Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume XX (x) (2019), 1 – 15
DOI : 10.15672/HJMS.xx

Research Article

Some bounds for Casorati curvatures on Golden Riemannian space forms with SSM connection

Jae Won Lee[1], Majid Ali Choudhary[2], Aliya Naaz Siddiqui*[3]

[1] *Department of Mathematics Education and RINS, Gyeongsang National University, Jinju, Republic of Korea*

[2] *Department of Mathematics, School of Sciences, Maulana Azad National Urdu University, Hyderabad, India*

[3] *Division of Mathematics, School of Basic Sciences, Galgotias University, Greater Noida 203201, Uttar Pradesh, India*

Abstract

In this article, we derive some sharp inequalities for slant submanifolds immersed into golden Riemannian space forms with a semi-symmetric metric connection. Also, we characterize submanifolds for the case of equalities. Lastly, we discuss these inequalities for some special submanifolds.

Mathematics Subject Classification (2020). 53B05, 53B20, 53C25, 53C40

Keywords. Golden Riemannian manifolds, δ-Casorati curvature, Slant submanifolds

1. Introduction

The golden ratio has attracted the attention of many researchers of diverse interests for more than 2000 years. Interestingly, this attraction is not only limited to mathematicians, but people from other backgrounds such as biology, arts, music, history, architecture and even psychology which have pondered and debated this ratio.

Hretcanu and Crasmareanu [13] focused on various properties of the induced structure for an invariant submanifold immersed in a golden Riemannian manifold. They also worked that a structure of a golden type on every invariant submanifold inherits a golden structure from a total manifold [14]. In 2013, Gezer et al [9] worked on integrabilities for golden structures. Golden structures on semi-Riemannian manifolds have also been of high interest. Poyraz and Erol [29] obtained several results for lightlike hypersurfaces in a golden semi-Riemannian manifold. Ozkan [28] also studied the complete and horizontal lifts of the golden structure, that is, a polynomial structure with the structure polynomial $\mathcal{W}(X) = X^2 - X - I$, in the tangent bundle. Recently, Bahadr and Uddin [2] studied slant submanifolds of a Riemannian manifold endowed with a golden structure

In 1993, Chen [4] established an inequality involving intrinsic invariants and extrinsic invariants; his pioneering work emerged as one of the most applicable topics in differential geometry. Later, Chen's work was considered in different ambient spaces ([21], [22], [25],

*Corresponding Author.
Email addresses: leejaew@gnu.ac.kr (J.W. Lee), majid_alichoudhary@yahoo.co.in (M.A. Choudhary), aliyanaazsiddiqui9@gmail.com (A.N. Siddiqui)
Received: 01.08.2022; Accepted: 10.09.2023

[30], [33]). In stead of Chen's extrinsic invariants, new extrinsic invariants, called Casorati curvatures, expose optimizations for different submanifolds in an ambient Riemannian manifold ([1], [6], [7], [10], [18], [19], [31], [32]).

The idea of a semi-symmetric linear connection on a differentiable manifold was initiated by Friedmann and Schouten [8]. Later, Hayden [12] introduced a metric connection with a torsion on a Riemannian manifold. Yano [34] proved that a Riemannian manifold is conformally flat if and only if the Riemannian manifold admits a semi-symmetric metric connection with a vanishing curvature tensor. Nakao [26] showed that any submanifold of a Riemannian manifold with a semi-symmetric connection has an induced semi-symmetric connection. Further, Imai ([15], [16]) generalized equations of Gauss and Codazzi-Mainardi were derived by Nakao. Chen-like inequalities for submanifolds of real, complex and Sasakian space forms endowed with a semi-symmetric metric connections were established ([23], [24]). Moreover, some optimal inequalities for submanifolds in a Riemannian manifold of a quasi-constant curvature with a semi-symmetric metric connection were derived by using different algebraic approach in [35]. Recently, optimal inequalities for submanifolds in real space forms, generalized space forms and generalized Sasakian space forms endowed with a semi-symmetric metric connection were established ([17], [20], [31]).

Inspired by all the above developments, we have dedicated this study to derive optimizations for slant submanifolds in golden Riemannian space forms equipped with a semi-symmetric metric connection. From Oprea's optimization method [27], we prove the following result:

Theorem 1.1. *In a θ-slant proper submanifold $\mathcal{N}^n$ of a locally golden space form ($\overline{\mathcal{N}} = \mathcal{N}_p(c_p) \times \mathcal{N}_q(c_q), g, \varphi$) with a semi-symmetric metric connection, we have the following relations for the normalized δ-Casorati curvatures:*

(i) *for $\delta_c(n-1)$*

$$\begin{aligned} \rho \leq\ & \delta_c(n-1) + \Big(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\Big)\Big\{1 + \frac{1}{n(n-1)}tr^2\varphi\Big\} \\ & -\Big(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\Big)cos^2\theta\Big\{\frac{1}{n-1} + \frac{1}{n(n-1)}trP\Big\} \\ & +\Big(-\frac{(1-\psi)c_p + \psi c_q}{4}\Big)\frac{2}{n}tr\varphi - \frac{2}{n}tr\alpha \end{aligned} \tag{1.1}$$

(ii) *for $\widehat{\delta}_c(n-1)$*

$$\begin{aligned} \rho \leq\ & \widehat{\delta}_c(n-1) + \Big(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\Big)\Big\{1 + \frac{1}{n(n-1)}tr^2\varphi\Big\} \\ & -\Big(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\Big)cos^2\theta\Big\{\frac{1}{n-1} + \frac{1}{n(n-1)}trP\Big\} \\ & +\Big(-\frac{(1-\psi)c_p + \psi c_q}{4}\Big)\frac{2}{n}tr\varphi - \frac{2}{n}tr\alpha \end{aligned} \tag{1.2}$$

In addition, (1.1) and (1.2) also hold for equality if and only if $\mathcal{N}^n$ is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{\mathcal{N}}$, such that the shape operators $A_r, r \in \{n+1, \ldots, m\}$ with respect to orthonormal tangent and orthonormal normal frames $\{E_1, \ldots, E_n\}$ and $\{E_{n+1}, \ldots, E_m\}$ satisfy:

$$A_{n+1} = \begin{pmatrix} t_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & t_1 & 0 & \dots & 0 & 0 \\ 0 & 0 & t_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & t_1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 2t_1 \end{pmatrix}, \qquad A_{n+2} = \cdots = A_m = 0. \tag{1.3}$$

and

$$A_{n+1} = \begin{pmatrix} 2t_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 2t_1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 2t_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2t_1 & 0 \\ 0 & 0 & 0 & \dots & 0 & t_1 \end{pmatrix}, \qquad A_{n+2} = \cdots = A_m = 0. \tag{1.4}$$

Moreover, we derived optimal inequalities for φ-invariant and φ-anti-invariant submanifolds in the same entire space.

2. Preliminaries

2.1. Riemannian structures

Let $\overline{\mathcal{N}}$ be Riemannian manifold of dimension m with a linear connection $\overline{\nabla}$ and a torsion tensor $\overline{T}$ such that

$$\overline{T}(\overline{X},\overline{Y}) = \overline{\nabla}_{\overline{X}}\overline{Y} - \overline{\nabla}_{\overline{Y}}\overline{X} - [\overline{X},\overline{Y}], \qquad \forall \ \overline{X},\overline{Y} \in \Gamma(T\overline{\mathcal{N}}).$$

If ϕ represents a 1-form such that the following relation holds

$$\overline{T}(\overline{X},\overline{Y}) = \phi(\overline{Y})\overline{X} - \phi(\overline{X})\overline{Y},$$

$\overline{\nabla}$ is called *a semi-symmetric connection*. In addition, for a Riemannian metric $\overline{g}$ on $\overline{\mathcal{N}}$, $\overline{\nabla}$ represents a semi-symmetric metric connection if $\overline{\nabla}\overline{g} = 0$.

When $\overline{\nabla}'$ represents the Levi-Civita connection and B is the dual vector field associated with the 1-form ϕ (i.e., $\overline{g}(B,\overline{X}) = \phi(\overline{X})$, $\overline{X} \in \Gamma(T\overline{\mathcal{N}})$), the semi-symmetric metric connection $\overline{\nabla}$ on $\overline{\mathcal{N}}$ is given by [34]

$$\overline{\nabla}_{\overline{X}}\overline{Y} = \overline{\nabla}'_{\overline{X}}\overline{Y} + \phi(\overline{Y})\overline{X} - \overline{g}(\overline{X},\overline{Y})B, \qquad \forall\overline{X},\overline{Y} \in \Gamma(T\overline{\mathcal{N}}).$$

Let $\mathcal{N}$ be a submanifold of a Riemannian manifold $\overline{\mathcal{N}}$, and let $\overline{\nabla}$ and $\overline{\nabla}'$ represent a semi-symmetric metric connection and a Levi-Civita connection of $\overline{\mathcal{N}}$, respectively and ∇ and ∇' represent the induced semi-symmetric metric connection and the induced Levi-Civita connection of $\mathcal{N}$, respectively. Further, we denote the curvature tensors of $\overline{\mathcal{N}}$ (respectively $\mathcal{N}$) by $\overline{R}$ and $\overline{R}'$ (respectively, R and R') with respect to $\overline{\nabla}$ and $\overline{\nabla}'$ (respectively, ∇ and ∇'). Let us also denote h' by the second fundamental form of $\mathcal{N}$ in $\overline{\mathcal{N}}$ and h as $(0,2)$-tensor on $\mathcal{N}$. Then, in the light of ∇ and ∇', we have

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \ \ \overline{\nabla}'_X Y = \nabla'_X Y + h'(X,Y),$$

for any vector fields X, Y on $\mathcal{N}$. We have the Gauss equation by [26]

$$\begin{aligned} \overline{R}'(X,Y,Z,W) &= R'(X,Y,Z,W) \\ &\quad - g(h'(X,W),h'(Y,Z)) + g(h'(X,Z),h'(Y,W)). \end{aligned} \tag{2.1}$$

In view of the semi-symmetric metric connection $\overline{\nabla}$, the curvature tensor $\overline{R}$ is expressed as follows [16]:

$$\begin{aligned}\overline{R}(X,Y,Z,W) &= \overline{R}'(X,Y,Z,W) - \alpha(Y,Z)g(X,W)\\ &+ \alpha(X,Z)g(Y,W) - \alpha(X,W)g(Y,Z)\\ &+ \alpha(Y,W)g(X,Z), \qquad \forall X,Y,Z,W \in \Gamma(T\mathcal{N}),\end{aligned} \tag{2.2}$$

where α is a $(0,2)$-tensor field satisfying the following relation

$$\alpha(X,Y) = (\overline{\nabla}'_X\phi)Y - \phi(X)\phi(Y) + \frac{1}{2}\phi(B)g(X,Y). \tag{2.3}$$

Let $\{E_1,\dots,E_n\}$ and $\{E_{n+1},\dots,E_m\}$ be local orthonormal tangent and local orthonormal normal frames of $\mathcal{N}$, respectively. Then at any $p \in \mathcal{N}$, we represent the scalar curvature τ with SSMC by

$$\tau = \sum_{1\le i<j\le n} R(E_i,E_j,E_j,E_i)$$

and the normalized scalar curvature is expressed as

$$\rho = \frac{2\tau}{n(n-1)}.$$

We write the mean curvature vector by

$$\mathcal{H} = \sum_{i=1}^{n} \frac{1}{n} h(E_i,E_i).$$

Let us put

$$h_{ij}^r = g(h(E_i,E_j),E_r), \qquad 1 \le i,j \le n \text{ and } n+1 \le r \le m.$$

We view the squared norm of mean curvature vector as

$$||\mathcal{H}||^2 = \frac{1}{n^2}\sum_{r=n+1}^{m}\left(\sum_{i=1}^{n} h_{ii}^r\right)^2$$

and the Casorati curvature is defined by

$$\begin{aligned}\mathcal{C} &= \frac{1}{n}||h||^2\\ &= \frac{1}{n}\sum_{r=n+1}^{m}\sum_{i,j=1}^{n}(h_{ij}^r)^2.\end{aligned} \tag{2.4}$$

Let $\mathcal{L}$ be a s-dimensional subspace of $T\mathcal{N}$, $s \ge 2$ and $\{E_1,\dots,E_s\}$ be an orthonormal basis of $\mathcal{L}$. Then the scalar curvature of the $s-$plane section $\mathcal{L}$ is given by

$$\tau(\mathcal{L}) = \sum_{1\le i<j\le s} R(E_i,E_j,E_j,E_i)$$

and the Casorati curvature of the subspace $\mathcal{L}$ is defined by

$$\mathcal{C}(\mathcal{L}) = \frac{1}{s}\sum_{r=n+1}^{m}\sum_{i,j=1}^{s}(h_{ij}^r)^2$$

The normalized δ-Casorati curvatures $\delta_c(n)$ and $\widehat{\delta}_c(n)$ are defined as

$$[\delta_c(n-1)]_p = \frac{1}{2}\mathcal{C}_p + \frac{n+1}{2n}\inf\{\mathcal{C}(\mathcal{L})|\mathcal{L} : \text{a hyperplane of } T_p\mathcal{N}\}$$

and

$$[\widehat{\delta}_c(n-1)]_p = 2\mathcal{C}_p + \frac{2n-1}{2n}\sup\{\mathcal{C}(\mathcal{L})|\mathcal{L} : \text{a hyperplane of } T_p\mathcal{N}\}.$$

We note that the submanifold $\mathcal{N}$ is said to be *invariantly quasi-umbilical* if there exist $m-n$ mutually orthogonal unit normal vectors $E_{n+1},\ldots,E_m$ such that the shape operator with respect to the direction E_r has an eigenvalue of multiplicity $n-1$ and that for each E_r the distinguished eigen-direction is the same (Details in [3]).

2.2. Golden Riemannian manifolds

Let $(\overline{\mathcal{N}},\overline{g})$ be an m-dimensional Riemannian manifold and let $\mathcal{B}$ be a $(1,1)$-tensor field on $\overline{\mathcal{N}}$. If $\mathcal{B}$ satisfies the following equation ([2], [5], [11])

$$\mathcal{W}(X)=X^n+a_nX^{n-1}+\ldots+a_2X+a_1I=0,$$

where I is the identity transformation and (for $X=\mathcal{B}$) $\mathcal{B}^{n-1}(p),\mathcal{B}^{n-2}(p),\ldots,\mathcal{B}(p),I$ are linearly independent at each point $p\in\overline{\mathcal{N}}$. Then, the polynomial $\mathcal{W}(X)$ is called *the structure polynomial.* If we select the structure polynomial $\mathcal{W}(X)=X^2+I$, we get an almost complex structure. On the other hand, the structure polynomial $\mathcal{W}(X)=X^2-I$ produces an almost product structure.

Let $(\overline{\mathcal{N}},\overline{g})$ be an m-dimensional Riemannian manifold and let φ be a $(1,1)$-tensor field on $\overline{\mathcal{N}}$. If φ satisfies the following equation [2, 11]

$$\varphi^2-\varphi-I=0,$$

where I is the identity transformation. Then the tensor field φ is called *a golden structure* on $\overline{\mathcal{N}}$. The golden structure φ also satisfies the following recurrence relation:

$$\varphi^{n+1}=f_{n+1}\,.\,\varphi+f_n\,.\,I,$$

where $(f_n)_{n\in\mathbb{N}}$ is the Fibonacci sequence defined by $f_{n+2}=f_{n+1}+f_n$,$f_1=f_2=1$.

We note that $\overline{g}$ is said to be *φ-compatible* if it satisfies

$$\overline{g}(\varphi X,Y)=\overline{g}(X,\varphi Y)\qquad\forall X,Y\in\Gamma(T\overline{\mathcal{N}}),\tag{2.5}$$

where $\Gamma(T\overline{\mathcal{N}})$ is the set of all vector fields on $\overline{\mathcal{N}}$.

Let $(\overline{\mathcal{N}},\overline{g})$ be a Riemannian manifold endowed with a golden structure φ and φ-compatible Riemannian metric $\overline{g}$. Then $(\overline{\mathcal{N}},\overline{g},\varphi)$ is called *a golden Riemannian manifold* ([2], [5]).

Replacing X by φX in (2.5), we get

$$\overline{g}(\varphi X,\varphi Y)=\overline{g}(\varphi^2X,Y)=\overline{g}(\varphi X,Y)+\overline{g}(X,Y)\qquad\forall X,Y\in\Gamma(T\overline{\mathcal{N}}).$$

Let $(\overline{\mathcal{N}},\overline{g})$ be an m-dimensional differentiable manifold with a tensor field $\mathcal{B}$ of type $(1,1)$ on $\overline{\mathcal{N}}$ such that $\mathcal{B}^2=I$. Then $\mathcal{B}$ is called *an almost product structure* [2]. If the almost product structure $\mathcal{B}$ admits the Riemannian metric $\overline{g}$ such that

$$\overline{g}(\mathcal{B}X,Y)=\overline{g}(X,\mathcal{B}Y),\forall X,Y\in\Gamma(T\overline{\mathcal{N}}),$$

then $(\overline{\mathcal{N}},\overline{g})$ is called *an almost product Riemannian manifold.*

We remark that the almost product structure $\mathcal{B}$ induces the golden structure

$$\varphi=\frac{1}{2}(I+\sqrt{5}\mathcal{B})$$

and a golden structure φ produces an almost product structure

$$\mathcal{B}=\frac{1}{\sqrt{5}}(2\varphi-I).$$

One can also note that a golden Riemannian manifold $(\overline{\mathcal{N}},\overline{g})$ is said to be locally golden if φ is parallel with respect to Levi-Civita connection associated to $\overline{g}$.

2.3. Slant submanifolds of a golden Riemannian manifold

Let $(\mathcal{N}, g)$ be a submanifold of a Golden Riemannian manifold $(\overline{\mathcal{N}}, \overline{g}, \varphi)$, where g is the induced metric on $\mathcal{N}$. Then, for any $X \in \Gamma(T\mathcal{N})$ we can write

$$\varphi X = PX + QX,$$

where PX and QX are the tangential and normal components of φX, respectively. In this case, we have

$$g(PX, Y) = g(X, PY).$$

A submanifold $(\mathcal{N}, g)$ of a golden Riemannian manifold $(\overline{\mathcal{N}}, \overline{g}, \varphi)$ is said to be *slant* if for any nonzero vector $X \in T_p\mathcal{N}$, $p \in \mathcal{N}$, the angle $\theta(X)$ between φX and tangent space $T_p\mathcal{N}$ is independent of the choice of $p \in \mathcal{N}$ and $X \in T_p\mathcal{N}$. If the slant angle $\theta = 0$ ($\theta = \frac{\pi}{2}$, respectively), then $\mathcal{N}$ is known as φ-invariant (φ-anti-invariant, respectively) submanifold. A slant submanifold which is neither invariant nor anti-invariant is called a proper slant (or θ-slant proper) submanifold.

We have the following characterization of slant submanifolds in a golden Riemannian manifolds.

Lemma 2.1 ([2])**.** *Let $(\mathcal{N}, g)$ be a submanifold of a golden Riemannian manifold $(\overline{\mathcal{N}}, \overline{g}, \varphi)$. Then, the following relations hold:*

(1) *$\mathcal{N}$ is a slant submanifold if and only if there exists a constant $\lambda \in [0, 1]$ such that $P^2 = \lambda(\varphi + I)$. Furthermore, if θ is the slant angle of $\mathcal{N}$, then $\lambda = \cos^2\theta$.*
(2) *$\mathcal{N}$ is a slant submanifold if and only if there exists a constant $\lambda \in [0, 1]$ such that $\varphi^2 = \frac{1}{\lambda}P^2$, where $\lambda = \cos^2\theta$ for the slant angle θ of $\mathcal{N}$.*
(3) $g(PX, PY) = \cos^2\theta\,(g(X, Y) + g(X, PY))$, $\quad \forall X, Y \in \Gamma(T\mathcal{N})$
(4) $g(QX, QY) = \sin^2\theta\,(g(X, Y) + g(PX, Y))$, $\quad \forall X, Y \in \Gamma(T\mathcal{N})$

Example 1 ([2])**.** Consider the Euclidean 4-space $\mathbb{E}^4$ with standard coordinates (a_1, a_2, a_3, a_4). Let φ be an $(1, 1)$-tensor field on $\mathbb{E}^4$, defined by

$$\varphi(a_1, a_2, a_3, a_4) = ((1-\psi)a_1, \psi a_2, (1-\psi)a_3, \psi a_4)$$

for any vector field $(a_1, a_2, a_3, a_4) \in \mathbb{E}^4$, where $\psi = \frac{1+\sqrt{5}}{2}$ and $1 - \psi = \frac{1-\sqrt{5}}{2}$ are the roots of the equation $a^2 = a + 1$. Then we obtain

$$\begin{aligned} \varphi^2(a_1, a_2, a_3, a_4) &= ((1-\psi)^2 a_1, \psi^2 a_2, (1-\psi)^2 a_3, \psi^2 a_4) \\ &= ((1-\psi)a_1, \psi a_2, (1-\psi)a_3, \psi a_4) + (a_1, a_2, a_3, a_4). \end{aligned}$$

Thus, we have $\varphi^2 = \varphi + I$. Moreover, we get

$$< \varphi(a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4) >=< (a_1, a_2, a_3, a_4), \varphi(b_1, b_2, b_3, b_4) >$$

for each vector fields $(a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4) \in \mathbb{E}^4$, where $<, >$ is the standard metric on $\mathbb{E}^4$. Hence, $(\mathbb{E}^4, <, >, \varphi)$ is a golden Riemannian manifold. Let us consider a submanifold $\mathcal{N}$ of $\mathbb{E}^4$, given by

$$a(u_1, u_2) = (\psi u_1, k(1-\psi)u_1, \psi u_2, k(1-\psi)u_2)$$

for any $k \neq 0, 1$. Then, we have $E_1 = (\psi, k(1-\psi), 0, 0)$, $E_2 = (0, 0, \psi, k(1-\psi))$ and $\varphi E_1 = (-1, -k, 0, 0)$, $\varphi E_2 = (0, 0, -1, -k)$. so, we derive

$$< \varphi E_1, E_1 >=< \varphi E_2, E_2 >= (-k^2 + 1)\psi - 1 \quad \text{and} \quad < \varphi E_1, E_2 >= 0.$$

Therefore, $\mathcal{N}$ is a slant submanifold with the slant angle $\theta = \cos^{-1}\left(\frac{-1+\psi-k^2\psi}{\sqrt{k^2+1}}\right)$.

Example 2 Let $f : \mathcal{N} \longrightarrow \mathbb{E}^{2n}$ be an immersion given by

$$f(u_1, u_2, \cdots, u_n) = (k\psi u_1, k\psi u_2, \cdots, k\psi u_n, (1-\psi)u_1, (1-\psi)u_2, \cdots, (1-\psi)u_n)$$

for any $k \neq 0, 1$. We can find a local vector fields on $\mathcal{N}$:

$$E_i = k\psi \frac{\partial}{\partial x_i} + (1-\psi)\frac{\partial}{\partial x_{n+i}}$$

for any $i \in \{1, 2, \cdots, n\}$. We remark that $E_i \perp E_j$ for $i \neq j$, where $i, j \in \{1, 2, \cdots, n\}$. Let $\varphi : \mathbb{E}^{2n} \longrightarrow \mathbb{E}^{2n}$ be the $(1,1)$-tensor field defined by

$$\varphi(x_1, x_2, \cdots, x_n, x_{n+1}, \cdots, x_{2n}) = ((1-\psi)x_1, \cdots, (1-\psi)x_n, \psi x_{n+1}, \cdots, \psi x_{2n}),$$

where $\psi = \frac{1+\sqrt{5}}{2}$ and $1 - \psi = \frac{1-\sqrt{5}}{2}$. It is easy to verify that φ is a golden structure on $\mathbb{E}^{2n}$ (i.e., $\varphi^2 - \varphi - I = 0$) with φ-compatible metric $<,>$. Therefore, $(\mathbb{E}^{2n}, <>, \varphi)$ is a golden Riemannian manifold. Moreover, $\varphi E_i = -k\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{n+i}}$ for any $i \in \{1, 2, \cdots, n\}$. We can verify

$$< \varphi E_i, E_i >= -1 + \psi - k^2\psi \quad \text{and} \quad \varphi E_i \perp E_j \quad (i \neq j),$$

where $i, j \in \{1, 2, \cdots, n\}$. Hence, $\mathcal{N}$ is a slant submanifold with the slant angle $\theta = \cos^{-1}\left(\frac{-1+\psi-k^2\psi}{\sqrt{k^2+1}}\right)$.

Example 3 Let $\mathcal{N}_2$ be a submanifold which is spanned by $\{E_1, E_2\}$ (see *Example 1*). Since $\overline{\nabla}'$ is the Levi-Civita connection, we have

$$\begin{aligned}
&\overline{\nabla}'_{E_1}E_1 = E_3, \quad \overline{\nabla}'_{E_1}E_2 = 0, \quad \overline{\nabla}'_{E_1}E_3 = -E_1, \\
&\overline{\nabla}'_{E_1}E_4 = 0, \quad \overline{\nabla}'_{E_2}E_1 = 0, \quad \overline{\nabla}'_{E_2}E_2 = E_4, \\
&\overline{\nabla}'_{E_2}E_3 = 0, \quad \overline{\nabla}'_{E_2}E_4 = -E_2.
\end{aligned}$$

By using the definition of SSMC $\overline{\nabla}$, we get

$$\begin{aligned}
\overline{\nabla}_{E_1}E_1 &= E_3 + \phi(E_1)E_1, \\
\overline{\nabla}_{E_1}E_2 &= \phi(E_2)E_1 - [(1-\psi)^2 + k^2\psi^2]\phi(E_1), \\
\overline{\nabla}_{E_2}E_1 &= \phi(E_1)E_2 - [(1-\psi)^2 + k^2\psi^2]\phi(E_2), \\
\overline{\nabla}_{E_2}E_2 &= E_4 + \phi(E_1)E_1.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&\nabla_{E_1}E_1 = \phi(E_1)E_1, \quad \nabla_{E_1}E_2 = \phi(E_2)E_1 - [(1-\psi)^2 + k^2\psi^2]\phi(E_1), \\
&\nabla_{E_2}E_1 = \phi(E_1)E_2 - [(1-\psi)^2 + k^2\psi^2]\phi(E_2), \quad \nabla_{E_2}E_2 = \phi(E_1)E_1.
\end{aligned}$$

And the second fundamental forms with respect to SSMC are given as

$$h(E_1, E_1) = E_3, \quad h(E_1, E_2) = h(E_2, E_1) = 0, \quad h(E_2, E_2) = E_4.$$

From this, we can easily compute the mean curvature with respect to SSMC as

$$\mathcal{H} = \frac{1}{2}[h(E_1, E_1) + h(E_2, E_2)] = \frac{1}{2}[E_3 + E_4].$$

And the Casorati curvature with SSMC is given by

$$\mathcal{C} = \frac{1}{2}||h||^2 = 1.$$

Now, we suppose that $\mathcal{N}_p$ and $\mathcal{N}_q$ are two real space forms with constant sectional curvatures c_p and c_q, respectively. Then, the Riemannian curvature tensor $\overline{R}'$ of a locally golden space form $(\overline{\mathcal{N}} = \mathcal{N}_p(c_p) \times \mathcal{N}_q(c_q), g, \varphi)$ is given by [29]

$$\begin{aligned}\overline{R}'(X,Y)Z = &\Big(-\frac{(1-\psi)c_p-\psi c_q}{2\sqrt{5}}\Big)\Big\{g(Y,Z)X - g(X,Z)Y + g(\varphi Y,Z)\varphi X\\ &- g(\varphi X,Z)\varphi Y\Big\} + \Big(-\frac{(1-\psi)c_p+\psi c_q}{4}\Big)\Big\{g(\varphi Y,Z)X\\ &- g(\varphi X,Z)Y + g(Y,Z)\varphi X - g(X,Z)\varphi Y\Big\}\end{aligned} \tag{2.6}$$

Further, we assume that $\overline{\mathcal{N}}$ is a locally golden space form with a semi-symmetric metric connection $\overline{\nabla}$. Then in the light of (2.2) and (2.6), the curvature tensor $\overline{R}$ of $\overline{\mathcal{N}}$ can be written as

$$\begin{aligned}\overline{R}(X,Y,Z,W) = &\Big(-\frac{(1-\psi)c_p-\psi c_q}{2\sqrt{5}}\Big)\Big\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\\ &+ g(\varphi Y,Z)g(\varphi X,W) - g(\varphi X,Z)g(\varphi Y,W)\Big\}\\ &+ \Big(-\frac{(1-\psi)c_p+\psi c_q}{4}\Big)\Big\{g(\varphi Y,Z)g(X,W) - g(\varphi X,Z)g(Y,W)\\ &+ g(Y,Z)g(\varphi X,W) - g(X,Z)g(\varphi Y,W)\Big\}\\ &- \alpha(Y,Z)g(X,W) + \alpha(X,Z)g(Y,W)\\ &- \alpha(X,W)g(Y,Z) + \alpha(Y,W)g(X,Z).\end{aligned} \tag{2.7}$$

We recall the following result, which is useful in proving the main result.

Lemma 2.2 ([27]). *Let $(\mathcal{N}, g)$ be a Riemannian submanifold of a Riemannian manifold $(\overline{\mathcal{N}}, \overline{g})$ with the induced metric g and $f : \overline{\mathcal{N}} \to \mathbb{R}$ be a differentiable function. Let h be the second fundamental form of $\mathcal{N}$ and $y \in \mathcal{N}$ be the solution of the constrained extremum problem $min_{x\in\mathcal{N}} f(x)$, then*

(i) $(grad\ f)(y) \in T_y^{\perp}\mathcal{N}$;
(ii) *the bilinear form* $L : T_y\mathcal{N} \times T_y\mathcal{N} \to \mathbb{R}$;
$L(X,Y) = \overline{g}(h(X,Y), (grad(f))(y)) + Hess_f(X,Y)$
is positive semi-definite, where grad(f) is the gradient of f.

3. Inequalities for the Casorati curvatures

In this section, we give the proof of Theorem 1.1 in which we derive optimal Casorati inequalities for slant submanifold $\mathcal{N}$ in $\overline{\mathcal{N}}$.

Proof. (i) Since $\overline{\mathcal{N}}$ is a locally golden space form, then from (2.7) and the Gauss equation with respect to semi-symmetric metric connection, we have

$$\begin{aligned}2\tau(p) = &\Big(-\frac{(1-\psi)c_p-\psi c_q}{2\sqrt{5}}\Big)\Big\{n(n-1) + tr^2\varphi\Big\}\\ &- \Big(-\frac{(1-\psi)c_p-\psi c_q}{2\sqrt{5}}\Big)\cos^2\theta(n + trP)\\ &+ \Big(-\frac{(1-\psi)c_p+\psi c_q}{4}\Big)2(n-1)tr\varphi\\ &+ n^2||\mathcal{H}||^2 - n\mathcal{C} - 2(n-1)tr(\alpha).\end{aligned} \tag{3.1}$$

We define a quadratic polynomial, denoted by $\mathcal{Q}$ as follows:

$$\begin{aligned}\mathcal{Q} &= \frac{1}{2}n(n-1)\mathcal{C} + \frac{n^2-1}{2}\mathcal{C}(\mathcal{L}) - 2\tau(p) \\ &+ \Big(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\Big)\Big\{n(n-1) + tr^2\varphi\Big\} \\ &- \Big(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\Big)\cos^2\theta(n + trP) \\ &+ \Big(-\frac{(1-\psi)c_p + \psi c_q}{4}\Big)2(n-1)tr\varphi - 2(n-1)tr(\alpha),\end{aligned} \tag{3.2}$$

where $\mathcal{L}$ is a hyperplane of $T_p\mathcal{N}$. We can assume without loss of generality that $\mathcal{L}$ is spanned by $\{E_1,\dots,E_{n-1}\}$. Then we have

$$\begin{aligned}\mathcal{Q} &= \frac{1}{2}(n-1)\sum_{\alpha=n+1}^{m}\sum_{i,j=1}^{n}(h_{ij}^{\alpha})^2 + \frac{n+1}{2}\sum_{\alpha=n+1}^{m}\sum_{i,j=1}^{n-1}(h_{ij}^{\alpha})^2 - 2\tau(p) \\ &+ \Big(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\Big)\Big\{n(n-1) + tr^2\varphi\Big\} \\ &- \Big(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\Big)\cos^2\theta(n + trP) \\ &+ \Big(-\frac{(1-\psi)c_p + \psi c_q}{4}\Big)2(n-1)tr\varphi - 2(n-1)tr(\alpha).\end{aligned} \tag{3.3}$$

From (3.1) and (3.3), we obtain

$$\begin{aligned}\mathcal{Q} \;=\; & \frac{n+1}{2}\sum_{\alpha=n+1}^{m}\sum_{i,j=1}^{n}(h_{ij}^{\alpha})^2 + \frac{n+1}{2}\sum_{\alpha=n+1}^{m}\sum_{i,j=1}^{n-1}(h_{ij}^{\alpha})^2 \\ & - \sum_{\alpha=n+1}^{m}\Big(\sum_{i=1}^{n}h_{ii}^{\alpha}\Big)^2.\end{aligned}$$

Now, we can easily derive the following relation:

$$\begin{aligned}\mathcal{Q} &= \sum_{\alpha=n+1}^{m}\sum_{i=1}^{n-1}\Big[n(h_{ii}^{\alpha})^2 + (n+1)(h_{in}^{\alpha})^2\Big] \\ &+ \sum_{\alpha=n+1}^{m}\Big[2(n+1)\sum_{i<j=1}^{n-1}(h_{ij}^{\alpha})^2 - 2\sum_{i<j=1}^{n}h_{ii}^{\alpha}h_{jj}^{\alpha} \\ &+ \frac{n-1}{2}(h_{nn}^{\alpha})^2\Big] \\ &\geq \sum_{\alpha=n+1}^{m}\sum_{i=1}^{n-1}\Big[n(h_{ii}^{\alpha})^2 - 2\sum_{i<j=1}^{n}h_{ii}^{\alpha}h_{jj}^{\alpha} \\ &+ \frac{n-1}{2}(h_{nn}^{\alpha})^2\Big].\end{aligned} \tag{3.4}$$

For $\alpha \in \{n+1,\dots,m\}$, let us consider the quadratic form $f_\alpha : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f_\alpha(h_{11}^{\alpha},\dots,h_{nn}^{\alpha}) = \sum_{i=1}^{n-1}n(h_{ii}^{\alpha})^2 - 2\sum_{i<j=1}^{n}h_{ii}^{\alpha}h_{jj}^{\alpha} + \frac{n-1}{2}(h_{nn}^{\alpha})^2,$$

and the constrained extremum problem $minf_\alpha$ subject to $F : h_{11}^{\alpha} + \cdots + h_{nn}^{\alpha} = \beta^{\alpha}$, where β^{α} is a real constant.

We observe that the function f_α has the following partial derivatives:

$$\begin{aligned}
\frac{\partial f_\alpha}{\partial h^\alpha_{11}} &= 2nh^\alpha_{11} - 2\sum_{l=2}^{n} h^\alpha_{ll},\\
\frac{\partial f_\alpha}{\partial h^\alpha_{22}} &= 2nh^\alpha_{22} - 2h^\alpha_{11} - 2\sum_{l=3}^{n} h^\alpha_{ll},\\
&\vdots\\
\frac{\partial f_\alpha}{\partial h^\alpha_{n-1\,n-1}} &= 2nh^\alpha_{n-1\,n-1} - 2\sum_{l=1}^{n-2} h^\alpha_{ll} - 2h^\alpha_{nn},\\
\frac{\partial f_\alpha}{\partial h^\alpha_{nn}} &= -2\sum_{l=1}^{n-1} h^\alpha_{ll} + (n-1)h^\alpha_{nn}.
\end{aligned} \tag{3.5}$$

For an optimal solution $(h^\alpha_{11}, \ldots, h^\alpha_{nn})$ of the problem, $grad(f_\alpha)$ is normal at F. From (3.5), we have the following critical points of the considered problem:

$$h^\alpha_{11} = h^\alpha_{22} = \ldots h^\alpha_{n-1n-1} = \frac{1}{n+1}\beta^\alpha, h^\alpha_{nn} = \frac{2}{n+1}\beta^\alpha \tag{3.6}$$

Now, let us fix an arbitrary point $y \in F.$, From lemma 2.2, the bilinear form $L : T_yF \times T_yF \to \mathbb{R}$ is defined by

$$L(X,Y) = \overline{g}(h(X,Y), (gradf_\alpha)(y)) + Hess_{f_\alpha}(X,Y),$$

where h is the second fundamental form of F in $\mathbb{R}^n$. Now, as F is totally geodesic in $\mathbb{R}^n$, considering a vector $X = (X_1, X_2, \ldots, X_n)$ tangent to F at the arbitrary point y on F (that is, verifying the relation $\sum_{l=1}^{n} X_i = 0$), we obtain the following:

$$\begin{aligned}
L(X,X) &= -2(X_1, ..., X_{n-1}, X_n)\begin{pmatrix} -n & 1 & \ldots & 1 & 1\\ 1 & -n & \ldots & 1 & 1\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 1 & 1 & \ldots & -n & 1\\ 1 & 1 & \ldots & 1 & \frac{1-n}{2}\end{pmatrix}\begin{pmatrix} X_1\\ X_2\\ \vdots\\ X_{n-1}\\ X_n\end{pmatrix}\\
&= 2(n+1)\sum_{l=1}^{n-1} X_i^2 + (n+1)X_n^2 - 2(\sum_{l=1}^{n} X_i)^2\\
&= 2(n+1)\sum_{l=1}^{n-1} X_i^2 + (n+1)X_n^2\\
&\geq 0,
\end{aligned}$$

Hence, in the light of lemma 2.2 the point $(h^\alpha_{11}, \ldots, h^\alpha_{nn})$ (see (3.6)) is a global minimum point. Moreover, we have $f_\alpha(h^\alpha_{11}, \ldots, h^\alpha_{nn}) = 0$. Thus, we arrive at

$$\mathfrak{Q} \geq 0, \tag{3.7}$$

which implies that

$$\begin{aligned}2\tau(p) \quad \leq \quad & \frac{1}{2}n(n-1)\mathcal{C}+\frac{1}{2}(n^2-1)\mathcal{C}(\mathcal{L})\\ & +\Big(-\frac{(1-\psi)c_p-\psi c_q}{2\sqrt{5}}\Big)\Big\{n(n-1)+tr^2\varphi\Big\}\\ & -\Big(-\frac{(1-\psi)c_p-\psi c_q}{2\sqrt{5}}\Big)cos^2\theta(n+trP)\\ & +\Big(-\frac{(1-\psi)c_p+\psi c_q}{4}\Big)2(n-1)tr\varphi-2(n-1)tr(\alpha),\end{aligned}$$

whereby, we obtain

$$\begin{aligned}\rho \leq \ & \frac{1}{2}\mathcal{C}+\frac{n+1}{2n}\mathcal{C}(\mathcal{L})\\ & +\Big(-\frac{(1-\psi)c_p-\psi c_q}{2\sqrt{5}}\Big)\Big\{1+\frac{1}{n(n-1)}tr^2\varphi\Big\}\\ & -\Big(-\frac{(1-\psi)c_p-\psi c_q}{2\sqrt{5}}\Big)\cos^2\theta\Big\{\frac{1}{n-1}+\frac{1}{n(n-1)}trP\Big\}\\ & +\Big(-\frac{(1-\psi)c_p+\psi c_q}{4}\Big)\frac{2}{n}tr\varphi-\frac{2}{n}tr(\alpha)\end{aligned} \tag{3.8}$$

for every tangent hyperplane $\mathcal{L}$ of $T_p\mathcal{N}$ and inequality (1.1) obviously follow from (3.8).

Moreover, the equality sign holds in (1.1) if and only if

$$h^{\alpha}_{ij}=0,\ \forall\ i,j\in\{1,\dots,n\},\ i\neq j,\alpha\in\{n+1,\dots,m\} \tag{3.9}$$

and

$$h^{\alpha}_{nn}=2h^{\alpha}_{11}=2h^{\alpha}_{22}\cdots=2h^{\alpha}_{n-1n-1},\ \forall\ \alpha\in\{n+1,\dots,m\}. \tag{3.10}$$

From (3.9) and (3.10), we conclude that the equality sign holds in the inequality (1.1) if and only if the submanifold $\mathcal{N}$ is invariantly quasi-umbilical with trivial normal connection in $\overline{\mathcal{N}}$, such that with respect to suitable orthonormal tangent and normal orthonormal frames, the shape operators take the form of (1.3).

(ii) In the same manner, we can establish an inequality in the second part of the theorem. □

4. Applications of Theorem 1.1 for different kind of submanifolds

As an application of Theorem 1.1, we give sharp inequalities for φ-invariant and φ-anti-invariant submanifolds in the same ambient space.

Theorem 4.1. *In an invariant submanifold $\mathcal{N}^n$ of a locally golden space form $(\overline{\mathcal{N}}=\mathcal{N}_p(c_p)\times\mathcal{N}_q(c_q),g,\varphi)$ with semi-symmetric metric connection, we have the following relations for the normalized δ-Casorati curvatures:*

(i) *for $\delta_c(n-1)$*

$$\begin{aligned}\rho \leq \ & \delta_c(n-1)+\Big(-\frac{(1-\psi)c_p-\psi c_q}{2\sqrt{5}}\Big)\Big\{1+\frac{1}{n(n-1)}tr^2\varphi\Big\}\\ & -\Big(-\frac{(1-\psi)c_p-\psi c_q}{2\sqrt{5}}\Big)\Big\{\frac{1}{n-1}+\frac{1}{n(n-1)}trP\Big\}\\ & +\Big(-\frac{(1-\psi)c_p+\psi c_q}{4}\Big)\frac{2}{n}tr\varphi-\frac{2}{n}tr\alpha\end{aligned} \tag{4.1}$$

(ii) *for* $\widehat{\delta}_c(n-1)$

$$\begin{aligned}\rho \leq \widehat{\delta}_c(n-1) &+ \Big(-\frac{(1-\psi)c_p-\psi c_q}{2\sqrt{5}}\Big)\Big\{1+\frac{1}{n(n-1)}tr^2\varphi\Big\}\\ &-\Big(-\frac{(1-\psi)c_p-\psi c_q}{2\sqrt{5}}\Big)\Big\{\frac{1}{n-1}+\frac{1}{n(n-1)}trP\Big\}\\ &+\Big(-\frac{(1-\psi)c_p+\psi c_q}{4}\Big)\frac{2}{n}tr\varphi-\frac{2}{n}tr\alpha.\end{aligned} \tag{4.2}$$

In addition, (4.1) and (4.2) also hold for equality if and only if $\mathcal{N}^n$ *is an invariantly quasi-umbilical with trivial normal connection in* $\overline{\mathcal{N}}$*, such that* $A_r, r \in \{n+1,\ldots,m\}$ *with respect to orthonormal tangent and orthonormal normal frames* $\{E_1,\ldots,E_n\}$ *and* $\{E_{n+1},\ldots,E_m\}$ *satisfy:*

$$A_{n+1}=\begin{pmatrix} t_1 & 0 & 0 & \ldots & 0 & 0\\ 0 & t_1 & 0 & \ldots & 0 & 0\\ 0 & 0 & t_1 & \ldots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \ldots & t_1 & 0\\ 0 & 0 & 0 & \ldots & 0 & 2t_1\end{pmatrix}, \qquad A_{n+2}=\cdots=A_m=0 \tag{4.3}$$

and

$$A_{n+1}=\begin{pmatrix} 2t_1 & 0 & 0 & \ldots & 0 & 0\\ 0 & 2t_1 & 0 & \ldots & 0 & 0\\ 0 & 0 & 2t_1 & \ldots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \ldots & 2t_1 & 0\\ 0 & 0 & 0 & \ldots & 0 & t_1\end{pmatrix}, \qquad A_{n+2}=\cdots=A_m=0. \tag{4.4}$$

Next, we have

Theorem 4.2. *In an anti-invariant submanifold* $\mathcal{N}^n$ *of a locally golden space form* $(\overline{\mathcal{N}} = \mathcal{N}_p(c_p)\times\mathcal{N}_q(c_q), g, \varphi)$ *with semi-symmetric metric connection, we have the following relations for the normalized* δ*-Casorati curvatures:*

(i) *for* $\delta_c(n-1)$

$$\begin{aligned}\rho \leq \delta_c(n-1) &+ \Big(-\frac{(1-\psi)c_p-\psi c_q}{2\sqrt{5}}\Big)\Big\{1+\frac{1}{n(n-1)}tr^2\varphi\Big\}\\ &+\Big(-\frac{(1-\psi)c_p+\psi c_q}{4}\Big)\frac{2}{n}tr\varphi-\frac{2}{n}tr\alpha\end{aligned} \tag{4.5}$$

(ii) *for* $\widehat{\delta}_c(n-1)$

$$\begin{aligned}\rho \leq \widehat{\delta}_c(n-1) &+ \Big(-\frac{(1-\psi)c_p-\psi c_q}{2\sqrt{5}}\Big)\Big\{1+\frac{1}{n(n-1)}tr^2\varphi\Big\}\\ &+\Big(-\frac{(1-\psi)c_p+\psi c_q}{4}\Big)\frac{2}{n}tr\varphi-\frac{2}{n}tr\alpha.\end{aligned} \tag{4.6}$$

In addition, (4.5) and (4.6) also hold for equality if and only if $\mathcal{N}^n$ *is an invariantly quasi-umbilical with trivial normal connection in* $\overline{\mathcal{N}}$*, such that* $A_r, r \in \{n+1,\ldots,m\}$ *with respect to orthonormal tangent and orthonormal normal frames* $\{E_1,\ldots,E_n\}$ *and* $\{E_{n+1},\ldots,E_m\}$ *satisfy:*

$$A_{n+1} = \begin{pmatrix} t_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & t_1 & 0 & \dots & 0 & 0 \\ 0 & 0 & t_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & t_1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 2t_1 \end{pmatrix}, \qquad A_{n+2} = \cdots = A_m = 0 \tag{4.7}$$

and

$$A_{n+1} = \begin{pmatrix} 2t_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 2t_1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 2t_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2t_1 & 0 \\ 0 & 0 & 0 & \dots & 0 & t_1 \end{pmatrix}, \qquad A_{n+2} = \cdots = A_m = 0. \tag{4.8}$$

Any Riemannian manifold $\mathcal{N}$ immersed in a Riemannian manifold $\overline{\mathcal{N}}$ is named as C-totally real if $\varphi(T\mathcal{N}) \subset T^{\perp}\mathcal{N}$. We have the following result.

Theorem 4.3. *In a C-totally real submanifold $\mathcal{N}^n$ of a locally golden space form $(\overline{\mathcal{N}} = \mathcal{N}_p(c_p) \times \mathcal{N}_q(c_q), g, \varphi)$ with semi-symmetric metric connection, we have the following relations for the normalized δ-Casorati curvatures:*

(i) *for $\delta_c(n-1)$*

$$\rho \;\leq\; \delta_c(n-1) + \Big(- \frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}} \Big) - \frac{2}{n} tr\alpha \tag{4.9}$$

(ii) *for $\widehat{\delta}_c(n-1)$*

$$\rho \;\leq\; \widehat{\delta}_c(n-1) + \Big(- \frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}} \Big) - \frac{2}{n} tr\alpha \tag{4.10}$$

In addition, (4.9) and (4.10) also hold for equality if and only if $\mathcal{N}^n$ is an invariantly quasi-umbilical with trivial normal connection in $\overline{\mathcal{N}}$, such that $A_r, r \in \{n+1, \dots, m\}$ with respect to orthonormal tangent and orthonormal normal frames $\{E_1, \dots, E_n\}$ and $\{E_{n+1}, \dots, E_m\}$ satisfy:

$$A_{n+1} = \begin{pmatrix} t_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & t_1 & 0 & \dots & 0 & 0 \\ 0 & 0 & t_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & t_1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 2t_1 \end{pmatrix}, \qquad A_{n+2} = \cdots = A_m = 0. \tag{4.11}$$

and

$$A_{n+1} = \begin{pmatrix} 2t_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 2t_1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 2t_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2t_1 & 0 \\ 0 & 0 & 0 & \dots & 0 & t_1 \end{pmatrix}, \qquad A_{n+2} = \dots = A_m = 0. \tag{4.12}$$

References

[1] M. Aslam and A.N. Siddiqui, *Bounds for generalized normalized δ-Casorati curvature for submanifold in real space forms endowed with a quarter-symmetric connection*, Balkan J. Geom. Appl., **27** (2), 1-12, 2022.

[2] O. Bahadr and S. Uddin, *Slant submanifolds of Golden Riemannian manifolds*, J. Math. Ext. **13** (4), 2019.

[3] D. Blair, *Quasi-umbilical, minimal submanifolds of Euclidean space*, Simon Stevin **51**, 3-22, 1977.

[4] B.Y. Chen, *Some pinching and classification theorems for minimal submanifolds*, Arch. Math. **60**, 568-578, 1993.

[5] M. Crasmareanu and C. Hretcanu, *Golden differential geometry*, Chaos Solitons Fractals, **38** (5), 1229-1238, 2008.

[6] S. Decu, S. Haesen and L. Verstraelen, *Optimal inequalities involving Casorati curvatures*, Bull. Transylv. Univ. Braÿsov, Ser. B **14** (49), suppl., 85-93, 2007.

[7] S. Decu, S. Haesen and L. Verstraelen, *Optimal inequalities characterising quasi-umbilical submanifolds*, J. Inequal. Pure Appl. Math. **9** (3), Article ID 79, 2008.

[8] A. Friedmann and A. Schouten, *Uber die Geometrie der halbsymmetrischen Ubertragungen*, Mathematische Zeitschrift **21**, 211-223, 1924.

[9] A. Gezer, N. Cengiz and A. Salimov, *On integrability of Golden Riemannian structures*, Turkish J. Math. **37**, 693-703, 2013.

[10] V. Ghisoiu, *Inequalities for the Casorati curvatures of slant submanifolds in complex space forms*, In: Riemannian Geometry and Applications. Proceedings RIGA 2011, . Ed. Univ. Bucureÿsti, Bucharest 145-150, 2011.

[11] S.I. Goldberg and K. Yano, *Polynomial structures on manifolds*, Kodai Math. Sem. Rep. **22**, 199-218, 1970.

[12] H. A. Hayden, *Subspaces of a space with torsion*, Proc. London Math. Soc. **34**, 27-50, 1932.

[13] C. Hretcanu and M. Crasmareanu, *On some invariant submanifolds in a Riemannian manifold with golden structure*, An. Stiins. Univ. Al. I. Cuza Iasi. Mat. (N.S.) **53**, 199-211, 2007.

[14] C. Hretcanu and M. Crasmareanu, *Applications of the golden ratio on Riemannian manifolds*, Turkish J. Math. **33**, 179-191, 2009.

[15] T. Imai, *Hypersurfaces of a Riemannian manifold with semi-symmetric metric connection*, Tensor(N.S.), **23**, 300-306, 1972.

[16] T. Imai, *Notes on semi-symmetric metric connection*, Tensor(N.S.), **24**, 293-296, 1972.

[17] C.W. Lee, D.W. Yoon and J.W. Lee, *Optimal inequalities for the Casorati curvatures of submanifolds of real space forms endowed with semi-symmetric metric connections*, J. Inequal. Appl. 2014:327, 9 pp. MR 3344114, 2014.

[18] C.W. Lee, J.W. Lee and G.E. Vilcu, *Optimal inequalities for the normalized δ-Casorati curvatures of submanifolds in Kenmotsu space forms*, Advances in Geom. **17**, 2017.

[19] J.W. Lee and G.E. Vilcu, *Inequalities for generalized normalized Casorati curvatures of slant submanifolds in quaternion space forms*, Taiwanese J. Math. **19**, 691-702, 2015.

[20] C.W. Lee, J.W. Lee, G.E. Vilcu, and D.W. Yoon, *Optimal Inequalities for the Casorati curvatures of submanifolds of generalized space forms endowed with semi-symmetric metric connections*, Bull. Korean Math. Soc. **52**, 16311647, 2015.

[21] X. Liu, *On Ricci curvature of totally real submanifolds in a quaternion projective space*, Arch. Math. **38** (4), 297-305, 2002.

[22] X. Liu and W. Dai, *Ricci curvature of submanifolds in a quaternion projective space*, Commun. Korean Math. Soc. **17** (4), 625-633, 2002.

[23] A. Mihai and C. Ozgur, *Chen inequalities for submanifolds of real space forms with a semi-symmetric metric connection*, Taiwanese J. Math. **14**, 1465-1477, 2010.

[24] A. Mihai and C. Ozgur, *Chen inequalities for submanifolds of complex space forms and Sasakian space forms endowed with semi-symmetric metric connection*, Rocky Mountain J. Math., **41**, 1653-1673, 2011.

[25] I. Mihai, F.R. Al-Solamy and M.H. Shahid, *On Ricci curvature of a quaternion CR-submanifold in a quaternion space form*, Rad. Mat. **12** (1), 91-98, 2003.

[26] Z. Nakao, *Submanifolds of a Riemannian manifold with semi-symmetric metric connections*, Proc. Amer. Math. Soc. **54**, 261-266, 1976.

[27] T. Oprea, *Optimization methods on Riemannian submanifolds*, An. Univ. Bucuresti Math. **54**, 127-136, 2005.

[28] M. Ozkan, *Prolongations of golden structures to tangent bundles*, Differential Geometry - Dynamical Sytem. **16**, 227-23, 2014.

[29] N.O. Poyraz and Y. Erol, *Lightlike Hypersurfaces of a Golden Semi-Riemannian Manifold*, Mediterr. J. Math. 14:204, DOI 10.1007/s00009-017-0999-2, 2017.

[30] S.S. Shukla and P.K. Rao, *Ricci curvature of quaternion slant submanifolds in quaternion space forms*, Acta Math. Acad. Paedagog. Nyházi **28** (1), 69-81, 2012.

[31] A.N. Siddiqui, *Upper bound inequalities for δ-Casorati curvatures of submanifolds in generalized Sasakian space forms admitting a semi-symmetric metric connection*, Inter. Elec. J. Geom. **11** (1), 57-67, 2018.

[32] M.M. Tripathi, *Inequalities for algebraic Casorati curvatures and their applications*, arXiv:1607.05828v1 [math.DG] 20 Jul 2016.

[33] G.E. Vilcu, *On Chen invariant and inequalities in quaternionic geometry*, J. Inequal. Appl. **2013**, Article ID 66, 2013

[34] K. Yano, *On semi-symmetric metric connection*, Rev. roumaine Math. Pure Appl. **15**, 1579-1586, 1970.

[35] P. Zhang, L. Zhang and W. Song, *Inequalities for submanifolds of a Riemannian manifold of quasi-constant curvature with a semi-symmetric metric connection*, Taiwanese J. Math. **18**, 1841-1862, 2014.