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Weighted Ostrowski's Type Integral Inequalities for Mapping Whose Second Derivative is Bounded

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Abstract

The aim of this paper is to concentrate on the domain of L_∞, L_p, and L_1 norms of inequalities and their applications for some special weight functions. For different weights some previous results are recaptured. Applications are also discussed.

1. Introduction

In 1938, Ostrowski [1] established the interesting integral inequality for differentiable mappings with bounded derivative.

Lemma 1.1. *Let $f : [\breve{a}, \hat{c}] \to \mathbb{R}$ be continuous on $[\breve{a}, \hat{c}]$ and differentiable on $(\breve{a}, \hat{c})$ and assume $\left|f'(\hat{s})\right| \leq M$ for all $\hat{s} \in (\breve{a}, \hat{c})$. Then the inequality*

$$|S(f;\breve{a},\hat{c})| \leq \left[\left(\frac{\breve{a}-\hat{c}}{2}\right)^2 + \left(\hat{s} - \frac{\breve{a}+\hat{c}}{2}\right)^2\right]\frac{M}{\breve{a}-\hat{c}} \tag{1.1}$$

holds for all $\hat{s} \in [\breve{a}, \hat{c}]$. The constant $\frac{1}{4}$ is the best possible.

Then Cerone [2], Dragomir et al. [3] and Sarıkaya et al. [4] also worked on this inequality. A. Qayyum et al. [5–9] worked on generalization of Ostrowski's type inequalities. Different authors worked on the generalization of Ostrowski's type inequalities that are [10], [11] and [12]. Some latest work done by S. Fahad et al. [13]. Further works done by Iftikhar et al. [14], Mustafa et al. [15] and J. Amjad et al. [16].
Let the functional $S(f;\omega;\breve{a},\hat{c})$ via weighted version represent the deviation of $f(\hat{s})$ over $[\breve{a},\hat{c}]$ defined as:

$$S(f;\omega;\breve{a},\hat{c}) = f(\hat{s}) - M(f;\omega;\breve{a},\hat{c}), \tag{1.2}$$

where $f(\hat{s})$ is continuous function and $M(f;\omega;\breve{a},\hat{c})$ is weighted integral mean defined as:

$$M(f;\omega;\breve{a},\hat{c}) = \frac{1}{\hat{c}-\breve{a}}\int_{\breve{a}}^{\hat{c}} f(\hat{y})\omega(\hat{y})d\hat{y}. \tag{1.3}$$

We suppose a weight function $\omega : (\breve{a},\hat{c}) \to [0,\infty)$ is integrable on $[0,\infty)$ such that

$$\int_{\breve{a}}^{\hat{c}} \omega(\hat{y})d\hat{y} < \infty. \tag{1.4}$$

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We define m, m_1, m_2 and notations μ and σ as:

$$m(\breve{a},\hat{c}) = \int_{\breve{a}}^{\hat{c}} \omega(\hat{y})d\hat{y}, \quad m_1(\breve{a},\hat{c}) = \int_{\breve{a}}^{\hat{c}} \hat{y}\omega(\hat{y})d\hat{y}, \tag{1.5}$$

$$m_2(\breve{a},\hat{c}) = \int_{\breve{a}}^{\hat{c}} \hat{y}^2\omega(\hat{y})d\hat{y}, \quad \mu(\breve{a},\hat{c}) = \frac{m_1(\breve{a},\hat{c})}{m(\breve{a},\hat{c})}, \tag{1.6}$$

$$\sigma^2(\breve{a},\hat{c}) = \frac{m_2(\breve{a},\hat{c})}{m(\breve{a},\hat{c})} - \mu^2(\breve{a},\hat{c}) \tag{1.7}$$

2. Main Result

Lemma 2.1. *Let $f : [\breve{a},\hat{c}] \to R$ be continuous on $[\breve{a},\hat{c}]$ and twice differentiable mapping on $(\breve{a},\hat{c})$, then the following weighted Peano kernel, define $\Bbbk(.,.) : [\breve{a},\hat{c}]^2 \to \mathbb{R}$ as:*

$$\Bbbk(\hat{s},\hat{y}) = \begin{cases} \frac{\Phi}{\Phi+\Psi}\frac{1}{\hat{s}-\breve{a}}\int_{\breve{a}}^{\hat{y}}(\hat{y}-\breve{u})\,\omega(\breve{u})\,d\breve{u}, & if \quad \breve{a} \le \hat{y} \le \hat{s} \\ \\ \frac{\Psi}{\Phi+\Psi}\frac{1}{\hat{c}-\hat{s}}\int_{\hat{c}}^{\hat{y}}(\hat{y}-\breve{u})\,\omega(\breve{u})\,d\breve{u}, & if \quad \hat{s} < \hat{y} \le \hat{c}, \end{cases} \tag{2.1}$$

where $\Phi,\Psi \in \mathbb{R}$ are non negative and both are non zero at the same time, $\forall\, \hat{y} \in [\breve{a},\hat{c}]$, $\hat{s} \in [\breve{a},\hat{c}]$ and ω is weight function as stated in (1.4). Before we state and prove our main result, we will prove the following identity by using integration by parts techniques, moments and notations. Then the following weighted integral identity

$$\tau(\omega;\hat{s};\Phi,\Psi) = \int_{\breve{a}}^{\hat{c}} \Bbbk(\hat{s},\hat{y})f''(\hat{y})d\hat{y} \tag{2.2}$$

$$= \breve{a}_1 f(\hat{s}) + \frac{1}{\Phi+\Psi} \times \left[\left(\frac{\Phi m(\breve{a},\hat{s})}{\hat{s}-\breve{a}}[\hat{s}-\mu(\breve{a},\hat{s})] + \frac{\Psi m(\hat{s},\hat{c})}{\hat{c}-\hat{s}}[\hat{s}-\mu(\hat{s},\hat{c})]\right)f'(\hat{s}) + (\{\Phi M(f;\omega,\breve{a},\hat{s}) + \Psi M(f;\omega,\hat{s},\hat{c})\})\right] \tag{2.3}$$

holds, here

$$\breve{a}_1 = \frac{-1}{\Phi+\Psi}\left(\frac{\Phi m(\breve{a},\hat{s})}{(\hat{s}-\breve{a})} + \frac{\Psi m(\hat{c},\hat{s})}{(\hat{c}-\hat{s})}\right),$$

and $M(f;\omega,\breve{a},\hat{c})$ is weighted integral mean as defined in (1.3).

Proof. From (2.1), we have

$$\int_{\breve{a}}^{\hat{c}} \Bbbk(\hat{s},\hat{y})f''(\hat{y})d\hat{y} = \frac{\Phi}{(\Phi+\Psi)(\hat{s}-\breve{a})}\int_{\breve{a}}^{\hat{s}}\int_{\breve{a}}^{\hat{y}}(\hat{y}-\breve{u})\omega(\breve{u})d\breve{u}f''(\hat{y})d\hat{y} + \frac{\Psi}{(\Phi+\Psi)(\hat{c}-\hat{s})}\int_{\hat{s}}^{\hat{c}}\int_{\hat{c}}^{\hat{y}}(\hat{y}-\breve{u})\omega(\breve{u})d\breve{u}f''(\hat{y})d\hat{y}.$$

After some calculations, we get

$$\int_{\breve{a}}^{\hat{c}} \Bbbk(\hat{s},\hat{y})f''(\hat{y})d\hat{y} = \frac{1}{\Phi+\Psi}\left[-\left(\frac{\Phi m(\breve{a},\hat{s})}{(\hat{s}-\breve{a})} + \frac{\Psi m(\hat{s},\hat{c})}{(\hat{c}-\hat{s})}\right)f(\hat{s}) + \left(\frac{\Phi m(\breve{a},\hat{s})}{\hat{s}-\breve{a}}[\hat{s}-\mu(\breve{a},\hat{s})] + \frac{\Psi m(\hat{s},\hat{c})}{\hat{c}-\hat{s}}[\hat{s}-\mu(\hat{s},\hat{c})]\right)f'(\hat{s}) + \frac{\Phi}{\hat{s}-\breve{a}}\int_{\breve{a}}^{\hat{s}}\omega(\hat{y})f(\hat{y})d\hat{y} + \frac{\Psi}{(\hat{c}-\hat{s})}\int_{\hat{s}}^{\hat{c}}\omega(\hat{y})f(\hat{y})d\hat{y}\right],$$

here the integration by parts formula has been utilised on the separate interval $[\breve{a},\hat{s}]$ and $(\hat{s},\hat{c}]$.
Simlification of the expressions readily produces the identity as stated in (2.2), $\forall\, \hat{s} \in [\breve{a},\hat{c}]$. □

Theorem 2.2. *Let $f : [\breve{a},\hat{c}] \to R$ be continuous on $[\breve{a},\hat{c}]$ and twice differentiable mapping on $(\breve{a},\hat{c})$, whose second derivative $f'' : [\breve{a},\hat{c}]^2 \to R$ is bounded on $(\breve{a},\hat{c})$, then following weighted integral inequalities*

$$|\tau(\omega;\hat{s};\Phi,\Psi)| \le \begin{cases} \frac{\|f''\|_\infty}{2(\Phi+\Psi)}\left[\frac{\Phi m(\breve{a},\hat{s})}{\hat{s}-\breve{a}}\left([\hat{s}-\mu(\breve{a},\hat{s})]^2 + \sigma^2(\breve{a},\hat{s})\right) + \frac{\Psi m(\hat{s},\hat{c})}{\hat{c}-\hat{s}}\left([\hat{s}-\mu(\hat{s},\hat{c})]^2 + \sigma^2(\hat{s},\hat{c})\right)\right] & for\ f'' \in L_\infty[\breve{a},\hat{c}] \\ \\ \frac{\omega(\hat{s})\|f''\|_p}{2(2\hat{q}+1)^{\frac{1}{\hat{q}}}(\Phi+\Psi)}\left[\Phi^{\hat{q}}(\hat{s}-\breve{a})^{\hat{q}+1} + \Psi^{\hat{q}}(\hat{c}-\hat{s})^{\hat{q}+1}\right]^{\frac{1}{\hat{q}}} & for\ f'' \in L_p[\breve{a},\hat{c}] \\ \\ \frac{\gamma\|f''\|_1}{2(\Phi+\Psi)}\left[1+\frac{|\varpi|}{\gamma}\right] & for\ f'' \in L_1[\breve{a},\hat{c}], \end{cases}$$

hold for $\forall\, \hat{y} \in [\breve{a}, \hat{c}]$, $\hat{s} \in [\breve{a}, \hat{c}]$ *and* ω *is weight function as stated in* (1.4)*, and* $\Phi, \Psi \in \mathbb{R}$ *are non negative and both are non zero at the same time. Here* $\frac{1}{p} + \frac{1}{\hat{q}} = 1,\ (p > 1)$,

$$\gamma = \frac{1}{(\hat{s} - \breve{a})(\hat{c} - \hat{s})} \left[\Phi m(\breve{a}, \hat{s})(\hat{c} - \hat{s})[\hat{s} - \mu(\breve{a}, \hat{s})] + \Psi m(\hat{s}, \hat{c})(\hat{s} - \breve{a})[\hat{s} - \mu(\hat{s}, \hat{c})]\right]$$

and

$$\varpi = \frac{1}{(\hat{s} - \breve{a})(\hat{c} - \hat{s})} \left[\Phi m(\breve{a}, \hat{s})(\hat{c} - \hat{s})[\hat{s} - \mu(\breve{a}, \hat{s})] - \Psi m(\hat{s}, \hat{c})(\hat{s} - \breve{a})[\hat{s} - \mu(\hat{s}, \hat{c})]\right].$$

Proof. Take the modulus of (2.2)

$$|\tau(\omega; \hat{s}; \Phi, \Psi)| = \left| \int_{\breve{a}}^{\hat{c}} \Bbbk(\hat{s}, \hat{y}) f''(\hat{y}) d\hat{y} \right| \leq \int_{\breve{a}}^{\hat{c}} |\Bbbk(\hat{s}, \hat{y})| \left| f''(\hat{y}) \right| d\hat{y}, \tag{2.4}$$

here we use properties of the integral and modulus

$$|\tau(\varpi; \ddot{z}; \varepsilon, \delta)| \leq \left\| f'' \right\|_{\infty} \int_{\breve{a}}^{\hat{c}} |\Bbbk(\hat{s}, \hat{y})| \, d\hat{y}.$$

By using (2.1) we prove

$$\int_{\breve{a}}^{\hat{c}} |\Bbbk(\hat{s}, \hat{y})| \, d\hat{y} = \int_{\breve{a}}^{\hat{s}} \int_{\breve{a}}^{\hat{y}} (\hat{y} - \breve{u}) \omega(\breve{u}) d\breve{u} d\hat{y} + \int_{\hat{s}}^{\hat{c}} \int_{\hat{c}}^{\hat{y}} (\hat{y} - \breve{u}) \omega(\breve{u}) d\breve{u} d\hat{y}$$

by using the techniques of J. Roummeliotis et al. [17],

$$\int_{\breve{a}}^{\hat{s}} \int_{\breve{a}}^{\hat{y}} (\hat{y} - \breve{u}) \omega(\breve{u}) d\breve{u} d\hat{y} + \int_{\hat{s}}^{\hat{c}} \int_{\hat{c}}^{\hat{y}} (\hat{y} - \breve{u}) \omega(\breve{u}) d\breve{u} d\hat{y} = \frac{1}{2} \int_{\breve{a}}^{\hat{c}} (\hat{s} - \hat{y})^2 \omega(\hat{y}) d\hat{y}$$

after some calculation we get

$$\int_{\breve{a}}^{\hat{c}} |\Bbbk(\hat{s}, \hat{y})| \, d\hat{y} = \frac{\Phi m(\breve{a}, \hat{s})}{2(\Phi + \Psi)(\hat{s} - \breve{a})} \left[[\hat{s} + \mu(\breve{a}, \hat{s})]^2 + \sigma^2(\breve{a}, \hat{s})\right] + \frac{\Psi m(\hat{s}, \hat{c})}{2(\Phi + \Psi)(\hat{c} - \hat{s})} \left[[\hat{s} + \mu(\hat{s}, \hat{c})]^2 + \sigma^2(\hat{s}, \hat{c})\right].$$

From above, first inequality is obtained.
Further, using Holder's Inequality, we have for $f'' \in L_p[\breve{a}, \hat{c}]$, from (2.4)

$$|\tau(\omega; \hat{s}; \Phi, \Psi)| \leq \left\| f'' \right\|_p \left(\int_{\breve{a}}^{\hat{c}} |\Bbbk(\hat{s}, \hat{y})|^{\hat{q}} d\hat{y} \right)^{\frac{1}{\hat{q}}},$$

here $\frac{1}{p} + \frac{1}{\hat{q}} = 1,\ (p > 1)$,
by using Mean Value Theorem, we get

$$\left(\int_{\breve{a}}^{\hat{c}} |\Bbbk(\hat{s}, \hat{y})|^{\hat{q}} d\hat{y} \right)^{\frac{1}{\hat{q}}} = \frac{1}{2(\Phi + \Psi)(2\hat{q} + 1)^{\frac{1}{\hat{q}}}} \left(\Phi^{\hat{q}} (\hat{s} - \hat{y})^{\hat{q}+1} \omega(\hat{s}) \right)^{\frac{1}{\hat{q}}} + \frac{1}{2(\Phi + \Psi)(2\hat{q} + 1)^{\frac{1}{\hat{q}}}} \left(\Psi^{\hat{q}} (\hat{s} - \hat{y})^{\hat{q}+1} \omega(\hat{s}) \right)^{\frac{1}{\hat{q}}},$$

so the second inequality is obtained.
Finally, for $f'' \in L_1[\breve{a}, \hat{c}]$ we have from (2.4)

$$|\tau(\omega; \hat{s}; \Phi, \Psi)| \leq \sup_{\hat{y} \in [\breve{a}, \hat{c}]} |\Bbbk(\hat{s}, \hat{y})| \left\| f'' \right\|_1.$$

By using (2.1), we prove

$$\begin{aligned}
\sup_{\hat{y} \in [\breve{a}, \hat{c}]} |\Bbbk(\hat{s}, \hat{y})| &= \frac{1}{(\Phi + \Psi)} \max \left\{ \frac{\Phi m(\breve{a}, \hat{s})}{\hat{s} - \breve{a}} [\hat{s} - \mu(\breve{a}, \hat{s})], \frac{\Psi m(\hat{s}, \hat{c})}{\hat{c} - \hat{s}} [\hat{s} - \mu(\hat{s}, \hat{c})] \right\} \\
&= \frac{1}{2(\Phi + \Psi)(\hat{s} - \breve{a})(\hat{c} - \hat{s})} \left(\Phi m(\breve{a}, \hat{s})(\hat{c} - \hat{s})[\hat{s} - \mu(\breve{a}, \hat{s})] \right. \\
&\quad \left. + \Psi m(\hat{s}, \hat{c})(\hat{s} - \breve{a})[\hat{s} - \mu(\hat{s}, \hat{c})] \right) \left[1 + \frac{\left| \frac{1}{(\hat{s} - \breve{a})(\hat{c} - \hat{s})} [\Phi m(\breve{a}, \hat{s})(\hat{c} - \hat{s})[\hat{s} - \mu(\breve{a}, \hat{s})] - \Psi m(\hat{s}, \hat{c})(\hat{s} - \breve{a})[\hat{s} - \mu(\hat{s}, \hat{c})]] \right|}{\frac{1}{(\hat{s} - \breve{a})(\hat{c} - \hat{s})} [\Phi m(\breve{a}, \hat{s})(\hat{c} - \hat{s})[\hat{s} - \mu(\breve{a}, \hat{s})] + \Psi m(\hat{s}, \hat{c})(\hat{s} - \breve{a})[\hat{s} - \mu(\hat{s}, \hat{c})]]} \right].
\end{aligned}$$

Hence (2.2) is proved. □

Remark 2.3. *From* (2.2) *and* (1.2)

$$(\Phi+\Psi)\,\tau(\omega;\hat{s};\Phi,\Psi)=\Phi S(f;\varpi;\breve{a},\hat{s})+\Psi S(f;\varpi;\hat{s},\hat{c})$$

and using triangular inequality in (2.2), *we get*

$$|(\varepsilon+\delta)\,\tau(\varpi;\ddot{z};\varepsilon,\delta)|\leq\begin{cases}\frac{\Phi m(\breve{a},\hat{s})\|f''\|_{\infty,[\breve{a},\hat{s}]}}{2(\hat{s}-\breve{a})}\left([\hat{s}-\mu(\breve{a},\hat{s})]^2+\sigma^2(\breve{a},\hat{s})\right)\\ +\frac{\Psi m(\hat{s},\hat{c})\|f''\|_{\infty,[\hat{s},\hat{c}]}}{2(\hat{c}-\hat{s})}\left([\hat{s}-\mu(\hat{s},\hat{c})]^2+\sigma^2(\hat{s},\hat{c})\right) & \text{for } f''\in L_\infty[\breve{a},\hat{c}]\\ \\ \frac{\omega(\hat{s})\left\|f''\right\|_{p,[\breve{a},\hat{s}]}}{2}\left(\frac{\Phi^{\hat{q}}(\hat{s}-\breve{a})^{\hat{q}+1}}{2\hat{q}+1}\right)^{\frac{1}{\hat{q}}}\\ +\frac{\omega(\hat{s})\left\|f''\right\|_{p,[\hat{s},\hat{c}]}}{2}\left(\frac{\Psi^{\hat{q}}(\hat{c}-\hat{s})^{\hat{q}+1}}{2\hat{q}+1}\right)^{\frac{1}{\hat{q}}} & \text{for } f''\in L_p[\breve{a},\hat{c}]\\ \\ \frac{\gamma}{2}\left\|f''\right\|_{1,[\breve{a},\hat{s}]}+\frac{|\varpi|}{2}\left\|f''\right\|_{1,[\hat{s},\hat{c}]} & \text{for } f''\in L_1[\breve{a},\hat{c}].\end{cases}\tag{2.5}$$

Remark 2.4. *Since we may write* (2.2) *as*

$$\begin{aligned}\Phi M(f;\omega;\breve{a},\hat{s})+\Psi M(f;\omega;\hat{s},\hat{c})&=\Phi M(f;\omega;\breve{a},\hat{s})+\frac{\Psi}{\hat{c}-\hat{s}}\left(\int_{\breve{a}}^{\hat{c}}\omega(\breve{u})f(\breve{u})d\breve{u}-\int_{\breve{a}}^{\hat{s}}\omega(\breve{u})f(\breve{u})d\breve{u}\right)\\&=\left[\Phi+\Psi\left(\frac{\hat{s}-\breve{a}}{\hat{c}-\hat{s}}\right)\right]M(f;\omega;\breve{a},\hat{s})+\Psi\left(\frac{\hat{c}-\breve{a}}{\hat{c}-\hat{s}}\right)M(f;\omega;\breve{a},\hat{c}).\end{aligned}$$

Thus, the identity

$$\begin{aligned}&\tau(\omega;\hat{s};\Phi,\Psi)=\\&\breve{a}_1 f(\hat{s})+\frac{1}{\Phi+\Psi}\left[\left(\frac{\Phi m(\breve{a},\hat{s})}{\hat{s}-\breve{a}}[\hat{s}-\mu(\breve{a},\hat{s})]^2+\frac{\Psi m(\hat{s},\hat{c})}{\hat{c}-\hat{s}}[\hat{s}-\mu(\hat{s},\hat{c})]^2\right)f'(\hat{s})+\left(1-\frac{\Psi\lambda}{\Phi+\Psi}\right)M(f;\omega;\breve{a},\hat{s})+\frac{\Psi\lambda}{\Phi+\Psi}M(f;\omega;\breve{a},\hat{c})\right],\end{aligned}$$

same as $[\breve{a},\hat{c}]$ and $M(f;\omega;\breve{a},\hat{c})$ is also fixed.

Corollary 2.5. *Let the conditions of Theorem 2 hold. Then the results for* $\Phi=\Psi$

$$|\tau(\omega;\hat{s};\Phi,\Psi)|\leq\begin{cases}\frac{\left\|f''\right\|_\infty}{4}\left[\frac{m(\breve{a},\hat{s})}{\hat{s}-\breve{a}}\left([\hat{s}-\mu(\breve{a},\hat{s})]^2f'(\hat{s})\;-f(\hat{s})+\sigma^2(\breve{a},\hat{s})\right)\right.\\ \left.+\frac{m(\hat{s},\hat{c})}{\hat{c}-\hat{s}}\left([\hat{s}-\mu(\hat{s},\hat{c})]^2f'(\hat{s})\;-f(\hat{s})+\sigma^2(\hat{s},\hat{c})\right)\right] & \text{for } f''\in L_\infty[\breve{a},\hat{c}]\\ \\ \frac{\omega(\hat{s})\left\|f''\right\|_p}{4(2\hat{q}+1)^{\frac{1}{\hat{q}}}}\left[(\hat{s}-\breve{a})^{\hat{q}+1}+(\hat{c}-\hat{s})^{\hat{q}+1}\right]^{\frac{1}{\hat{q}}} & \text{for } f''\in L_p[\breve{a},\hat{c}]\\ \\ \frac{\zeta\left\|f'\right\|_1}{4}\left[1+\frac{|\eta|}{\zeta}\right] & \text{for } f''\in L_1[\breve{a},\hat{c}],\end{cases}\tag{2.6}$$

here

$$\begin{aligned}\tau(\omega;\hat{s};\Phi,\Phi)=&\frac{-1}{2}\left(\frac{m(\breve{a},\hat{s})}{(\hat{s}-\breve{a})}+\frac{m(\hat{c},\hat{s})}{(\hat{c}-\hat{s})}\right)f(\hat{s})+\frac{1}{2}\left[\left(\frac{m(\breve{a},\hat{s})}{\hat{s}-\breve{a}}[\hat{s}-\mu(\breve{a},\hat{s})]^2+\frac{m(\hat{s},\hat{c})}{\hat{c}-\hat{s}}[\hat{s}-\mu(\hat{s},\hat{c})]^2\right)f'(\hat{s})\right.\\&\left.+(\{M(f;\omega,\breve{a},\hat{s})+M(f;\omega,\hat{s},\hat{c})\})\right],\end{aligned}$$

$$\zeta=\frac{1}{(\hat{s}-\breve{a})(\hat{c}-\hat{s})}\left[m(\breve{a},\hat{s})(\hat{c}-\hat{s})[\hat{s}-\mu(\breve{a},\hat{s})]\;+m(\hat{s},\hat{c})(\hat{s}-\breve{a})[\hat{s}-\mu(\hat{s},\hat{c})]\right]$$

and

$$\eta=\frac{1}{(\hat{s}-\breve{a})(\hat{c}-\hat{s})}\left[m(\breve{a},\hat{s})(\hat{c}-\hat{s})[\hat{s}-\mu(\breve{a},\hat{s})]\;-\;m(\hat{s},\hat{c})(\hat{s}-\breve{a})[\hat{s}-\mu(\hat{s},\hat{c})]\right].$$

Proof. The result is readily obtained on allowing $\Phi=\Psi$ in (2.2) so that the left hand side is $\tau(\omega;\hat{s};\Phi,\Phi)$ from (2.2). □

Corollary 2.6. *According to Theorem 2, then mid point* $\left(\hat{s}=\breve{A}=\frac{\breve{a}+\hat{c}}{2}\right)$, *inequality from* (2.2)

$$\left|\tau\left(\omega;\breve{A};\Phi,\Psi\right)\right|\leq\begin{cases}\frac{\left\|f''\right\|_\infty}{2(\Phi+\Psi)}\left[\frac{\Phi m(\breve{a},\breve{A})}{\breve{A}-\breve{a}}\left(\left[\breve{A}-\mu(\breve{a},\breve{A})\right]^2+\sigma^2(\breve{a},\breve{A})\right)\right.\\ \left.+\frac{\Psi m(\breve{A},\hat{c})}{\hat{c}-\breve{A}}\left(\left[\breve{A}-\mu(\breve{A},\hat{c})\right]^2+\sigma^2(\breve{A},\hat{c})\right)\right] & \text{for } f''\in L_\infty[\breve{a},\hat{c}]\\ \\ \frac{\omega(\breve{A})\left\|f''\right\|_p}{2(2\hat{q}+1)^{\frac{1}{\hat{q}}}(\Phi+\Psi)}\left[\Phi^{\hat{q}}\left(\breve{A}-\breve{a}\right)^{\hat{q}+1}+\Psi^{\hat{q}}\left(\hat{c}-\breve{A}\right)^{\hat{q}+1}\right]^{\frac{1}{\hat{q}}} & \text{for } f''\in L_p[\breve{a},\hat{c}]\\ \\ \frac{\psi\left\|f''\right\|_1}{2(\Phi+\Psi)}\left[1+\frac{|\varkappa|}{\psi}\right] & \text{for } f''\in L_1[\breve{a},\hat{c}],\end{cases}\tag{2.7}$$

here

$$\psi=\frac{1}{2(\Phi+\Psi)\left(\breve{A}-\breve{a}\right)(\hat{c}-\breve{a})}\left[\Phi m\left(\breve{a},\breve{A}\right)(\hat{c}-\breve{a})\ \left[\breve{A}-\mu(\breve{a},\breve{A})\right]\ +\Psi m\left(\breve{A},\hat{c}\right)(\hat{c}-\breve{a})\left[\breve{A}-\mu(\breve{A},\hat{c})\right]\right]$$

and

$$\varkappa=\frac{1}{\left(\breve{A}-\breve{a}\right)(\hat{c}-\breve{a})}\left[\left|\Phi m\left(\breve{a},\breve{A}\right)(\hat{c}-\breve{a})\ \left[\breve{A}-\mu(\breve{a},\breve{A})\right]\ -\Psi m\left(\breve{A},\hat{c}\right)(\hat{c}-\breve{a})\left[\breve{A}-\mu(\breve{A},\hat{c})\right]\right|\right].$$

Proof. Placing $\left(\hat{s}=\breve{A}=\frac{\breve{a}+\hat{c}}{2}\right)$ in (2.2) and (2.2) produces the results as stated in (2.7). □

Corollary 2.7. *When the conditions of Theorem 2 hold and (2.7) is evaluated at $\Phi=\Psi$, then we get*

$$\left|\tau\left(\omega;\breve{A};\Phi,\Phi\right)\right|\leq\begin{cases}\frac{\left\|f''\right\|_{\infty}}{4}\left[\frac{m\left(\breve{a},\breve{A}\right)}{\breve{A}-\breve{a}}\left(\left[\breve{A}-\mu(\breve{a},\breve{A})\right]^{2}+\sigma^{2}\left(\breve{a},\breve{A}\right)\right)+\frac{m\left(\breve{A},\hat{c}\right)}{\hat{c}-\breve{A}}\left(\left[\breve{A}-\mu(\breve{A},\hat{c})\right]^{2}+\sigma^{2}\left(\breve{A},\hat{c}\right)\right)\right] & \text{for } f''\in L_{\infty}[\breve{a},\hat{c}]\\ \frac{\omega(\breve{A})\left\|f''\right\|_{p}}{4(2\hat{q}+1)^{\frac{1}{\hat{q}}}}\left[\left(\breve{A}-\breve{a}\right)^{\hat{q}+1}+\left(\hat{c}-\breve{A}\right)^{\hat{q}+1}\right]^{\frac{1}{\hat{q}}} & \text{for } f''\in L_{p}[\breve{a},\hat{c}]\\ \frac{\psi\left\|f''\right\|_{1}}{4}\left[1+\frac{|\varkappa|}{\psi}\right] & \text{for } f''\in L_{1}[\breve{a},\hat{c}],\end{cases}\tag{2.8}$$

here

$$\psi=\frac{1}{\left(\breve{A}-\breve{a}\right)(\hat{c}-\breve{a})}\left[m\left(\breve{a},\breve{A}\right)(\hat{c}-\breve{a})\ \left[\breve{A}-\mu(\breve{a},\breve{A})\right]+m\left(\breve{A},\hat{c}\right)(\hat{c}-\breve{a})\left[\breve{A}-\mu(\breve{A},\hat{c})\right]\right]$$

and

$$\varkappa=\frac{1}{\left(\breve{A}-\breve{a}\right)(\hat{c}-\breve{a})}\left[m\left(\breve{a},\breve{A}\right)(\hat{c}-\breve{a})\ \left[\breve{A}-\mu(\breve{a},\breve{A})\right]-m\left(\breve{A},\hat{c}\right)(\hat{c}-\breve{a})\left[\breve{A}-\mu(\breve{A},\hat{c})\right]\right].$$

Proof. Putting $\Phi=\Psi$; in (2.7) we get (2.8). □

Remark 2.8. *For $\varpi(\hat{s})=1$ in (2.2), (2.5), (2.6), (2.7), and in (2.8) we get A. Qayyum et al.'s result [7].*

2.1. Applications for some special means:

Now we discuss applications for some special means by taking different weight.

Remark 2.9. *For Uniform (legendre) mean:*

Let $\varpi(\hat{s})=1$ put in (2.2) and in (2.2), we get A. Qayyum et al.'s results [7].

Remark 2.10. *For Logarithm mean:*

Let

$$\omega(\hat{y})=\ln(1/\hat{y});\quad \breve{a}=0,\ \hat{c}=1,$$

put in (1.7), we get

$$\mu(0,1)=\frac{\int_{0}^{1}\hat{y}\ln(1/\hat{y})\,d\hat{y}}{\int_{0}^{1}\ln(1/\hat{y})\,d\hat{y}}=\frac{1}{4}$$

and

$$\sigma^{2}(0,1)=\frac{\int_{0}^{1}\hat{y}^{2}\ln(1/\hat{y})\,d\hat{y}}{\int_{0}^{1}\ln(1/\hat{y})\,d\hat{y}}-(\mu(0,1))^{2}=\frac{7}{144},$$

put in (2.2), then the inequalities are

$$\left|\frac{1}{\Phi+\Psi}\left(\frac{\Phi}{\hat{s}-\breve{a}}+\frac{\Psi}{\hat{c}-\hat{s}}\right)\left(\int_0^1 \ln\left(1/\hat{y}\right) f\left(\hat{y}\right) d\hat{y}-f\left(\hat{s}\right)+\left(\hat{s}-\frac{1}{4}\right) f^{'}\left(\hat{s}\right)\right)\right|$$

$$\leq \begin{cases} \frac{\ln(1/\hat{y})\left\|f''\right\|_\infty}{2(\Phi+\Psi)}\left(\frac{\Phi}{\hat{s}-\breve{a}}+\frac{\Psi}{\hat{c}-\hat{s}}\right)\left[\left(\hat{s}-\frac{1}{4}\right)^2+\frac{7}{144}\right] & for\ f'' \in L_\infty\left[\breve{a},\hat{c}\right] \\ \frac{\ln(1/\hat{y})\left\|f''\right\|_p}{2(2\hat{q}+1)^{\frac{1}{\hat{q}}}(\Phi+\Psi)}\left[\Phi^{\hat{q}}\left(\hat{s}-\breve{a}\right)^{\hat{q}+1}+\Psi^{\hat{q}}\left(\hat{c}-\hat{s}\right)^{\hat{q}+1}\right]^{\frac{1}{\hat{q}}} & for\ f'' \in L_p\left[\breve{a},\hat{c}\right] \\ \frac{\left\|f''\right\|_1}{2(\Phi+\Psi)}\left[\Phi\ln\left(1/\hat{y}\right)\left(\hat{s}-\frac{1}{4}\right)+\Psi\ln\left(1/\hat{y}\right)\left(\hat{s}-\frac{1}{4}\right)\ \left|\Phi\ln\left(1/\hat{y}\right)\left(\hat{s}-\frac{1}{4}\right)-\Psi\ln\left(1/\hat{y}\right)\left(\hat{s}-\frac{1}{4}\right)\right|\right] & for\ f'' \in L_1\left[\breve{a},\hat{c}\right]. \end{cases}$$

The mid point reflecting if the optimum point $\hat{s}=\mu\left(0,1\right)=\frac{1}{4}$ is near to the origin.

Remark 2.11. *For Jacobi mean:*

Let

$$\omega\left(\hat{y}\right)=1/\sqrt{\hat{y}};\quad \breve{a}=0,\quad \hat{c}=1,$$

we have

$$\mu\left(0,1\right)=\frac{\int_0^1 \sqrt{\hat{y}}d\hat{y}}{\int_0^1 1/\sqrt{\hat{y}}d\hat{y}}=\frac{1}{3}$$

and

$$\sigma^2\left(0,1\right)=\frac{\int_0^1 \hat{y}\sqrt{\hat{y}}d\hat{y}}{\int_0^1 1/\sqrt{\hat{y}}d\hat{y}}-\left(\frac{1}{3}\right)^2=\frac{4}{45}.$$

Then

$$\left|\frac{1}{\Phi+\Psi}\left(\frac{\Phi}{\hat{s}-\breve{a}}+\frac{\Psi}{\hat{c}-\hat{s}}\right)\left(\int_0^1 \frac{f\left(\hat{y}\right)}{\sqrt{\hat{y}}}d\hat{y}-f\left(\hat{s}\right)+\left(\hat{s}-1\right)f'\left(\hat{s}\right)\right)\right|$$

$$\leq \begin{cases} \frac{1/\sqrt{\hat{y}}\left\|f''\right\|_\infty}{2(\Phi+\Psi)}\left(\frac{\Phi}{\hat{s}-\breve{a}}+\frac{\Psi}{\hat{c}-\hat{s}}\right)\left[1+\left(\hat{s}-1\right)^2\right] & for\ f'' \in L_\infty\left[\breve{a},\hat{c}\right] \\ \frac{1/\sqrt{\hat{y}}\left\|f''\right\|_p}{2(2\hat{q}+1)^{\frac{1}{\hat{q}}}(\Phi+\Psi)}\left[\Phi^{\hat{q}}\left(\hat{s}-\breve{a}\right)^{\hat{q}+1}+\Psi^{\hat{q}}\left(\hat{c}-\hat{s}\right)^{\hat{q}+1}\right]^{\frac{1}{\hat{q}}} & for\ f'' \in L_p\left[\breve{a},\hat{c}\right] \\ \frac{\left\|f''\right\|_1}{2(\Phi+\Psi)}\left[\Phi/\sqrt{\hat{y}}\left(\hat{s}-\frac{1}{3}\right)+\Psi/\sqrt{\hat{y}}\left(\hat{s}-\frac{1}{3}\right)\ \left|\Phi/\sqrt{\hat{y}}\left(\hat{s}-\frac{1}{4}\right)-\Psi/\sqrt{\hat{y}}\left(\hat{s}-\frac{1}{4}\right)\right|\right] & for\ f'' \in L_1\left[\breve{a},\hat{c}\right]. \end{cases}$$

The optimum point $\hat{s}=\mu\left(0,1\right)=\frac{1}{3}$ is moved to the left of midpoint.

Remark 2.12. *For Chebyshev mean:*

Let

$$\omega\left(\hat{y}\right)=1/\sqrt{1-\hat{y}^2};\quad \breve{a}=-1,\ \hat{c}=1,$$

mean

$$\mu\left(-1,1\right)=0$$

and

$$\sigma^2\left(-1,1\right)=\frac{1}{2}.$$

Hence, Chebyshev weighted inequalities are

$$\left|\frac{1}{\Phi+\Psi}\left(\frac{\Phi}{\hat{s}-\breve{a}}+\frac{\Psi}{\hat{c}-\hat{s}}\right)\left(\frac{1}{\pi}\int_{-1}^{1}\frac{f(\hat{y})}{\sqrt{1-\hat{y}}}d\hat{y}-f(\hat{s})+\hat{s}f'(\hat{s})\right)\right|$$

$$\leq\begin{cases}\frac{1/\sqrt{1-\hat{y}^2}\left\|f''\right\|_\infty}{2(\Phi+\Psi)}\left(\frac{\Phi}{\hat{s}-\breve{a}}+\frac{\Psi}{\hat{c}-\hat{s}}\right)\left[\hat{s}^2+\frac{1}{2}\right] & \text{for } f''\in L_\infty[\breve{a},\hat{c}]\\ \frac{1/\sqrt{1-\hat{y}^2}\left\|f''\right\|_p}{2(2\hat{q}+1)^{\frac{1}{\hat{q}}}(\Phi+\Psi)}\left[\Phi^{\hat{q}}(\hat{s}-\breve{a})^{\hat{q}+1}+\Psi^{\hat{q}}(\hat{c}-\hat{s})^{\hat{q}+1}\right]^{\frac{1}{\hat{q}}} & \text{for } f''\in L_p[\breve{a},\hat{c}]\\ \frac{\left\|f''\right\|_1}{2(\Phi+\Psi)}\left[\hat{s}\Phi/\sqrt{1-\hat{y}^2}+\hat{s}\Psi/\sqrt{1-\hat{y}^2}\ \left|\Phi\hat{s}/\sqrt{1-\hat{y}^2}-\Psi\hat{s}/\sqrt{1-\hat{y}^2}\right|\right] & \text{for } f''\in L_1[\breve{a},\hat{c}].\end{cases}$$

The optimum point $\hat{s}=\mu(-1,1)=0$ is at the midpoint of the interval.

Remark 2.13. *For Laguerre mean:*

Let

$$\omega(\hat{y})=e^{-\hat{y}};\quad \breve{a}=0,\quad \hat{c}=\infty,$$

such that

$$\mu(0,\infty)=1$$

and

$$\sigma^2(0,\infty)=1$$

then inequalities are

$$\left|\frac{1}{\Phi+\Psi}\left(\frac{\Phi}{\hat{s}-\breve{a}}+\frac{\Psi}{\hat{c}-\hat{s}}\right)\left(\int_{0}^{\infty}e^{-\hat{y}}f(\hat{y})d\hat{y}-f(\hat{s})+(\hat{s}-1)f'(\hat{s})\right)\right|$$

$$\leq\begin{cases}\frac{e^{-\hat{y}}\left\|f''\right\|_\infty}{2(\Phi+\Psi)}\left(\frac{\Phi}{\hat{s}-\breve{a}}+\frac{\Psi}{\hat{c}-\hat{s}}\right)\left[1+(\hat{s}-1)^2\right] & \text{for } f''\in L_\infty[\breve{a},\hat{c}]\\ \frac{e^{-\hat{y}}\left\|f''\right\|_p}{2(2\hat{q}+1)^{\frac{1}{\hat{q}}}(\Phi+\Psi)}\left[\Phi^{\hat{q}}(\hat{s}-\breve{a})^{\hat{q}+1}+\Psi^{\hat{q}}(\hat{c}-\hat{s})^{\hat{q}+1}\right]^{\frac{1}{\hat{q}}} & \text{for } f''\in L_p[\breve{a},\hat{c}]\\ \frac{\left\|f''\right\|_1}{2(\Phi+\Psi)}\left[\Phi e^{-\hat{y}}(\hat{s}-1)+\Psi e^{-\hat{y}}(\hat{s}-1)\ \left|\Phi e^{-\hat{y}}(\hat{s}-1)-\Psi e^{-\hat{y}}(\hat{s}-1)\right|\right] & \text{for } f''\in L_1[\breve{a},\hat{c}].\end{cases}$$

The optimum sample point is deduced $\hat{s}=1$.

Remark 2.14. *For Hermite mean:*

Let

$$\omega(\hat{y})=e^{-\hat{y}^2};\quad \breve{a}=-\infty,\quad \hat{c}=\infty,$$

then

$$\mu(-\infty,\infty)=0$$

and

$$\sigma^2(-\infty,\infty)=\frac{1}{2}.$$

Then inequalities are

$$\left|\frac{1}{\Phi+\Psi}\left(\frac{\Phi}{\hat{s}-\breve{a}}+\frac{\Psi}{\hat{c}-\hat{s}}\right)\left(\frac{1}{\pi}\int_{-\infty}^{\infty}e^{-\hat{y}^2}f(\hat{y})d\hat{y}-f(\hat{s})+\hat{s}f'(\hat{s})\right)\right|$$

$$\leq\begin{cases}\frac{e^{-\hat{s}^2}\left\|f''\right\|_\infty}{2(\Phi+\Psi)}\left(\frac{\Phi}{\hat{s}-\breve{a}}+\frac{\Psi}{\hat{c}-\hat{s}}\right)\left[\frac{1}{2}+\hat{s}^2\right] & \text{for } f''\in L_\infty[\breve{a},\hat{c}]\\ \frac{e^{-\hat{y}^2}\left\|f''\right\|_p}{2(2\hat{q}+1)^{\frac{1}{\hat{q}}}(\Phi+\Psi)}\left[\Phi^{\hat{q}}(\hat{s}-\breve{a})^{\hat{q}+1}+\Psi^{\hat{q}}(\hat{c}-\hat{s})^{\hat{q}+1}\right]^{\frac{1}{\hat{q}}} & \text{for } f''\in L_p[\breve{a},\hat{c}]\\ \frac{\|f''\|_1}{2(\Phi+\Psi)}\left[\Phi\hat{s}e^{-\hat{y}^2}+\Psi\hat{s}e^{-\hat{y}^2}\ \left|\Phi\hat{s}e^{-\hat{y}^2}-\Psi\hat{s}e^{-\hat{y}^2}\right|\right] & \text{for } f''\in L_1[\breve{a},\hat{c}].\end{cases}$$

An optimum sampling point is $\hat{s}=0$.

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