\rho\right) - \left(\rho_s + v\tau\right)\right)T + \left(\mu_v\left(\rho_s + v\tau\right) - \rho_v\left(\mu_s - v\kappa\right)\right)N \\ +\left(\left(\mu_s - v\kappa\right) - \mu_v\left(\kappa\mu - \tau\rho\right)\right)B \end{bmatrix}}{\begin{bmatrix} \left(\rho_v\left(\kappa\mu - \tau\rho\right) - \left(\rho_s + v\tau\right)\right)^2 + \left(\left(\mu_s - v\kappa\right) - \mu_v\left(\kappa\mu - \tau\rho\right)\right)^2 \\ +\left(\mu_v\left(\rho_s + v\tau\right) - \rho_v\left(\mu_s - v\kappa\right)\right)^2 \end{bmatrix}^{\frac{1}{2}}}.$$

The coefficients of first and second fundamental form are calculated by using the expressions (6) and (7) as:

$$
\begin{aligned}
E_\varphi &= \frac{1}{2}\left((\mu_s - \upsilon\kappa)^2 + (\kappa\mu - \tau\rho)^2 + (\rho_s + \upsilon\tau)^2\right),\\
F_\varphi &= \frac{1}{2}\left(\mu_v\left(\mu_s - \upsilon\kappa\right) + (\kappa\mu - \tau\rho) + \rho_v\left(\rho_s + \upsilon\tau\right)\right),\\
G_\varphi &= 1
\end{aligned}
$$

and

$$
e_\varphi = \frac{\left[\begin{array}{l}
(\rho_v(\kappa\mu - \tau\rho) - (\rho_s + \upsilon\tau))\left(\mu_{ss} - \upsilon\kappa' - \kappa^2\mu + \tau\kappa\rho\right)\\
+\left((\mu_s - \upsilon\kappa) - \mu_v(\kappa\mu - \tau\rho)\right)\left(\rho_{ss} + \upsilon\tau' + \tau\kappa\mu - \tau^2\rho\right)\\
+\left(\mu_v(\rho_s + \upsilon\tau) - \rho_v(\mu_s - \upsilon\kappa)\right)\left(2\kappa\mu_s - 2\tau\rho_s + \kappa'\mu - \tau'\rho - \upsilon\kappa^2 - \upsilon\tau^2\right)
\end{array}\right]}{\sqrt{2}\left[\begin{array}{l}
(\rho_v(\kappa\mu - \tau\rho) - (\rho_s + \upsilon\tau))^2 + ((\mu_s - \upsilon\kappa) - \mu_v(\kappa\mu - \tau\rho))^2\\
+(\mu_v(\rho_s + \upsilon\tau) - \rho_v(\mu_s - \upsilon\kappa))^2
\end{array}\right]^{\frac{1}{2}}},
$$

$$
f_\varphi = \frac{\left[\begin{array}{l}
(\rho_v(\kappa\mu - \tau\rho) - (\rho_s + \upsilon\tau))(\mu_{sv} - \kappa)\\
+\left((\mu_s - \upsilon\kappa) - \mu_v(\kappa\mu - \tau\rho)\right)(\rho_{sv} + \tau)\\
+\left(\mu_v(\rho_s + \upsilon\tau) - \rho_v(\mu_s - \upsilon\kappa)\right)(\kappa\mu_v - \tau\rho_v)
\end{array}\right]}{\sqrt{2}\left[\begin{array}{l}
(\rho_v(\kappa\mu - \tau\rho) - (\rho_s + \upsilon\tau))^2 + ((\mu_s - \upsilon\kappa) - \mu_v(\kappa\mu - \tau\rho))^2\\
+(\mu_v(\rho_s + \upsilon\tau) - \rho_v(\mu_s - \upsilon\kappa))^2
\end{array}\right]^{\frac{1}{2}}},
$$

$$
g_\varphi = 0,
$$

respectively. Finally, from the expression (5), we have the Gaussian and mean curvatures as in the following:

$$K_\varphi = -\frac{\begin{bmatrix} (\rho_v(\kappa\mu - \tau\rho) - (\rho_s + \upsilon\tau))(\mu_{sv} - \kappa) \\ +((\mu_s - \upsilon\kappa) - \mu_v(\kappa\mu - \tau\rho))(\rho_{sv} + \tau) \\ +(\mu_v(\rho_s + \upsilon\tau) - \rho_v(\mu_s - \upsilon\kappa))(\kappa\mu_v - \tau\rho_v) \end{bmatrix}^2}{\begin{bmatrix} (\rho_v(\kappa\mu - \tau\rho) - (\rho_s + \upsilon\tau))^2 + ((\mu_s - \upsilon\kappa) - \mu_v(\kappa\mu - \tau\rho))^2 \\ +(\mu_v(\rho_s + \upsilon\tau) - \rho_v(\mu_s - \upsilon\kappa))^2 \end{bmatrix}^2},$$

$$H_\varphi = \frac{\begin{bmatrix} (\rho_v(\kappa\mu - \tau\rho) - (\rho_s + \upsilon\tau))\left(\mu_{ss} - \upsilon\kappa' - \kappa^2\mu + \tau\kappa\rho\right) \\ +((\mu_s - \upsilon\kappa) - \mu_v(\kappa\mu - \tau\rho))\left(\rho_{ss} + \upsilon\tau' + \tau\kappa\mu - \tau^2\rho\right) \\ +(\mu_v(\rho_s + \upsilon\tau) - \rho_v(\mu_s - \upsilon\kappa))\left(2\kappa\mu_s - 2\tau\rho_s + \kappa'\mu - \tau'\rho - \upsilon\kappa^2 - \upsilon\tau^2\right) \\ -(\rho_v(\kappa\mu - \tau\rho) - (\rho_s + \upsilon\tau)) \\ .(\mu_{sv} - \kappa)(\mu_v(\mu_s - \upsilon\kappa) + (\kappa\mu - \tau\rho) + \rho_v(\rho_s + \upsilon\tau)) \\ -((\mu_s - \upsilon\kappa) - \mu_v(\kappa\mu - \tau\rho))(\rho_{sv} + \tau) \\ .(\mu_v(\mu_s - \upsilon\kappa) + (\kappa\mu - \tau\rho) + \rho_v(\rho_s + \upsilon\tau)) \\ -(\mu_v(\rho_s + \upsilon\tau) - \rho_v(\mu_s - \upsilon\kappa))(\kappa\mu_v - \tau\rho_v) \\ .(\mu_v(\mu_s - \upsilon\kappa) + (\kappa\mu - \tau\rho) + \rho_v(\rho_s + \upsilon\tau)) \end{bmatrix}}{\sqrt{2}\begin{bmatrix} (\rho_v(\kappa\mu - \tau\rho) - (\rho_s + \upsilon\tau))^2 + ((\mu_s - \upsilon\kappa) - \mu_v(\kappa\mu - \tau\rho))^2 \\ +(\mu_v(\rho_s + \upsilon\tau) - \rho_v(\mu_s - \upsilon\kappa))^2 \end{bmatrix}^{\frac{3}{2}}}.$$

Definition 9. *Let's define a ruled surface generated by continuously moving the vector $B + C$ along the $BC-$ Smarandache curve. Thus, we provide its parametric form as*

$$\begin{aligned} \lambda(s,v) &= \frac{B + C}{\sqrt{2 + 2\cos\omega}} + v\frac{B + C}{\sqrt{2 + 2\cos\omega}}, \\ \lambda(s,v) &= \frac{1 + v}{\sqrt{2}}\left(\left(\sqrt{1 - \cos\omega}\right)T + \left(\sqrt{1 + \cos\omega}\right)B\right). \end{aligned}$$

Next, the first and second partial differentials of $\lambda(s,v)$ are

$$\lambda_s = \frac{1+v}{\sqrt{2}}\left(\omega' \cos\omega T - \tau N - \omega' \sin\omega B\right),$$

$$\lambda_v = \frac{1}{\sqrt{2}}\left(\sin\omega T + (1+\cos\omega) B\right),$$

$$\lambda_{vv} = 0,$$

$$\lambda_{sv} = \frac{1}{\sqrt{2}}\left(\omega' \cos\omega T - \tau N - \omega' \sin\omega B\right),$$

$$\lambda_{ss} = \frac{1+v}{\sqrt{2}}\left((\kappa\tau + \omega' \cos\omega) T + \left(\omega' \sqrt{\kappa^2+\tau^2} - \tau'\right) N - \left(\tau^2 + \omega' \sin\omega\right) B\right),$$

And the vectorial product of the vectors λ_s , λ_v and its norm are

$$\lambda_s \wedge \lambda_v = \frac{1+v}{2}\left(-\tau(1+\cos\omega) T - \omega'(1+\cos\omega) N + \tau \sin\omega B\right)$$

$$\|\lambda_s \wedge \lambda_v\| = \frac{1+v}{2}\sqrt{\left(\tau^2 + \omega'^2\right)(1+\cos\omega)^2 + \tau^2 \sin^2\omega}.$$

From the expression (4), the normal of this surface shown by N_λ is given

$$N_\lambda = \frac{-\tau(1+\cos\omega) T - \omega'(1+\cos\omega) N + \tau \sin\omega B}{\sqrt{\left(\tau^2 + \omega'^2\right)(1+\cos\omega)^2 + \tau^2 \sin^2\omega}}.$$

Next, from the expressions (6) and (7), the coefficients of the first and the second fundamental forms can be calculated as

$$E_\lambda = \frac{1}{2}\left((1+v)^2\left(\omega'^2 + \tau^2\right)\right),$$

$$F_\lambda = -\frac{1}{\sqrt{2}}\left(\omega'(1+v)\sin\omega\right),$$

$$G_\lambda = (1+\cos\omega)$$

and

$$e_\lambda = -\frac{\tau^2\left(\kappa + \sqrt{\kappa^2+\tau^2}\right) + (\tau\omega' + \omega')(1+\cos\omega)\left(\omega'\sqrt{\kappa^2+\tau^2} - \tau'\right)}{(1+v)^{-1}\sqrt{2}\sqrt{\left(\tau^2 + \omega'^2\right)(1+\cos\omega)^2 + \tau^2 \sin^2\omega}},$$

$$f_\lambda = 0,$$

$$g_\lambda = 0,$$

respectively. Finally, from the expression (5), we have the Gaussian and mean curvatures as

$$K_\lambda = 0,$$

$$H_\lambda = \frac{\left(-\kappa\tau^2 - \tau^2\sqrt{\kappa^2+\tau^2}\right)(1+\cos\omega) - (\tau\omega' + \omega')(1+\cos\omega)^2\left(\omega'\sqrt{\kappa^2+\tau^2} - \tau'\right)}{\left[\begin{array}{l}\sqrt{2}\,(1+v)\left(\left(\omega'^2+\tau^2\right)(1+\cos\omega) - \omega'^2\sin^2\omega\right)\\ \cdot\sqrt{\left(\tau^2+\omega'^2\right)(1+\cos\omega.)^2 + \tau^2\sin^2\omega}\end{array}\right]}.$$

Corollary 3. *The ruled surface $\lambda(s,v)$ is always developable.*

Example 1. *Let us consider the famous Viviani's curve whose parametric form is given by $\alpha(s) = \left(\cos^2(s), \cos(s)\sin(s), \sin(s)\right)$. The Frenet vectors $T(s), N(s), B(s)$ and the unit Darboux vector $C(s)$ are given in respective order as*

$$T(s) = \left(-\frac{2\cos(s)\sin(s)}{\sqrt{\cos(s)^2+1}}, \frac{2\cos(s)^2-1}{\sqrt{\cos(s)^2+1}}, \frac{\cos(s)}{\sqrt{\cos(s)^2+1}}\right),$$

$$N(s) = \left(\begin{array}{l} -\frac{2\left(\cos(s)^4+2\cos(s)^2-1\right)}{\sqrt{3\cos(s)^2+5}\sqrt{\cos(s)^2+1}}, -\frac{\cos(s)\sin(s)\left(2\cos(s)^2+5\right)}{\sqrt{\cos(s)^2+1}\sqrt{3\cos(s)^2+5}},\\ -\frac{\sin(s)}{\sqrt{\cos(s)^2+1}\sqrt{3\cos(s)^2+5}}\end{array}\right)$$

$$B(s) = \left(\frac{\left(2\cos(s)^2+1\right)\sin(s)}{\sqrt{3\cos(s)^2+5}}, -\frac{2\cos(s)^3}{\sqrt{3\cos(s)^2+5}}, \frac{2}{\sqrt{3\cos(s)^2+5}}\right),$$

$$C(s)=\left(\begin{array}{c}\dfrac{\left(-6\cos(s)^4+\cos(s)^2+5\right)\sin(s)\sqrt{\cos(s)^2+1}}{\sqrt{\begin{array}{l}\left(36\cos(s)^8+135\cos(s)^6+243\cos(s)^4+261\cos(s)^2+125\right)\\ .\left(\cos(s)^2+1\right)\end{array}}},\\ \dfrac{2\left(3\cos(s)^4-2\cos(s)^2-3\right)\cos(s)\sqrt{\cos(s)^2+1}}{\sqrt{\begin{array}{l}\left(36\cos(s)^8+135\cos(s)^6+243\cos(s)^4+261\cos(s)^2+125\right)\\ .\left(\cos(s)^2+1\right)\end{array}}},\\ \dfrac{2\left(3\cos(s)^4+6\cos(s)^2+5\right)\sqrt{\cos(s)^2+1}}{\sqrt{\begin{array}{l}\left(36\cos(s)^8+135\cos(s)^6+243\cos(s)^4+261\cos(s)^2+125\right)\\ .\left(\cos(s)^2+1\right)\end{array}}}.\end{array}\right)$$

The graphs of ruled surfaces, obtained using these vectors and definitions and given the parametric equations below, are presented in FIGURE 1, 2 and 3, respectively.

$$\Sigma(s,v)=\frac{1}{\sqrt{2}}(T+C)+\frac{v}{\sqrt{2}}(T+C),\quad \Delta(s,v)=\frac{1}{\sqrt{2}}(T+C)+\frac{v}{\sqrt{2}}(N+C),$$

$$\Upsilon(s,v)=\frac{1}{\sqrt{2}}(T+C)+\frac{v}{\sqrt{2}}(B+C),\quad \chi(s,v)=\frac{1}{\sqrt{2}}(N+C)+\frac{v}{\sqrt{2}}(T+C),$$

$$\mathrm{P}(s,v)=\frac{1}{\sqrt{2}}(N+C)+\frac{v}{\sqrt{2}}(N+C),\quad \delta(s,v)=\frac{1}{\sqrt{2}}(N+C)+\frac{v}{\sqrt{2}}(B+C),$$

$$\eta(s,v)=\frac{1}{\sqrt{2}}(B+C)+\frac{v}{\sqrt{2}}(T+C),\quad \varphi(s,v)=\frac{1}{\sqrt{2}}(B+C)+\frac{v}{\sqrt{2}}(N+C),$$

$$\lambda(s,v)=\frac{1}{\sqrt{2}}(B+C)+\frac{v}{\sqrt{2}}(B+C).$$

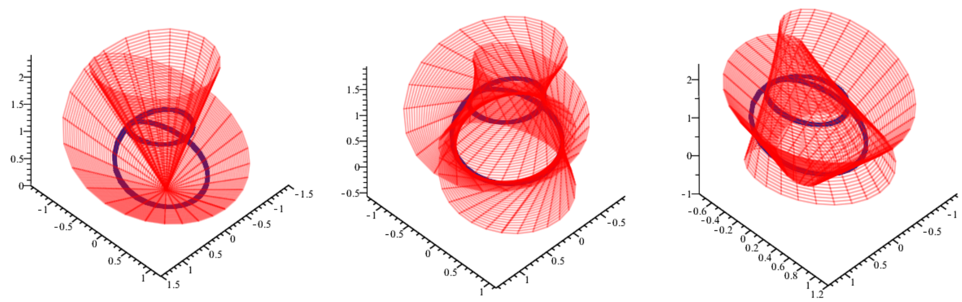

FIGURE 1. The ruled surfaces whose the base curve TC – Smarandache curve and the direction vector TC, NC, BC, respectively.

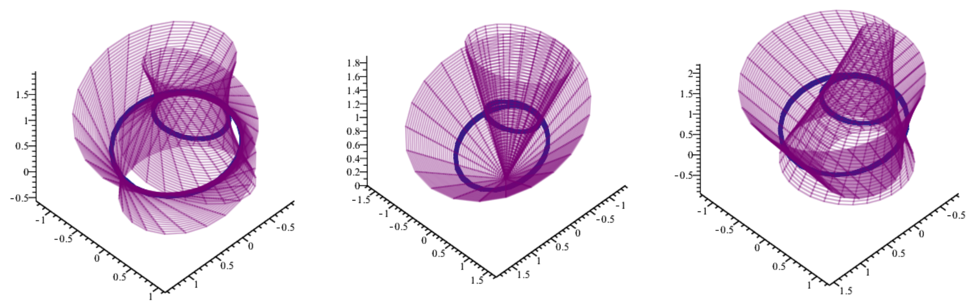

FIGURE 2. The ruled surfaces whose the base curve NC – Smarandache curve and the direction vector TC, NC, BC, respectively.

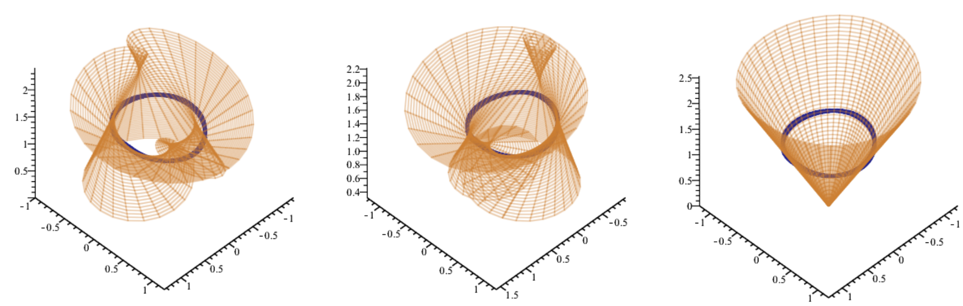

FIGURE 3. The ruled surfaces whose the base curve BC – Smarandache curve and the direction vector TC, NC, BC, respectively.

4. Conclusion

In this paper, Smarandache curves derived from Frenet vectors and Darboux vector of any curve are described. Then, by considering the direction vectors obtained from Frenet vectors and Darboux vectors, new ruled surfaces are obtained along these curves. Finally, the Gaussian and mean curvatures of these surfaces are given. This paper can also be studied by considering other frames defined on the curve, additionally it can be examined in the spaces other than Euclidean space.

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