AN APPLICATION OF GEOMETRIC PROGRAMMING

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Abstract: *A Geometric Program (GP) is a type of mathematical optimization problem characterized by objective and constraint functions that have a special form. The basic approach in GP modeling is to attempt to express a practical problem, such as an engineering analysis or design problem, in GP format. In this study, using Douglas production function, between the years 2006-2009 in the construction sector in Turkey estimates of labor and capital index values were obtained.*
Geometric Programming Algorithms for the calculation was carried out with the Matlab Toolbox software.

Keywords: Convex optimization, Geometric programming, Cobb-Douglas production function, Mathematical optimization.

1. INTRODUCTİON

In 1967, Duffin, Peterson, and Zener published the book *Geometric Programming: Theory and Applications* that started the field of Geometric Program (GP) as a branch of nonlinear optimization. Several important developments of GP took place in the late 1960s and 1970s. GP was tied with convex optimization and Lagrange duality, and was extended to include more general formulations beyond posynomials. Several numerical algorithms to solve GP were proposed and tested. And researchers in mechanical engineering, civil engineering, and chemical engineering found successful applications of GP to their problems.

However, as researchers felt that most of the theoretical, algorithmic and application aspects of GP had been exhausted by the early 1980s, the period of 1980−1998 was relatively quiet. Over the last few years, however, GP started to receive renewed attention from the operations research community, for example, in a special issue of *Annals of Operations Research* in 2000 and a special session in the INFORMS annual meeting in 2001. In particular, we now have very efficient algorithms to solve GP, either by general purpose convex optimization solvers, or by more specialized methods. Approximation methods for robust GP and distributed algorithms for GP have also appeared recently [2].

In the last four years, some studies with geometric programming as follows: Profit maximization [9], Profit maximization with quantity discount [10], Multi-objective marketing planning inventory model [6], Global optimization for signomial geometric programming [7], Optimizing the geometric programming problem with single-term exponents subject to max–min fuzzy relational equation constraints [13], Maximum likelihood estimation of ordered multinomial probabilities [3], Temperature-aware floorplanning [8], Fuzzy pricing and marketing planning model [14].

A GP is a type of mathematical optimization problem characterized by objective and constraint functions that have a special form. The importance of GPs comes from two relatively recent developments: New solution methods can solve even large-scale GPs extremely efficiently and reliably. A number of

practical problems, particularly in electrical circuit design, have recently been found to be equivalent to (or well approximated by) GPs. Putting these two together, we get effective solutions for the practical problems. Neither of these developments is widely known, at least not yet. Nor is the story over: Further improvements in GP solution methods will surely be developed, and, we believe, many more practical applications of GP will be discovered. Indeed, one of our principal aims is to broaden knowledge and awareness of GP among potential users, to help accelerate the hunt for new practical applications of GP.

Geometric programming is a mathematical technique for optimizing positive polynomials, which are called posynominials. This technique has many similarities to linear programming but has advantages in that:

- a non-linear objective function can be used;
- the constraints can be non-linear; and
- the optimal cost value can be determined with the dual without first determining the specific values of the primal variables [4].

2. METHOD

A function $f : R^n \to R$ with

$$\textbf{dom}\, f = R_{++}^n, \qquad \text{defined as}$$

$$f(x) = c\, x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}, \qquad (1)$$

where $c > 0$ and $a_i \in R$, is called a *monomial function*, or simply, a *monomial*. The exponents a_i of a monomial can be any real numbers, including fractional or negative, but the coefficient c an only be positive. A sum of monomials, *i.e.*, a function of the form

$$f(x) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \dots x_n^{a_{nk}}, \qquad (2)$$

where $c_k > 0$, is called a *posynomial function* (with K terms), or simply, a *posynomial*.

Posynomials are closed under addition, multiplication, and nonnegative scaling. Monomials are closed under multiplication and division. If a posynomial is multiplied by a monomial, the result is a posynomial; similarly, a posynomial can be divided by a

monomial, with the result a posynomial. An optimization problem of the form

$$\begin{aligned} minimize \quad & f_0(x) \\ subject\ to \quad & f_i(x) \le 1, \\ & i = 1, \dots, m \end{aligned} \qquad (2)$$

$$h_i(x) = 1, \qquad i = 1, \dots, p$$

where $f_0, \dots, f_m$ are posynomials and $h_1, \dots, h_p$ are monomials, is called a *geometric program* (GP). The domain of this problem is $D = R_{++}^n$; the constraint $x > 0$ is implicit [1].

3. APPLICATION

The firm uses n inputs (labor, coal, iron ore, etc.) to produce a single output, let x be the column vector of inputs,

$$x' = (x_1, x_2, \dots, x_n)',$$

and let q be the output. The firm's production function is represented by

$$q = f(x) = f(x_1, x_2, \dots, x_n), \qquad (3)$$

giving output as a function of its inputs. Equation (3) assumes nothing but the existence of a maximum output corresponding to any combination of inputs.

Let r be a row vector of (positive) given prices of the inputs,

$$r = (r_1, r_2, \dots, r_n),$$

and p the (positive) given price of the output.

A firm is in a competitive situation if it can buy and sell in any quantities at exogenously given prices, which are independent of its production decisions. In other words, the competitive firm is a price taker [11].

The firm behaves so as to maximize the profit π, given as the difference between revenue, pq, and cost, given as the total expenditure on all inputs,

$$rx = \sum_{j=1}^{n} r_j x_j.$$

The problem of the (competitive) firm can then be stated as the following mathematical programming problem:

$$maximize_x \ \pi(x) = pf(x) - rx,$$
$$subject \ to \ x \geq 0.$$

Another version of this problem often used in production theory is based on the assumption of the given output level q^*. The firm is trying to minimize its cost, M, of the inputs used to produce q^*. The expenditure of the firm is given by

$$M(x) = r_1 x_1 + r_2 x_2 + \cdots + r_n x_n$$

and the mathematical programming problem is then

$$minimize_x \ M(x)$$
$$subject \ to \ f(x) = q^* \ and \ x \geq 0. \quad (4)$$

Closely interrelated to this formulation is the problem of the firm with a given level of expenditure or prespecified budget, M^*, and an objective function that maximizes the production q. Thus the firm chooses levels of inputs so as to maximize output, subject to a budget constraint:

$$maximize_x \ f(x)$$
$$subject \ to \ r_1 x_1 + r_2 x_2 + \cdots + r_n x_n \leq$$
$$and \ x \geq 0. \quad (5)$$

For illustration of models (4) and (5), we consider the technology defined by the Cobb–Douglas production function (for simplicity, but without loss of generality, with just two inputs):

$$q = f(x_1, x_2) = ax_1^\alpha x_2^\beta$$
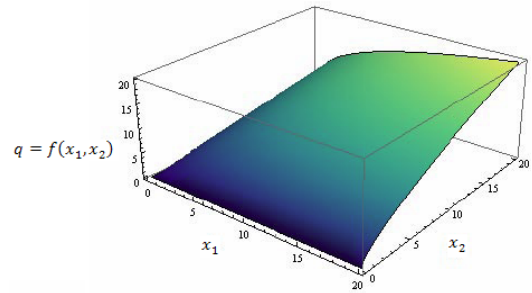
with $a > 0, 0 < \alpha < 1$, and $0 < \beta < 1$.

Figure 1 A two-input Cobb-Douglas production function [12].

In this case, problem (4) leads to the following mathematical programming model:

$$minimize_{x_1,x_2} \ r_1 x_1 + r_2 x_2$$
$$subject \ to \ ax_1^\alpha x_2^\beta \geq q^* \quad and$$
$$x_1 > 0, x_2 > 0. \quad (6)$$

The Minimization of Cost

It is to find, for a particular output level q^* and with a given structure of input prices, what input levels would constitute the cheapest way of producing this output and what would be the minimum cost. This question can be answered for all possible levels of output and the minimum cost would depend on the level of output to be produced [11].
Assuming that the technology is defined by the Cobb–Douglas production function (for simplicity, but without loss of generality, with two inputs only), the solution of model (6) will yield the cost function $C(r, q)$, expressing minimum cost as a function of input prices r and output level q. A slight modification of the constraint in model (6) leads to the following geometric programming problem:

$$minimize_{x_1,x_2} \ M(x) = r_1 x_1 + r_2 x_2$$
$$subject \ to \ \frac{q^*}{a} x_1^{-\alpha} x_2^{-\beta} \leq 1, \quad (7)$$
$$x_1 > 0, x_2 > 0.$$

The following notations about this model (7) are described as:

M: cost of inputs used to produce q^*,
x_1: first input (labor input),
x_2: second input (capital input),
r_1: first prices of the first input

r_2: second prices of the second input,

q^*: output level,

a: total factor productivity,

α and β: The output elasticities of labor and capital, respectively. These values are constants determined by available technology.

4. DATA

The data used in the study were taken from the construction sector in Turkey between the years 2006- 2009. They are included in Table 1 below.

Table 1 Construction sector in Turkey between 2006 and 2009 years. [16].

Year	Classification of Types of Constructions (CC)	Yearly Average		
		Production Index	Labour Total Index	Materials Total Index
2006	**Total: Building, Civil engineering**	**118,4**	**121,88**	**114,32**
2007		**124,9**	**137,80**	**122,32**
2008		**115,6**	**153,85**	**140,06**
2009		**96,4**	**158,53**	**131,48**

q^*: production index; r_1: labour total index; r_2: materials total index are in table 1.

α and β output elasticities coefficients are taken as:

$$(\alpha, \beta) = (0.53, 0.47).$$

a total factor productivity is taken as 1

because q^* output level coefficients explains the model sufficiently.

5.RESULT AND CONCLUSION

Geometric programming is not used very often in the areas related to the production function. The indexes in labor and capital in the construction sector in Turkey were estimated as follows:

Table 2 Estimated values of labor and capital index.

Year	Classification of Types of Constructions (CC)	Yearly Average			Estimates	
		Production Index	Labour Total Index	Materials Total Index	x_1	x_2
2006	**Total: Building, Civil engineering**	**118,4**	**121,88**	**114,32**	121,56	114,93
2007		**124,9**	**137,80**	**122,32**	124,96	124,84
2008		**115,6**	**153,85**	**140,06**	117,03	114
2009		**96,4**	**158,53**	**131,48**	93,414	99,882

Cobb-Douglas production function of the geometric programming on the calculated values to would shed light on the forthcoming studies in this field.

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