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# The Number of Representatıons of Positive Integers by Some Direct Sum Of Binary Quadratic Forms With Discriminant -103

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**Abstract**
A basis of a subspace of $S_4(\Gamma_0(103))$ has been found by means of generalized theta series associated to the direct sums of the quadratic forms $x_1^2+x_1x_2+26x_2^2$, $2x_1^2\pm x_1x_2+13x_2^2$, $4x_1^2\pm 3x_1x_2+7x_2^2$ .

***Keywords:*** *Quadratic forms, representation number*

## 1. Introduction

Let's obtain a formula for the number of representation

$$r(Q,n)=\#\{(x,y)\in \mathbb{Z}x\mathbb{Z}: x^2+y^2=n\}$$

of a positive integer $n$ by quadratic form $Q(x,y)=x^2+y^2$. For this, we calculate the theta series

$$\Theta_Q(q)=\sum_{n\in\mathbb{Z}} q^{n^2}\sum_{m\in\mathbb{Z}} q^{m^2} = \sum_{n,m\in\mathbb{Z}} q^{n^2+m^2} \quad (1.1)$$

$$=1+\sum_{n\in\mathbb{Z}} r(Q,n)q^n,\ q=e^{2\pi i\tau},\ \mathrm{Im}\tau>0.$$

It is known that it is a modular form of weight 1 for the group

$\Gamma_0(4)$ with character $\chi_{-4}$.

Here $\chi_{-4}$ is the Dirichlet character modulo 4 determined by

$$1\to 1, 3\to -1.$$

On the other hand, it is known that the space of modular forms of weight 1 with character $\chi_{-4}$ is one dimensional and generated by the Eisenstein series

$$\frac{1}{2}L(0,\chi_{-4})+\sum_{n=1}\left(\sum_{d|n}\chi_{-4}(d)\right)q^n,$$

where the L-function

$$L(s,\chi_{-4})=\sum_{n=1}^{\infty}\frac{\chi_{-4}(n)}{n^s}$$

defined for $Res>0$ and can be analytically continued to the whole plane as holomorphic function. It can be shown that $L(0,\chi_{-4})=\frac{1}{2}$. Hence

$$\Theta_Q(q)=4\cdot\left(\frac{1}{2}\cdot\frac{1}{2}+\sum_{n=1}\left(\sum_{d|n \text{ d is odd}}(-1)^{\frac{d-1}{2}}\right)q^n\right)$$

$$=1+\sum_{n=1}4\left(\sum_{d|n \text{ d is odd}}(-1)^{\frac{d-1}{2}}\right)q^n,$$

consequently, we obtain the simple formula

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$$r(Q,n) = 4\left(\sum_{d|n \text{ d is odd}} (-1)^{\frac{d-1}{2}}\right). \quad (1.2)$$

The immediate generalization to positive definite binary quadratic form $Q$ of this formula is given by Dirichlet for discriminants $\Delta$ with class number $\mathrm{h}(\Delta) = 1$ as follows:

$$r(Q,n) = w_\Delta\left(\sum_{d|n}\left(\frac{\Delta}{d}\right)\right), w_\Delta = \begin{cases} 6 \text{ for } \Delta = -3 \\ 4 \text{ for } \Delta = -4. \\ 2 \text{ for } \Delta < -4 \end{cases}$$

The determination of $r(Q,n)$ can be obtained by convolution products as in [1], [2] and [3]. But in this paper, we will obtain similar simple formulas as extensions of the work [4], [5] and [6] for direct sums of four positive definite quadratic forms of discriminmant $-103$.

Since the class number of $\mathbb{Q}(\sqrt{-103})$ is 5, there exist 5 inequivalent classes of binary quadratic forms of discriminant $-103$ whose reduced forms, the ideals their norms and the corresponding complex numbers in the standard fundamental domain of $SL(2,\mathbb{Z})$ can be calculated easily as

$$F_1 = x_1^2 + x_1x_2 + 26x_2^2, \left[1, \frac{-1+i\sqrt{103}}{2}\right] = \left[1, \frac{1+i\sqrt{103}}{2}\right] = O_{-103},$$

$$N\left(\left[1, \frac{1+i\sqrt{103}}{2}\right]\right) = 1, \frac{-1}{2} + \frac{i\sqrt{103}}{2}$$

$$\Phi_1 = 2x_1^2 + x_1x_2 + 13x_2^2, \left[2, \frac{-1+i\sqrt{103}}{2}\right] = \left[2, 1 + \frac{1+i\sqrt{103}}{2}\right],$$

$$N\left(\left[2, 1 + \frac{1+i\sqrt{103}}{2}\right]\right) = 2, \ \frac{-1}{4} + \frac{i\sqrt{103}}{4}$$

$$\Psi_1 = 4x_1^2 + 3x_1x_2 + 7x_2^2, \left[4, \frac{-3+i\sqrt{103}}{2}\right] = \left[4, -2 + \frac{1+i\sqrt{103}}{2}\right],$$

$$N\left(\left[4, 2 + \frac{1+i\sqrt{103}}{2}\right]\right) = 4, \ \frac{-3}{8} + \frac{i\sqrt{103}}{8}$$

$$\Psi_1' = 4x_1^2 - 3x_1x_2 + 7x_2^2, \left[4, \frac{3+i\sqrt{103}}{2}\right] = \left[4, 1 + \frac{1+i\sqrt{103}}{2}\right]$$

$$N\left(\left[4, 1 + \frac{1+i\sqrt{103}}{2}\right]\right) = 4, \ \frac{3}{8} + \frac{i\sqrt{103}}{8}$$

$$\Phi_1' = M_1 = 2x_1^2 - x_1x_2 + 13x_2^2, \left[2, \frac{1+i\sqrt{103}}{2}\right],$$

$$N\left(\left[2, \frac{1+i\sqrt{103}}{2}\right]\right) = 2, \ \frac{1}{4} + \frac{i\sqrt{103}}{4}.$$

Here $\Phi_1'$ is the inverse of $\Phi_1$, and they represent the same integers. Similarly, $\Psi_1'$ is the inverse of $\Psi_1$ and they represent the same integers. Therefore, the theta series of $\Phi_1$ and $\Phi_1'$ are the same with the theta series of $\Psi_1$ and $\Psi_1'$ respectively. $F_1$ is the identity element. The group of these quadratic forms is a group of order 5 and can be described in the following way.

Consider

$$\Phi_1 = 2x_1^2 + x_1x_2 + 13x_2^2.$$
$$\gcd(2,1,2) = 1,$$

so by Dirichlet composition formula, there exists unique

$$b \ mod \ 2.2.2 = 8$$

such that

$$b \equiv 5 mod 4, b^2 \equiv -103 mod 4.2.2 = 16.$$

$$c = \frac{103 + 5^2}{16} = 8$$

So,

$$\varphi(x_1, x_2) = 2 \cdot 2x_1^2 + 5x_1x_2 + 8x_2^2 = 4x_1^2 + 5x_1x_2 + 8x_2^2$$

which is equivalent to

$$4x_1^2 - 3x_1x_2 + 7x_2^2.$$

So,

$$\Phi_1^2 = \Psi_1' \text{ and } \Phi_1^3 = \Phi_1\Psi_1' = 8x_1^2 + 5x_1x_2 + 4x_2^2$$

which is equivalent to

$$4x_1^2 - 5x_1x_2 + 8x_2^2$$

, i.e.,

$$\Phi_1^3 = \Psi_1.$$

Therefore,

$$\Phi_1, \Phi_1^2 = \Psi_1', \Phi_1^3 = \Psi_1, \Phi_1^4 = \Phi_1', \Phi_1^5 = F_1.$$

Since $103$ is prime, there is only one genus, i.e., principal genus.

Obviously, there are only two inequivalent cusps $i\infty$ and $0$ for $\Gamma_0(103)$. We have the following important Theorem for the Eisenstein part of theta series associated to the quadratic form.

**Theorem 1.** Let $Q$ be a positive definite form of $2k$ variables,

$$k = 4,6,8,\dots,$$

whose theta series $\Theta_Q$ is in $M_k(\Gamma_0(p))$, $p$ prime, then the Eisenstein part of $\Theta_Q$ is

$$E(q:Q) = 1 + \sum_{n=1}^{\infty} (\alpha\sigma_{k-1}(n)q^n + \beta\sigma_{k-1}(n)q^{pn})$$

where

$$\alpha = \frac{i^k}{\rho_k}\frac{p^{k/2} - i^k}{p^k - 1}, \beta = \frac{1}{\rho_k}\frac{p^k - i^k p^{k/2}}{p^k - 1}, \rho_k = (-1)^{k/2}\frac{(k-1)!}{(2\pi)^k}\zeta(k).$$

**Proof.** See [4].

We immediately obtain the following Corollary.

**Corollary 2.** Let $Q$ be a positive definite form of 8 variables whose theta series $\Theta_Q$ is in

$$M_4(\Gamma_0(103)),$$

then the Eisenstein part of $\Theta_Q$ is

$$E(q:Q) = 1 + \sum_{n=1}^{\infty} (\alpha\sigma_3(n)q^n + \beta\sigma_3(n)q^{79n})$$

where

$$\rho_4 = \frac{3!}{(2\pi)^4}\zeta(4) = \frac{1}{240}$$

$$\alpha = 240\frac{103^2 - 1}{103^4 - 1} = 240\frac{1}{103^2 + 1} = \frac{24}{1061},$$

$$\beta = 240\frac{103^4 - 103^2}{103^4 - 1} = 240\frac{103^2}{103^2 + 1} = 103^2\frac{24}{1061}$$

and

$$E(q:F_4) = E(q:F_3 \oplus \Phi_1) = E(q:F_2 \oplus \Phi_2) = E(q:F_1 \oplus \Phi_3) = E(q:\Phi_4)$$

$$= E(q:F_3 \oplus \Psi_1) = E(q:F_2 \oplus \Psi_2) = E(q:F_1 \oplus \Psi_3) = E(q:\Psi_4)$$

$$= E(q:\Phi_3 \oplus \Psi_1) = E(q:\Phi_2 \oplus \Psi_2) = E(q:\Phi_1 \oplus \Psi_3) = \cdots = E(q:F_1 \oplus \Phi_1 \oplus \Psi_2) =$$

$$= 1 + \frac{24}{1061}\sum_{n=1}^{\infty}(q^n + 103^2 q^{103n})\sigma_3(n)$$

$$= 1 + \frac{24}{1061}\sum_{n=1}^{\infty}\sigma_3^*(n)\, q^n$$

$$= 1 + \frac{24}{1061} q + \frac{24 \cdot 9}{1061} q^2 + \frac{24 \cdot 28}{1061} q^3 + \frac{24 \cdot 73}{1061} q^4 + \frac{24 \cdot 126}{1061} q^5 + \frac{24 \cdot 252}{1061} q^6$$

$$+ \frac{24 \cdot 344}{1061} q^7 + \frac{24 \cdot 585}{1061} q^8 + \frac{24 \cdot 757}{1061} q^9 + \frac{24 \cdot 1134}{1061} q^{10} + \frac{24 \cdot 1332}{1061} q^{11}$$

$$+ \frac{24 \cdot 2044}{1061} q^{12} + \frac{24 \cdot 2198}{1061} q^{13} + \frac{24 \cdot 3096}{1061} q^{14} + \frac{24 \cdot 3528}{1061} q^{15} + \frac{24 \cdot 4681}{1061} q^{16}$$

$$+ \frac{24 \cdot 4914}{1061} q^{17} + \frac{24 \cdot 6813}{1061} q^{18} + \frac{24 \cdot 6860}{1061} q^{19} + \frac{24 \cdot 9198}{1061} q^{20} + \frac{24 \cdot 9632}{1061} q^{21}$$

$$+ \frac{24 \cdot 11988}{1061} q^{22} + \frac{24 \cdot 12168}{1061} q^{23} + \frac{24 \cdot 16380}{1061} q^{24} + \frac{24 \cdot 15751}{1061} q^{25} + \cdots$$

where $\sigma_3^*(n) = \begin{cases} \sigma_3(n) \text{ if } n \geq 1 \text{ and } 103 \nmid n \\ \sigma_3(n) + 103^2 \sigma_3(n/103) \text{ if } 103 | n \end{cases}$.

## 1. The Positive Definite Forms

Now we will give some definitions, an important Theorem and evaluation of our quadratic forms.

**Definition 1.** Let $Q: \mathbb{Z}^{2k} \to \mathbb{Z}$ be a positive definite integer-valued form of $2k$ variables

$$Q = \sum_{1 \leq i \leq j \leq 2k}^{2k} b_{ij} x_i x_j, b_{ij} \in \mathbb{Z}$$

and the matrix $A$ is defined by

$$a_{ii} = 2b_{ii}, a_{ji} = a_{ij} = b_{ij} \text{ for } i < j.$$

Let $D$ be the determinant of the quadratic form

$$2Q = \sum_{i,j=1}^{2k} a_{ij} x_i x_j$$

, $i.e.$, the determinant of the matrix $A$. Let $A_{ij}$ be the cofactors of $a_{ij}$ for $1 \leq i,j \leq 2k$. If $\delta = \gcd\left(\frac{A_{ii}}{2}, A_{ij} \text{ for } 1 \leq i,j \leq 2k\right)$, then $N := \frac{D}{\delta}$ is the smallest positive integer, called the level of $Q$, for which $NA^{-1}$ is again an even integral matrix like $A$. $\Delta = (-1)^k D$ is called the discriminant of the form $Q$.

**Theorem 1.** Let $Q: \mathbb{Z}^{2k} \to \mathbb{Z}$ be a positive definite integer-valued form of $2k$ variables of level $N$ and discriminant $\Delta$. Then 1-the theta function

$$\Theta_Q(q) = \sum_{(n_1, n_2, \cdots, n_k) \in \mathbb{Z} \times \mathbb{Z} \cdots \times \mathbb{Z}} q^{Q(n_1, n_2, \cdots, n_k)} = 1 + \sum_{n=1}^{\infty} r(n; Q) q^n, q = e^{2\pi i z}$$

is a modular form on $\Gamma_0(N)$ of weight $k$ and character $\chi_\Delta$, i.e., $\Theta_Q \in M_k(\Gamma_0(N), \chi_\Delta)$, where

$$\chi_\Delta(d) := \left(\frac{\Delta}{d}\right), d \in (\mathbb{Z}/N\mathbb{Z})^\times, \left(\frac{\Delta}{d}\right)$$ is the Kronecker character.

2 −The homogeneous quadratic polynomials in $2k$ variables

$$\varphi_{ij} = x_i x_j - \frac{1}{2k}\frac{A_{ij}}{D}2Q, \quad 1 \leq i,j \leq 2k \qquad (1.1)$$

are spherical functions of second order with respect to $Q$.

3-The generalized theta series

$$\Theta_{Q,\varphi_{ij}}(q) = \sum_{n=1}^{\infty}\left(\sum_{Q=n}\varphi_{ij}\right)q^n \qquad (1.2)$$

is a cusp form in $S_{k+2}(\Gamma_0(N),\chi_\Delta)$.

4-If two quadratic forms $Q_1, Q_2$ have the same level $N$ and the characters $\chi_1(d)$, $\chi_2(d)$ respectively, then the direct sum $Q_1 \oplus Q_2$ of the quadratic forms has the same level $N$ and the character $\chi_1(d)\cdot\chi_2(d)$ [4].

Let

$$F_2 = F_1 \oplus F_1,\ \Phi_2 = \Phi_1 \oplus \Phi_1,\ \Psi_2 = \Psi_1 \oplus \Psi_1, F_1 \oplus \Phi_1, F_1 \oplus \Psi_1, \Phi_1 \oplus \Psi_1$$

be direct sums of the quadratic forms. In this paper, we will obtain the generalized theta series determined by the spherical functions associated to these quadratic forms. Let

$$F_k = F_1 + \cdots + F_1(k \text{ times}),\ \Phi_k = \Phi_1 + \cdots + \Phi_1(k \text{ times}),\ \Psi_k = \Psi_1 + \cdots + \Psi_1(k \text{ times})$$

are $k$ direct sums of the quadratic forms.

By Theorem 1, $F_1$, $\Phi_1$, $\Phi'_1, \Psi_1$, $\Psi'_1$are quadratic forms whose theta series are in

$$M_1\left(\Gamma_0(103),\left(\frac{-103}{d}\right)\right).$$

Therefore, again by Theorem 1

$F_4, \Phi_4, \Phi_3 \oplus \Phi'_1, \Phi_2 \oplus \Phi'_2, \Phi_1 \oplus \Phi'_3, \Phi'_4, \Psi_4, \Psi_3 \oplus \Psi'_1, \Psi_2 \oplus \Psi'_2, \Psi_1 \oplus \Psi'_3, \Psi'_4,$

$F_3 \oplus \Phi_1, F_3 \oplus \Phi'_1, F_2 \oplus \Phi_2, F_2 \oplus \Phi_1 \oplus \Phi'_1, F_2 \oplus \Phi'_2, F_1 \oplus \Phi_3, F_1 \oplus \Phi_2 \oplus \Phi'_1,$

$F_1 \oplus \Phi_1 \oplus \Phi'_2, F_1 \oplus \Phi'_3, F_3 \oplus \Psi_1, F_3 \oplus \Psi'_1, F_2 \oplus \Psi_2, F_2 \oplus \Psi_1 \oplus \Psi'_1, F_2 \oplus \Psi'_2, F_1 \oplus \Psi_3,$

$F_1 \oplus \Psi_2 \oplus \Psi'_1, F_1 \oplus \Psi_1 \oplus \Psi'_2, F_1 \oplus \Psi'_3, \Phi_3 \oplus \Psi_1, \Phi_2 \oplus \Phi'_1 \oplus \Psi_1, \Phi_1 \oplus \Phi'_2 \oplus \Psi_1,$

$\Phi'_3 \oplus \Psi_1, \Phi_3 \oplus \Psi'_1, \Phi_2 \oplus \Phi'_1 \oplus \Psi'_1, \Phi_1 \oplus \Phi'_2 \oplus \Psi'_1,\ \Phi'_3 \oplus \Psi'_1, \Phi_2 \oplus \Psi_2, \Phi'_2 \oplus \Psi_2,$

$\Phi_2 \oplus \Psi'_2, \Phi'_2 \oplus \Psi'_2, \Phi_1 \oplus \Phi'_1 \oplus \Psi_2, \Phi_1 \oplus \Phi'_1 \oplus \Psi_1 \oplus \Psi'_1, \Phi_1 \oplus \Phi'_1 \oplus \Psi'_2, \Phi_2 \oplus \Psi_1 \oplus \Psi'_1,$

$\Phi'_2 \oplus \Psi_1 \oplus \Psi'_1, \Psi_3 \oplus \Phi_1, \Psi_2 \oplus \Psi'_1 \oplus \Phi_1, \Psi_1 \oplus \Psi'_2 \oplus \Phi_1, \Psi'_3 \oplus \Phi_1, \Psi_3 \oplus \Phi'_1, \Psi_2 \oplus \Psi'_1 \oplus \Phi'_1,$

$\Psi_1 \oplus \Psi'_2 \oplus \Phi'_1$, or $\Psi'_3 \oplus \Phi'_1, F_2 \oplus \Phi_1 \oplus \Psi_1,\ F_2 \oplus \Phi'_1 \oplus \Psi_1,\ F_2 \oplus \Phi_1 \oplus \Psi'_1,\ F_2 \oplus \Phi'_1 \oplus \Psi'_1,$

$F_1 \oplus \Phi_2 \oplus \Psi_1,\ F_1 \oplus \Phi'_2 \oplus \Psi_1,\ F_1 \oplus \Phi_2 \oplus \Psi'_1,\ F_1 \oplus \Phi'_2 \oplus \Psi'_1, F_1 \oplus \Phi_1 \oplus \Phi'_1 \oplus \Psi_1,$

$F_1 \oplus \Phi_1 \oplus \Phi'_1 \oplus \Psi'_1, F_1 \oplus \Phi_1 \oplus \Psi_2,\ F_1 \oplus \Phi'_1 \oplus \Psi_2,\ F_1 \oplus \Phi_1 \oplus \Psi'_2,\ F_1 \oplus \Phi'_1 \oplus \Psi'_2,$

$F_1 \oplus \Phi_1 \oplus \Psi_1 \oplus \Psi'_1, F_1 \oplus \Phi'_1 \oplus \Psi_1 \oplus \Psi'_1,$ are quadratic forms whose theta series are in

$$M_4(\Gamma_0(103)).$$

In this paper, we will obtain the formulas of $r(n;Q)$ for these quadratic forms

Now let's look at the positive definite quadratic forms of discriminant -103:

1-For the quadratic form $F_1 = x_1^2 + x_1x_2 + 26x_2^2$,

$$2F_1 = 2x_1^2 + 2x_1x_2 + 52x_2^2 = (x_1, x_2)\begin{pmatrix} 2 & 1 \\ 1 & 52 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

the determinant of the matrix and a cofactor are

$$D = 103, A_{22} = 2.$$

So $\delta = 1, N = D = 103$ and the discriminant is

$$\Delta = (-1)^{2/2}103 = -103.$$

The character of $F_1$ is the Kronecker symbol

$$\chi(d) = \left(\frac{-103}{d}\right).$$

2-For the quadratic form $\Phi_1 = 2x_1^2 + x_1x_2 + 13x_2^2$,

$$2\Phi_1 = 4x_1^2 + 2x_1x_2 + 26x_2^2 = (x_1, x_2)\begin{pmatrix} 4 & 1 \\ 1 & 26 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

the determinant of the matrix and a cofactor are

$$D = 103, A_{21} = -1.$$

So $\delta = 1, N = D = 103$ and the discriminant is

$$\Delta = (-1)^{2/2}103 = -103.$$

The character of $\Phi_1$ is the Kronecker symbol

$$\chi(d) = \left(\frac{-103}{d}\right).$$

3-For the quadratic form $\Psi_1 = 4x_1^2 + 3x_1x_2 + 7x_2^2$,

$$2\Psi_1 = 8x_1^2 + 6x_1x_2 + 14x_2^2 = (x_1, x_2)\begin{pmatrix} 8 & 3 \\ 3 & 14 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

the determinant of the matrix and some cofactors are

$$D = 103, A_{21} = -3, \frac{A_{11}}{2} = 7.$$

So $\delta = 1, N = D = 103$ and the discriminant is

$$\Delta = (-1)^{2/2}103 = -103.$$

The character of $\Psi_1$ is the Kronecker symbol

$$\chi(d) = \left(\frac{-103}{d}\right),$$

Consequently, $F_1, \Phi_1, \Phi_1', \Psi_1, \Psi_1'$are quadratic forms whose theta series are in

$$M_1\left(\Gamma_0(103), \left(\frac{-103}{d}\right)\right),$$

hence by Theorem 1,

$$F_2 = F_1 + F_1,\ \Phi_2 = \Phi_1 + \Phi_1,\ \Psi_2 = \Psi_1 + \Psi_1, F_1 + \Phi_1, F_1 + \Psi_1, \Phi_1 + \Psi_1$$

are quadratic forms whose theta series are in

$$M_2\big(\Gamma_0(103)\big).$$

## 1. Spherical Functions

Here we will find all spherical functions such that the corresponding generalized theta series span all the generalized theta series of the form 2.4 induced by spherical functions of the form 2.3.

1-For quadratic form

$$2F_2 = 2x_1^2 + 2x_1x_2 + 52x_2^2 + 2x_3^2 + 2x_3x_4 + 52x_4^2$$

$$= (x_1, x_2, x_3, x_4)\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 52 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 52 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix},$$

the determinant and the matrix of cofactors are

$$D = 103^2, \begin{pmatrix} 5356 & -103 & 0 & 0 \\ -103 & 206 & 0 & 0 \\ 0 & 0 & 5356 & -103 \\ 0 & 0 & -103 & 206 \end{pmatrix}$$

By putting, $2k = 4, Q = F_2$, and appropriate $i, j$ in Theorem 1, we get

$$\varphi_{11} = x_1^2 - \frac{1}{4}\frac{5356}{103^2}2F_2 = x_1^2 - \frac{26}{103}F_2,$$

$$\varphi_{12} = x_1x_2 + \frac{1}{4}\frac{103}{103^2}2F_2 = x_1x_2 + \frac{1}{206}F_2,$$

which will be a spherical function of second order with respect to $F_2$.

2-For quadratic form

$$2\Phi_2 = 4x_1^2 + 2x_1x_2 + 26x_2^2 + 4x_3^2 + 2x_3x_4 + 26x_4^2$$

$$= (x_1, x_2, x_3, x_4)\begin{pmatrix} 4 & 1 & & \\ 1 & 26 & & \\ & & 4 & 1 \\ & & 1 & 26 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix},$$

By putting $2k = 4, Q = \Phi_2$, and appropriate $i, j$ in Theorem 1,we get

$$\varphi_{11} = x_1^2 - \frac{1}{4}\frac{2678}{103^2}2\Phi_2 = x_1^2 - \frac{13}{103}\Phi_2,$$

$$\varphi_{12} = x_1x_2 + \frac{1}{4}\frac{103}{103^2}2\Phi_2 = x_1x_2 + \frac{1}{206}\Phi_2,$$

$$\varphi_{22} = x_2^2 - \frac{1}{4}\frac{412}{103^2}2\Phi_2 = x_2^2 - \frac{2}{103}\Phi_2,$$

which will be spherical functions of second order with respect to $\Phi_2$ .

3-For quadratic form

$$2(F_1 \oplus \Phi_1) = 2x_1^2 + 2x_1x_2 + 52x_2^2 + 4x_3^2 + 2x_3x_4 + 26x_4^2$$

$$= (x_1, x_2, x_3, x_4)\begin{pmatrix} 2 & 1 & & \\ 1 & 52 & & \\ & & 4 & 1 \\ & & 1 & 26 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix},$$

the determinant and the matrix of cofactors are

$$D = 103^2, \begin{pmatrix} 5356 & -103 & 0 & 0 \\ -103 & 206 & 0 & 0 \\ 0 & 0 & 2678 & -103 \\ 0 & 0 & -103 & 412 \end{pmatrix}$$

By putting $2k = 4, Q = F_1 \oplus \Phi_1$, and appropriate $i, j$ in Theorem 1, we get

$$\varphi_{11} = x_1^2 - \frac{1}{4}\frac{52 \cdot 103}{103^2}2(F_1 \oplus \Phi_1) = x_1^2 - \frac{26}{103}(F_1 \oplus \Phi_1),$$

$$\varphi_{12} = x_1x_2 + \frac{1}{4}\frac{103}{103^2}2(F_1 \oplus \Phi_1) = x_1x_2 + \frac{1}{206}(F_1 \oplus \Phi_1),$$

$$\varphi_{22} = x_2^2 - \frac{1}{4}\frac{206}{103^2}2(F_1 \oplus \Phi_1) = x_2^2 - \frac{1}{103}(F_1 \oplus \Phi_1),$$

$$\varphi_{33} = x_3^2 - \frac{1}{4}\frac{26 \cdot 103}{103^2}2(F_1 \oplus \Phi_1) = x_3^2 - \frac{13}{103}(F_1 \oplus \Phi_1),$$

$$\varphi_{34} = x_3x_4 + \frac{1}{4}\frac{103}{103^2}2(F_1 \oplus \Phi_1) = x_3x_4 + \frac{1}{206}(F_1 \oplus \Phi_1)$$

$$\varphi_{44} = x_4^2 - \frac{1}{4}\frac{4 \cdot 103}{103^2}2(F_1 \oplus \Phi_1) = x_4^2 - \frac{2}{103}(F_1 \oplus \Phi_1),$$

which will be spherical functions of second order with respect to $F_1 \oplus \Phi_1$.

4-For quadratic form

$$2\Psi_2 = 8x_1^2 + 6x_1x_2 + 14x_2^2 + 8x_3^2 + 6x_3x_4 + 14x_4^2$$

$$= (x_1, x_2, x_3, x_4)\begin{pmatrix} 8 & 3 & & \\ 3 & 14 & & \\ & & 8 & 3 \\ & & 3 & 14 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix},$$

the determinant and the matrix of cofactors are

$$D = 103^2, \begin{pmatrix} 1442 & -309 & 0 & 0 \\ -309 & 824 & 0 & 0 \\ 0 & 0 & 1442 & -309 \\ 0 & 0 & -309 & 824 \end{pmatrix}$$

By putting $2k = 4$, $Q = \Psi_2$, and appropriate $i, j$ in Theorem 1, we get

$$\varphi_{11} = x_1^2 - \frac{1}{4}\frac{1442}{103^2} 2\Psi_2 = x_1^2 - \frac{7}{103}\Psi_2,$$

$$\varphi_{12} = x_1x_2 + \frac{1}{4}\frac{3 \cdot 103}{103^2} 2\Psi_2 = x_1x_2 + \frac{3}{206}\Psi_2,$$

$$\varphi_{22} = x_2^2 - \frac{1}{4}\frac{8 \cdot 103}{103^2} 2\Psi_2 = x_2^2 - \frac{4}{103}\Psi_2$$

which will be spherical functions of second order with respect to $\Psi_2$ .

5-For quadratic form

$$2(F_1 \oplus \Psi_1) = 2x_1^2 + 2x_1x_2 + 52x_2^2 + 8x_3^2 + 6x_3x_4 + 14x_4^2$$

$$= (x_1, x_2, x_3, x_4) \begin{pmatrix} 2 & 1 & & \\ 1 & 52 & & \\ & & 8 & 3 \\ & & 3 & 14 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix},$$

the determinant and the matrix of cofactors are

$$D = 103^2, \begin{pmatrix} 5356 & -103 & 0 & 0 \\ -103 & 206 & 0 & 0 \\ 0 & 0 & 1442 & -309 \\ 0 & 0 & -309 & 824 \end{pmatrix}$$

By putting $2k = 4, Q = F_1 \oplus \Psi_1$, and appropriate $i, j$ in Theorem 1, we get

$$\varphi_{11} = x_1^2 - \frac{1}{4}\frac{52 \cdot 103}{103^2} 2(F_1 \oplus \Psi_1) = x_1^2 - \frac{26}{103}(F_1 \oplus \Psi_1),$$

$$\varphi_{12} = x_1x_2 + \frac{1}{4}\frac{103}{103^2} 2(F_1 \oplus \Psi_1) = x_1x_2 + \frac{1}{206}(F_1 \oplus \Psi_1),$$

$$\varphi_{22} = x_2^2 - \frac{1}{4}\frac{2 \cdot 103}{103^2} 2(F_1 \oplus \Psi_1) = x_2^2 - \frac{1}{103}(F_1 \oplus \Psi_1),$$

$$\varphi_{33} = x_3^2 - \frac{1}{4}\frac{14 \cdot 103}{103^2} 2(F_1 \oplus \Psi_1) = x_3^2 - \frac{7}{103}(F_1 \oplus \Psi_1),$$

$$\varphi_{34} = x_3x_4 + \frac{1}{4}\frac{103}{103^2} 2(F_1 \oplus \Psi_1) = x_3x_4 + \frac{1}{206}(F_1 \oplus \Psi_1),$$

$$\varphi_{44} = x_4^2 - \frac{1}{4}\frac{8 \cdot 103}{103^2} 2(F_1 \oplus \Psi_1) = x_4^2 - \frac{4}{103}(F_1 \oplus \Psi_1),$$

which will be spherical functions of second order with respect to $F_1 \oplus \Psi_1$.

6-For quadratic form

$$2(\Phi_1 \oplus \Psi_1) = 4x_1^2 + 2x_1x_2 + 26x_2^2 + 8x_3^2 + 6x_3x_4 + 14x_4^2$$

$$= (x_1, x_2, x_3, x_4) \begin{pmatrix} 4 & 1 & & \\ 1 & 26 & & \\ & & 8 & 3 \\ & & 3 & 14 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix},$$

the determinant and the matrix of cofactors are

$$D = 103^2, \begin{pmatrix} 2678 & -103 & 0 & 0 \\ -103 & 412 & 0 & 0 \\ 0 & 0 & 1442 & -309 \\ 0 & 0 & -309 & 824 \end{pmatrix}$$

By putting, $2k = 4, Q = \Phi_1 \oplus \Psi_1$, and appropriate $i, j$ in Theorem 1, we get

$$\varphi_{11}=x_1^2-\frac{1}{4}\frac{26\cdot\ 103}{103^2}2(\ \Phi_1\oplus\Psi_1)$$
$$=x_1^2-\frac{13}{103}(\ \Phi_1\oplus\Psi_1),$$

$$\varphi_{12}=x_1x_2+\frac{1}{4}\frac{103}{103^2}2(\ \Phi_1\oplus\Psi_1)$$
$$=x_1x_2+\frac{1}{206}(\ \Phi_1\oplus\Psi_1),$$

$$\varphi_{22}=x_2^2-\frac{1}{4}\frac{26\cdot\ 14}{103^2}2(\ \Phi_1\oplus\Psi_1)$$
$$=x_2^2-\frac{182}{103^2}(\ \Phi_1\oplus\Psi_1),$$

$$\varphi_{23}=x_2x_3+\frac{1}{4}\frac{4\cdot\ 14}{103^2}2(\ \Phi_1\oplus\Psi_1)$$
$$=x_2x_3+\frac{28}{103^2}(\ \Phi_1\oplus\Psi_1),$$

$$\varphi_{33}=x_3^2-\frac{1}{4}\frac{14\cdot\ 103}{103^2}2(\ \Phi_1\oplus\Psi_1)$$
$$=x_3^2-\frac{7}{103}(\ \Phi_1\oplus\Psi_1),$$

$$\varphi_{34}=x_3x_4+\frac{1}{4}\frac{103}{103^2}2(\ \Phi_1\oplus\Psi_1)$$
$$=x_3x_4+\frac{1}{206}(\ \Phi_1\oplus\Psi_1),$$

$$\varphi_{44}=x_4^2-\frac{1}{4}\frac{8\cdot 103}{103^2}2(\ \Phi_1\oplus\Psi_1)$$
$$=x_4^2-\frac{4}{103}(\ \Phi_1\oplus\Psi_1),$$

which will be spherical functions of second order with respect to $\Phi_1\oplus\Psi_1$.

Now we will determine a basis of the subspace of $S_4(\Gamma_0(103))$ spanned by the generalized theta series of the form 2.4 induced by spherical functions of the form 2.3. The dimension of $S_4(\Gamma_0(103))$ is 25[8].

**Theorem 1.** The following generalized 27 theta series

$$\Theta_{F_2,\varphi_{11}}(q)=\frac{1}{103}\sum_{n=1}^{\infty}\sum_{F_2=n}(103x_1^2-26F_2)\,q^n, \tag{1.1}$$

$$\Theta_{F_2,\varphi_{12}}(q)=\frac{1}{206}\sum_{n=1}^{\infty}\sum_{F_2=n}(206x_1x_2+F_2)\,q^n, \tag{1.2}$$

$$\Theta_{\Phi_2,\varphi_{11}}(q)=\frac{1}{103}\sum_{n=1}^{\infty}\sum_{\Phi_2=n}(103x_1^2-13\Phi_2)\,q^n \tag{1.3}$$

$$\Theta_{\Phi_2,\varphi_{12}}(q)=\frac{1}{2\cdot\ 103}\sum_{n=1}^{\infty}\sum_{\Phi_2=n}(2\cdot\ 103x_1x_2+\Phi_2)\,q^n \tag{0.1}$$

$$\Theta_{\Phi_2,\varphi_{22}}(q)=\frac{1}{103}\sum_{n=1}^{\infty}\sum_{\Phi_2=n}(\ 103x_2^2-2\Phi_2)\,q^n, \tag{0.2}$$

$$\Theta_{F_1\oplus\Phi_1,\varphi_{11}}(q)=\frac{1}{103}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Phi_1=n}(\ 103x_1^2-26F_1\oplus\Phi_1)\,q^n, \tag{0.3}$$

$$\Theta_{F_1\oplus\Phi_1,\varphi_{12}}(q)=\frac{1}{2\cdot 103}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Phi_1=n}(2\cdot\ 103x_1x_2+F_1\oplus\Phi_1)\,q^n, \tag{0.4}$$

$$\Theta_{F_1\oplus\Phi_1,\varphi_{22}}(q) = \frac{1}{103}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Phi_1=n}(103x_2^2 - F_1\oplus\Phi_1)q^n, \tag{0.5}$$

$$\Theta_{F_1\oplus\Phi_1,\varphi_{33}}(q) = \frac{1}{103}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Phi_1=n}(103x_3^2 - 13F_1\oplus\Phi_1)q^n, \tag{0.6}$$

$$\Theta_{F_1\oplus\Phi_1,\varphi_{34}}(q) = \frac{1}{2\cdot 103}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Phi_1=n}(2\cdot 103x_3x_4 + F_1\oplus\Phi_1)q^n, \tag{0.7}$$

$$\Theta_{F_1\oplus\Phi_1,\varphi_{44}}(q) = \frac{1}{103}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Phi_1=n}(103x_4^2 - 2F_1\oplus\Phi_1)q^n, \tag{0.8}$$

$$\Theta_{\Psi_2,\varphi_{11}}(q) = \frac{1}{103}\sum_{n=1}^{\infty}\sum_{\Psi_2=n}(103x_1^2 - 7\Psi_2)q^n, \tag{0.9}$$

$$\Theta_{\Psi_2,\varphi_{12}}(q) = \frac{1}{2\cdot 103}\sum_{n=1}^{\infty}\sum_{\Psi_2=n}(2\cdot 103x_1x_2 + 3\Psi_2)q^n, \tag{0.10}$$

$$\Theta_{\Psi_2,\varphi_{22}}(q) = \frac{1}{103}\sum_{n=1}^{\infty}\sum_{\Psi_2=n}(103x_2^2 - 4\Psi_2)q^n, \tag{0.11}$$

$$\Theta_{F_1\oplus\Psi_1,\varphi_{11}}(q) = \frac{1}{103}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Psi_1=n}(103x_1^2 - 26F_1\oplus\Psi_1)q^n, \tag{0.12}$$

$$\Theta_{F_1\oplus\Psi_1,\varphi_{12}}(q) = \frac{1}{2\cdot 103}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Psi_1=n}(2\cdot 103x_1x_2 + F_1\oplus\Psi_1)q^n, \tag{0.13}$$

$$\Theta_{F_1\oplus\Psi_1,\varphi_{22}}(q) = \frac{1}{103}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Psi_1=n}(103x_2^2 - F_1\oplus\Psi_1)q^n, \tag{0.14}$$

$$\Theta_{F_1\oplus\Psi_1,\varphi_{33}}(q) = \frac{1}{103}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Psi_1=n}\left(103x_3^2 - 7(F_1\oplus\Psi_1)\right)q^n, \tag{0.15}$$

$$\Theta_{F_1\oplus\Psi_1,\varphi_{34}}(q) = \frac{1}{2\cdot 103}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Psi_1=n}\left(2\cdot 103x_3x_4 + (F_1\oplus\Psi_1)\right)q^n, \tag{0.16}$$

$$\Theta_{F_1\oplus\Psi_1,\varphi_{44}}(q) = \frac{1}{103}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Psi_1=n}\left(103x_4^2 - 4(F_1\oplus\Psi_1)\right)q^n, \tag{0.17}$$

$$\Theta_{\Phi_1\oplus\Psi_1,\varphi_{11}}(q) = \frac{1}{103}\sum_{n=1}^{\infty}\sum_{\Phi_1\oplus\Psi_1=n}\left(103x_1^2 - 13(\Phi_1\oplus\Psi_1)\right)q^n, \tag{0.18}$$

$$\Theta_{\Phi_1\oplus\Psi_1,\varphi_{12}}(q) = \frac{1}{2\cdot 103}\sum_{n=1}^{\infty}\sum_{\Phi_1\oplus\Psi_1=n}\left(2\cdot 103x_1x_2 + (\Phi_1\oplus\Psi_1)\right)q^n, \quad (0.19)$$

$$\Theta_{\Phi_1\oplus\Psi_1,\varphi_{22}}(q) = \frac{1}{103^2}\sum_{n=1}^{\infty}\sum_{\Phi_1\oplus\Psi_1=n}\left(103^2x_2^2 - 182(\Phi_1\oplus\Psi_1)\right)q^n, \quad (0.20)$$

$$\Theta_{\Phi_1\oplus\Psi_1,\varphi_{23}}(q) = \frac{1}{103^2}\sum_{n=1}^{\infty}\sum_{\Phi_1\oplus\Psi_1=n}\left(103^2x_2x_3 + 28(\Phi_1\oplus\Psi_1)\right)q^n, \quad (0.21)$$

$$\Theta_{\Phi_1\oplus\Psi_1,\varphi_{33}}(q) = \frac{1}{103}\sum_{n=1}^{\infty}\sum_{\Phi_1\oplus\Psi_1=n}\left(103x_3^2 - 7(\Phi_1\oplus\Psi_1)\right)q^n, \quad (0.22)$$

$$\Theta_{\Phi_1\oplus\Psi_1,\varphi_{34}}(q) = \frac{1}{2\cdot 103}\sum_{n=1}^{\infty}\sum_{\Phi_1\oplus\Psi_1=n}\left(2\cdot 103x_3x_4 + (\Phi_1\oplus\Psi_1)\right)q^n, \quad (0.23)$$

$$\Theta_{\Phi_1\oplus\Psi_1,\varphi_{44}}(q) = \frac{1}{103}\sum_{n=1}^{\infty}\sum_{\Phi_1\oplus\Psi_1=n}\left(103x_4^2 - 4(\Phi_1\oplus\Psi_1)\right)q^n \quad (0.24)$$

has rank 20 in $S_4(\Gamma_0(103))$ and the set

$$\{\Theta_{F_2,\varphi_{11}},\Theta_{\Phi_2,\varphi_{11}},\Theta_{\Phi_2,\varphi_{12}},\Theta_{F_1\oplus\Phi_1,\varphi_{11}},\Theta_{F_1\oplus\Phi_1,\varphi_{12}},\Theta_{F_1\oplus\Phi_1,\varphi_{33}},\Theta_{F_1\oplus\Phi_1,\varphi_{34}},$$
$$\Theta_{\Psi_2,\varphi_{11}},\Theta_{\Psi_2,\varphi_{12}},\Theta_{F_1\oplus\Psi_1,\varphi_{11}},\Theta_{F_1\oplus\Psi_1,\varphi_{12}},\Theta_{F_1\oplus\Psi_1,\varphi_{33}},\Theta_{F_1\oplus\Psi_1,\varphi_{34}},\Theta_{\Phi_1\oplus\Psi_1,\varphi_{11}},$$
$$\Theta_{\Phi_1\oplus\Psi_1,\varphi_{12}},\Theta_{\Phi_1\oplus\Psi_1,\varphi_{22}},\Theta_{\Phi_1\oplus\Psi_1,\varphi_{23}},\Theta_{\Phi_1\oplus\Psi_1,\varphi_{33}},\Theta_{\Phi_1\oplus\Psi_1,\varphi_{34}},\Theta_{\Phi_1\oplus\Psi_1,\varphi_{44}}\}$$

is a basis of the subspace of $S_4(\Gamma_0(103))$ spanned by the generalized theta series of the form 2.4 induced by spherical functions of the form 2.3.

**Proof.** The series are cusp forms because of Theorem1, 2).

$$F_2 = x_1^2 + x_1x_2 + 26x_2^2 + x_3^2 + x_3x_4 + 26x_4^2 = n$$

has the following solutions for

$n = 1 \Longrightarrow x_1 = \pm 1, x_2 = 0, x_3 = 0, x_4 = 0$ and $x_1 = 0, x_2 = 0, x_3 = \pm 1, x_4 = 0,$

$n = 2 \Longrightarrow x_1 = \pm 1, x_2 = 0, x_3 = \pm 1, x_4 = 0,$

$n = 3 \Longrightarrow$ it has no integral solutions,

$n = 4 \Longrightarrow x_1 = \pm 2, x_2 = 0, x_3 = 0, x_4 = 0$ and $x_1 = 0, x_2 = 0, x_3 = \pm 2, x_4 = 0,$

$n = 5 \Longrightarrow x_1 = \pm 1, x_2 = 0, x_3 = \pm 2, x_4 = 0$ and $x_1 = \pm 2, x_2 = 0, x_3 = \pm 1, x_4 = 0,$

$n = 6 \Longrightarrow$ it has no integral solutions,

$n = 7 \Longrightarrow$ it has no integral solutions,

$n = 8 \Longrightarrow x_1 = \pm 2, x_2 = 0, x_3 = \pm 2, x_4 = 0,$

$n = 9 \Longrightarrow x_1 = \pm 3, x_2 = 0, x_3 = 0, x_4 = 0$ and $x_1 = 0, x_2 = 0, x_3 = \pm 3, x_4 = 0,$

$n = 10 \Longrightarrow x_1 = \pm 3, x_2 = 0, x_3 = \pm 1, x_4 = 0$ and $x_1 = \pm 1, x_2 = 0, x_3 = \pm 3, x_4 = 0,$

$n = 11 \Longrightarrow$ it has no integral solutions,

$n = 12 \Longrightarrow$ it has no integral solutions,

$n = 13 \Longrightarrow x_1 = \pm 2, x_2 = 0, x_3 = \pm 3, x_4 = 0$ and $x_1 = \pm 3, x_2 = 0, x_3 = \pm 2, x_4 = 0,$

$n = 14 \Longrightarrow$ it has no integral solutions,

$n = 15 \Longrightarrow$ it has no integral solutions,

$n = 16 \Longrightarrow x_1 = \pm 4, x_2 = 0, x_3 = 0, x_4 = 0$ and $x_1 = 0, x_2 = 0, x_3 = \pm 4, x_4 = 0,$

$n = 17 \Longrightarrow x_1 = \pm 1, x_2 = 0, x_3 = \pm 4, x_4 = 0$ and $x_1 = \pm 4, x_2 = 0, x_3 = \pm 1, x_4 = 0,$

$n = 18 \Longrightarrow x_1 = \pm 3, x_2 = 0, x_3 = \pm 3, x_4 = 0,$

$n = 19 \Longrightarrow$ it has no integral solutions,

$n = 20 \Longrightarrow x_1 = \pm 4, x_2 = 0, x_3 = \pm 2, x_4 = 0,$ and $x_1 = \pm 2, x_2 = 0, x_3 = \pm 4, x_4 = 0,$

$n = 21 \Longrightarrow$ it has no integral solutions,

$n = 22 \Longrightarrow$ it has no integral solutions,

$n = 23 \Longrightarrow$ it has no integral solutions,

$n = 24 \Longrightarrow$ it has no integral solutions,

$n = 25 \Longrightarrow x_1 = \pm 5, x_2 = 0, x_3 = 0, x_4 = 0$ and $x_1 = 0, x_2 = 0, x_3 = \pm 5, x_4 = 0,$

$x_1 = \pm 4, x_2 = 0, x_3 = \pm 3, x_4 = 0$ and $x_1 = \pm 3, x_2 = 0, x_3 = \pm 4, x_4 = 0.$

So

$$\Theta_{F_2,\varphi_{11}}(q) = \frac{1}{103} \sum_{n=1}^{\infty} \sum_{F_2 = n} (103x_1^2 - 26F_2) q^n \qquad (0.25)$$

$$= \frac{1}{103}\big(( 103 \cdot 2 - 26 \cdot 4)\, q + ( 103 \cdot 4 - 26 \cdot 2 \cdot 4)\, q^2 + ( 103 \cdot 4 \cdot 2 - 26 \cdot 4 \cdot 4)\, q^4$$

$$+( 103 \cdot 1 \cdot 4 + 103 \cdot 4 \cdot 4 - 26 \cdot 5 \cdot 8)\, q^5 + ( 103 \cdot 4 \cdot 4 - 26 \cdot 8 \cdot 4)\, q^8$$

$$+( 103 \cdot 9 \cdot 2 - 26 \cdot 9 \cdot 4)\, q^9 + ( 103 \cdot 9 \cdot 4 + 103 \cdot 1 \cdot 4 - 26 \cdot 10 \cdot 8)\, q^{10}$$

$$+( 103 \cdot 4 \cdot 4 + 103 \cdot 9 \cdot 4 - 26 \cdot 13 \cdot 8)\, q^{13} + ( 103 \cdot 16 \cdot 2 - 26 \cdot 16 \cdot 4)\, q^{16}$$

$$+( 103 \cdot 1 \cdot 4 + 103 \cdot 16 \cdot 4 - 26 \cdot 17 \cdot 8)\, q^{17} + ( 103 \cdot 9 \cdot 4 - 26 \cdot 18 \cdot 4)\, q^{18}$$

$$+( 103 \cdot 16 \cdot 4 + 103 \cdot 4 \cdot 4 - 26 \cdot 20 \cdot 8)\, q^{20}$$

$$+ ( 103 \cdot 25 \cdot 2 + 103 \cdot 16 \cdot 4 + 103 \cdot 9 \cdot 4 - 26 \cdot 25 \cdot 12)\, q^{25} + \cdots \big)$$

$$= \frac{1}{103}(102q + 204q^2 + 408q^4 + 1020q^5 + 816q^8 + 918q^9 + 2040q^{10}$$

$$+2652q^{13}+1632q^{16}+3468q^{17}+1836q^{18}+4080q^{20}+7650q^{25}\cdots),$$

and

$$\Theta_{F_2,\varphi_{12}}(q)=\frac{1}{206}\sum_{n=1}^{\infty}\sum_{F_2=n}(206x_1x_2+F_2)q^n \quad (0.26)$$

$$=\frac{1}{103}(2q+4q^2+8q^4+20q^5+16q^8+18q^9+40q^{10}+52q^{13}+32q^{16}$$

$$+68q^{17}+36q^{18}+80q^{20}+150q^{25}\cdots).$$

Similarly, by solving the equations

$$\Phi_2=2x_1^2+x_1x_2+13x_2^2+2x_3^2+x_3x_4+13x_4^2=n$$

for

$$n=1,2,\ldots,25,$$

we get

$$\Theta_{\Phi_2,\varphi_{11}}(q)=\frac{1}{103}\sum_{n=1}^{\infty}\sum_{\Phi_2=n}(103x_1^2-13\Phi_2)q^n \quad (0.27)$$

$$=\frac{1}{103}(102q^2+204q^4+408q^8+1020q^{10}-676q^{13}-522q^{14}-1148q^{15}$$

$$-650q^{16}-130q^{18}-164q^{19}+2040q^{20}-660q^{21}-228q^{22}-372q^{23}$$

$$-436q^{24}-540q^{25}+\cdots),$$

,

$$\Theta_{\Phi_2,\varphi_{12}}(q)=\frac{1}{2\cdot 103}\sum_{n=1}^{\infty}\sum_{\Phi_2=n}(2\cdot 103x_1x_2+\Phi_2)q^n \quad (0.28)$$

$$=\frac{1}{103}(4q^2+8q^4+16q^8+40q^{10}+26q^{13}-178q^{14}+60q^{15}$$

$$-78q^{16}+520q^{18}-374q^{19}+80q^{20}-656q^{21}-324q^{22}+458q^{23}$$

$$+508q^{24}+924q^{25}+\cdots),$$

,

$$\Theta_{\Phi_2,\varphi_{22}}(q)=\frac{1}{103}\sum_{n=1}^{\infty}\sum_{\Phi_2=n}(103x_2^2-2\Phi_2)q^n, \quad (0.29)$$

$$=\frac{1}{103}(-16q^2-32q^4-64q^8-160q^{10}+102q^{13}+94q^{14}+172q^{15}+106q^{16}$$

$$-20q^{18}+54q^{19}-320q^{20}+152q^{21}+60q^{22}+22q^{23}+28q^{24}+12q^{25}+\cdots).$$

Similarly, by solving the equations

$$F_1\oplus\Phi_1=x_1^2+x_1x_2+26x_2^2+2x_3^2+x_3x_4+13x_4^2=n$$

for

$$n=1,2,\ldots,25,$$

we get

$$\Theta_{F_1\oplus\Phi_1,\varphi_{11}}(q)=\frac{1}{103}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Phi_1=n}(103x_1^2-26F_1\oplus\Phi_1)q^n \quad (0.30)$$

$$=\frac{1}{103}(154q-104q^2+100q^3+616q^4+1024q^6-416q^8+862q^9+2564q^{11}+400q^{12}$$

$$-676q^{13}-1772q^{14}-1148q^{15}+1632q^{16}+464q^{17}+3560q^{18}-2552q^{19}$$

$$-2100q^{20}+780q^{22}-624q^{23}+2012q^{24}+4958q^{25}+\cdots),$$

$$\Theta_{F_1\oplus\Phi_1,\varphi_{12}}(q)=\frac{1}{2\cdot 103}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Phi_1=n}(2\cdot 103x_1x_2+F_1\oplus\Phi_1)\,q^n \quad (0.31)$$

$$=\frac{1}{103}(q+2q^2+6q^3+4q^4+12q^6+8q^8+27q^9+22q^{11}+24q^{12}$$

$$+13q^{13}+42q^{14}+30q^{15}+32q^{16}+102q^{17}+90q^{18}+57q^{19}$$

$$+80q^{20}+88q^{22}+115q^{23}+96q^{24}+75q^{25}+\cdots),$$

$$\Theta_{F_1\oplus\Phi_1,\varphi_{22}}(q)=\frac{1}{103}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Phi_1=n}(103x_2^2-F_1\oplus\Phi_1)\,q^n \quad (0.32)$$

$$=\frac{1}{103}(-2q-4q^2-12q^3-8q^4-24q^6-16q^8-54q^9-44q^{11}-48q^{12}$$

$$-26q^{13}-84q^{14}-60q^{15}-64q^{16}-204q^{17}-180q^{18}-114q^{19}$$

$$-160q^{20}-176q^{22}-230q^{23}-192q^{24}-150q^{25}+\cdots),$$

$$\Theta_{F_1\oplus\Phi_1,\varphi_{33}}(q)=\frac{1}{103}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Phi_1=n}(103x_3^2-13F_1\oplus\Phi_1)\,q^n \quad (0.33)$$

$$=\frac{1}{103}(-26q+154q^2+256q^3-104q^4+100q^6+616q^8+946q^9-160q^{11}+1024q^{12}$$

$$-338q^{13}-886q^{14}-368q^{15}-626q^{16}-592q^{17}+338q^{18}+3050q^{19}$$

$$-20q^{20}+1420q^{22}-106q^{23}+800q^{24}-1538q^{25}+\cdots),$$

$$\Theta_{F_1\oplus\Phi_1,\varphi_{34}}(q)=\frac{1}{2\cdot 103}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Phi_1=n}(2\cdot 103x_3x_4+F_1\oplus\Phi_1)\,q^n \quad (0.1)$$

$$=\frac{1}{103}(q+2q^2+6q^3+4q^4+12q^6+8q^8+27q^9+22q^{11}+24q^{12}$$

$$+13q^{13}-164q^{14}-382q^{15}+238q^{16}+514q^{17}-322q^{18}-355q^{19}$$

$$-332q^{20}+88q^{22}-709q^{23}+920q^{24}+487q^{25}+\cdots),$$

and

$$\Theta_{F_1\oplus\Phi_1,\varphi_{44}}(q)=\frac{1}{103}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Phi_1=n}(103x_4^2-2F_1\oplus\Phi_1)\,q^n \quad (0.2)$$

$$= \frac{1}{103}(-4q - 8q^2 - 24q^3 - 16q^4 - 48q^6 - 32q^8 - 108q^9 - 88q^{11} - 96q^{12} + 154q^{13}$$

$$+450q^{14} + 292q^{15} + 78q^{16} + 416q^{17} + 52q^{18} - 22q^{19} + 504q^{20}$$

$$+60q^{22} + 570q^{23} + 28q^{24} + 112q^{25} + \cdots).$$

Similarly, by solving the equations

$$\Psi_2 = 4x_1^2 + 3x_1x_2 + 7x_2^2 + 4x_3^2 + 3x_3x_4 + 7x_4^2 = n$$

for

$$n = 1,2, \dots, 25,$$

**Proof.** we get

$$\Theta_{\Psi_2,\varphi_{11}}(q) = \frac{1}{103}\sum_{n=1}^{\infty}\sum_{\Psi_2=n}(103x_1^2 - 7\Psi_2)\,q^n \qquad (0.3)$$

$$= \frac{1}{103}(94q^4 - 196q^7 + 170q^8 - 204q^{11} + 152q^{12} - 578q^{14}$$

$$-428q^{15} + 340q^{16} + 348q^{17} - 184q^{18} + 940q^{20}$$

$$+120q^{21} - 408q^{22} + 360q^{23} + 1020q^{24} + 660q^{25} + \cdots),$$

$$\Theta_{\Psi_2,\varphi_{12}}(q) = \frac{1}{2\cdot 103}\sum_{n=1}^{\infty}\sum_{\Psi_2=n}(2\cdot 103x_1x_2 + 3\Psi_2)\,q^n \qquad (0.4)$$

$$= \frac{1}{103}(24q^4 + 42q^7 - 110q^8 + 132q^{11} - 268q^{12} + 374q^{14}$$

$$-232q^{15} - 220q^{16} - 310q^{17} + 628q^{18} + 240q^{20}$$

$$+92q^{21} + 264q^{22} + 276q^{23} - 660q^{24} - 936q^{25} + \cdots),$$

$$\Theta_{\Psi_2,\varphi_{22}}(q) = \frac{1}{103}\sum_{n=1}^{\infty}\sum_{\Psi_2=n}(103x_2^2 - 4\Psi_2)\,q^n \qquad (0.5)$$

$$= \frac{1}{103}(-64q^4 + 94q^7 - 50q^8 + 60q^{11} + 28q^{12} + 170q^{14}$$

$$344q^{15} - 100q^{16} - 66q^{17} - 164q^{18} - 640q^{20}$$

$$-108q^{21} + 120q^{22} - 324q^{23} - 300q^{24} + 24q^{25} + \cdots).$$

Similarly, by solving the equations

$$F_1 \oplus \Psi_1 = x_1^2 + x_1x_2 + 26x_2^2 + 4x_3^2 + 3x_3x_4 + 7x_4^2 = n$$

for

$$n = 1,2, \dots, 25,$$

we get

$$\Theta_{F_1\oplus\Psi_1,\varphi_{11}}(q) = \frac{1}{103}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Psi_1=n}(103x_1^2 - 26F_1 \oplus \Psi_1)\,q^n \qquad (0.6)$$

$$= \frac{1}{103}(154q + 408q^4 - 108q^5 - 364q^7 - 20q^8 + 862q^9 + 504q^{11} + 400q^{12}$$

$$+2356q^{13} - 728q^{14} - 1148q^{15} + 3676q^{16} - 300q^{17} - 1684q^{18} + 4080q^{20}$$

$$-536q^{21}+5516q^{23}+4096q^{24}+4958q^{25}+\cdots),$$

$$\Theta_{F_1\oplus\Psi_1,\varphi_{12}}(q)=\frac{1}{2\cdot 103}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Psi_1=n}\left(2\cdot 103x_1x_2+F_1\oplus\Psi_1\right)q^n \qquad (0.7)$$

$$=\frac{1}{103}(q+8q^4+10q^5+7q^7+40q^8+27q^9+22q^{11}+24q^{12}+26q^{13}$$

$$+14q^{14}+30q^{15}+64q^{16}+85q^{17}+72q^{18}+80q^{20}+42q^{21}+92q^{23}$$

$$+48q^{24}+75q^{25}+\cdots),$$

$$\Theta_{F_1\oplus\Psi_1,\varphi_{22}}(q)=\frac{1}{103}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Psi_1=n}\left(103x_3^2-(F_1\oplus\Psi_1)\right) \qquad (0.8)$$

$$=\frac{1}{103}(-2q-16q^4-20q^5-14q^7-80q^8-54q^9-44q^{11}-48q^{12}$$

$$-52q^{13}-28q^{14}-60q^{15}-128q^{16}-170q^{17}-144q^{18}-160q^{20}$$

$$-84q^{21}-184q^{23}-96q^{24}-150q^{25}+\cdots),$$

$$\Theta_{F_1\oplus\Psi_1,\varphi_{33}}(q)=\frac{1}{103}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Psi_1=n}\left(103x_3^2-7(F_1\oplus\Psi_1)\right) \qquad (0.9)$$

$$=\frac{1}{103}(-14q+94q^4+272q^5-98q^7+58q^8+34q^9-308q^{11}+76q^{12}$$

$$+48q^{13}+10q^{14}-8q^{15}-72q^{16}+1694q^{17}+1052q^{18}+940q^{20}$$

$$+1060q^{21}-876q^{23}-260q^{24}+598q^{25}+\cdots),$$

$$\Theta_{F_1\oplus\Psi_1,\varphi_{34}}(q)=\frac{1}{2\cdot 103}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Psi_1=n}\left(2\cdot 103x_3x_4+(F_1\oplus\Psi_1)\right) \qquad (0.10)$$

$$=\frac{1}{103}(q+8q^4+10q^5+7q^7-166q^8-385q^9+22q^{11}-388q^{12}+26q^{13}$$

$$+220q^{14}+442q^{15}+64q^{16}-739q^{17}-340q^{18}+80q^{20}$$

$$-782q^{21}+504q^{23}-364q^{24}+75q^{25}+\cdots),$$

and

$$\Theta_{F_1\oplus\Psi_1,\varphi_{44}}(q)=\frac{1}{103}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Psi_1=n}\left(103x_4^2-4(F_1\oplus\Psi_1)\right) \qquad (0.11)$$

$$=\frac{1}{103}(-8q-64q^4-80q^5+150q^7+298q^8+196q^9+236q^{11}+220q^{12}-208q^{13}$$

$$+94q^{14}+172q^{15}-100q^{16}-62q^{17}+248q^{18}-640q^{20}$$

$$+76q^{21}+88q^{23}+28q^{24}-600q^{25}+\cdots).$$

Similarly, by solving the equations

$$\Phi_1 \oplus \Psi_1 = 2x_1^2 + x_1x_2 + 13x_2^2 + 4x_3^2 + 3x_3x_4 + 7x_4^2 = n$$

for

$$n = 1,2, \dots, 25,$$

we get

$$\Theta_{\Phi_1 \oplus \Psi_1, \varphi_{11}}(q) = \frac{1}{103}\sum_{n=1}^{\infty}\sum_{F_1 \oplus \Psi_1 = n}\left(103x_1^2 - 13(\Phi_1 \oplus \Psi_1)\right)q^n \quad (0.12)$$

$$= \frac{1}{103}(154q^2 - 104q^4 + 100q^6 - 182q^7 + 408q^8 - 56q^9 - 108q^{10} + 1024q^{12}$$
$$-338q^{13} - 522q^{14} + 868q^{15} - 230q^{16} - 1326q^{17} + 338q^{18} - 246q^{19} - 1668q^{20}$$
$$-1772q^{21} + 2336q^{22} - 106q^{23} - 436q^{24} + 2756q^{25} + \cdots),$$

$$\Theta_{\Phi_1 \oplus \Psi_1, \varphi_{12}}(q) = \frac{1}{2 \cdot 103}\sum_{n=1}^{\infty}\sum_{\Phi_1 \oplus \Psi_1 = n}\left(2 \cdot 103x_1x_2 + (\Phi_1 \oplus \Psi_1)\right)q^n \quad (0.13)$$

$$= \frac{1}{103}(2q^2 + 4q^4 + 12q^6 + 7q^7 + 16q^8 + 18q^9 + 20q^{10} + 24q^{12} + 13q^{13}$$
$$-178q^{14} + 30q^{15} + 302q^{16} + 51q^{17} - 322q^{18} - 355q^{19} + 492q^{20}$$
$$-328q^{21} - 280q^{22} + 115q^{23} + 508q^{24} + 100q^{25} + \cdots),$$

$$\Theta_{\Phi_1 \oplus \Psi_1, \varphi_{22}}(q) = \frac{1}{103^2}\sum_{n=1}^{\infty}\sum_{\Phi_1 \oplus \Psi_1 = n}\left(103^2x_2^2 - 182(\Phi_1 \oplus \Psi_1)\right)q^n \quad (0.14)$$

$$= \frac{1}{103^2}(-728q^2 - 1456q^4 - 4368q^6 - 2548q^7 - 5824q^8 - 6552q^9 - 7280q^{10} - 8736q^{12}$$
$$+16486q^{13} + 11026q^{14} - 10920q^{15} - 13726q^{16} + 23872q^{17} + 9676q^{18} + 470q^{19}$$
$$+55752q^{20} + 54296q^{21} - 5612q^{22} + 64230q^{23} + 7492q^{24} - 36400q^{25} + \cdots),$$

$$\Theta_{\Phi_1 \oplus \Psi_1, \varphi_{23}}(q) = \frac{1}{103^2}\sum_{n=1}^{\infty}\sum_{\Phi_1 \oplus \Psi_1 = n}\left(103^2x_2x_3 + 28(\Phi_1 \oplus \Psi_1)\right)q^n \quad (0.15)$$

$$= \frac{1}{103^2}(112q^2 + 224q^4 + 672q^6 + 392q^7 + 896q^8 + 1008q^9 + 1120q^{10} + 1344q^{12} + 728q^{13}$$
$$+1568q^{14} + 1680q^{15} + 5376q^{16} + 2856q^{17} + 5040q^{18} + 3192q^{19} + 4480q^{20}$$
$$+4704q^{21} + 7392q^{22} + 6440q^{23} + 5376q^{24} + 5600q^{25} + \cdots),$$

$$\Theta_{\Phi_1 \oplus \Psi_1, \varphi_{33}}(q) = \frac{1}{103}\sum_{n=1}^{\infty}\sum_{\Phi_1 \oplus \Psi_1 = n}\left(103x_3^2 - 7(\Phi_1 \oplus \Psi_1)\right)q^n \quad (0.16)$$

$$=\frac{1}{103}(-28q^2+150q^4+244q^6-98q^7-18q^8-252q^9+132q^{10}+76q^{12}$$

$$-182q^{13}-186q^{14}-420q^{15}+304q^{16}+522q^{17}+800q^{18}+850q^{19}$$

$$-708q^{20}-764q^{21}-612q^{22}-1198q^{23}+716q^{24}+248q^{25}+\cdots),$$

$$\Theta_{\Phi_1\oplus\Psi_1,\varphi_{34}}(q)=\frac{1}{2\cdot 103}\sum_{n=1}^{\infty}\sum_{\Phi_1\oplus\Psi_1=n}\left(2\cdot 103x_3x_4+(\Phi_1\oplus\Psi_1)\right)q^n \quad (0.17)$$

$$=\frac{1}{103}(2q^2+4q^4+12q^6+7q^7-190q^8+18q^9-392q^{10}+24q^{12}+13q^{13}$$

$$+234q^{14}+30q^{15}+96q^{16}-361q^{17}+90q^{18}-767q^{19}+80q^{20}$$

$$-328q^{21}+132q^{22}+115q^{23}-316q^{24}-724q^{25}+\cdots),$$

and

$$\Theta_{\Phi_1\oplus\Psi_1,\varphi_{44}}(q)=\frac{1}{103}\sum_{n=1}^{\infty}\sum_{\Phi_1\oplus\Psi_1=n}\left(103x_4^2-4(\Phi_1\oplus\Psi_1)\right)q^n \quad (0.18)$$

$$=\frac{1}{103}(-8q^2-8q^4-16q^6+198q^7+190q^8+396q^9+396q^{10}-16q^{12}$$

$$-8q^{13}+190q^{14}+396q^{15}+776q^{16}+182q^{17}-40q^{18}+388q^{19}+380q^{20}$$

$$+792q^{21}+776q^{22}+372q^{23}+380q^{24}+792q^{25}+\cdots).$$

The rank of the matrix of these coefficients of the theta series is 20 and the set of theta series in the Theorem is a basis of the subspace of $S_4(\Gamma_0(103))$ spanned by the generalized theta series of the form 2.4 induced by spherical functions of the form 2.3.

## 1. The solutions of Q=n and the Generalized Theta Series associated to the Quadratic forms

By solving the equations

$$F_1=x_1^2+x_1x_2+26x_2^2=n,$$

$$\Phi_1=2x_1^2+x_1x_2+13x_2^2=n,$$

and

$$\Psi_1=4x_1^2+3x_1x_2+7x_2^2=n$$

for

$$n=1,2,\ldots,25,$$

we get

$$\Theta_{F_1}(q)=1+2q+2q^4+2q^9+2q^{16}+2q^{25}+\cdots,$$

$$\Theta_{\Phi_1}(q)=1+2q^2+2q^8+2q^{13}+2q^{14}+2q^{16}+2q^{18}+2q^{19}+2q^{23}+\cdots$$

$$\Theta_{\Psi_1}(q)=1+2q^4+2q^7+2q^8+2q^{14}+2q^{16}+2q^{17}+\cdots,$$

$$\Theta_{F_2}(q)=\Theta_{F_1}(q)\Theta_{F_1}(q)=1+4q+4q^2+4q^4+8q^5+4q^8+4q^9+8q^{10}+8q^{13}$$

$$+4q^{16}+8q^{17}+4q^{18}+8q^{20}+12q^{25}+\cdots,$$

$$\Theta_{F_3}(q)=\Theta_{F_2}(q)\Theta_{F_1}(q)=1+6q+12q^2+8q^3+6q^4+24q^5+24q^6+12q^8+30q^9+24q^{10}+24q^{11}$$

$$+8q^{12}+24q^{13}+48q^{14}+6q^{16}+48q^{17}+36q^{18}+24q^{19}+24q^{20}+48q^{21}+24q^{22}+24q^{24}+30q^{25}+\cdots,$$

$$\Theta_{F_4}(q)=\Theta_{F_2}(q)\Theta_{F_2}(q)=1+8q+24q^2+32q^3+24q^4+48q^5+96q^6+64q^7+24q^8$$

$$104q^9+144q^{10}+96q^{11}+96q^{12}+112q^{13}+192q^{14}+192q^{15}+24q^{16}+144q^{17}$$

$$+312q^{18}+160q^{19}+144q^{20}+256q^{21}+288q^{22}+192q^{23}+96q^{24}+248q^{25}+\cdots,$$

$$\Theta_{\Phi_2}(q)=\Theta_{\Phi_1}(q)\Theta_{\Phi_1}(q)=1+4q^2+4q^4+4q^8+8q^{10}+4q^{13}+4q^{14}$$

$$+8q^{15}+16q^{16}+12q^{18}+4q^{19}+8q^{20}+16q^{21}+8q^{22}+4q^{23}+8q^{24}+8q^{25}+\cdots,$$


$$\Theta_{F_4}(q)=\Theta_{F_2}(q)\Theta_{F_2}(q)=1+8q+24q^2+32q^3+24q^4+48q^5+96q^6+64q^7+24q^8$$

$$104q^9+144q^{10}+96q^{11}+96q^{12}+112q^{13}+192q^{14}+192q^{15}+24q^{16}+144q^{17}$$

$$+312q^{18}+160q^{19}+144q^{20}+256q^{21}+288q^{22}+192q^{23}+96q^{24}+248q^{25}+\cdots,$$

$$\Theta_{\Phi_2}(q)=\Theta_{\Phi_1}(q)\Theta_{\Phi_1}(q)=1+4q^2+4q^4+4q^8+8q^{10}+4q^{13}+4q^{14}$$

$$+8q^{15}+16q^{16}+12q^{18}+4q^{19}+8q^{20}+16q^{21}+8q^{22}+4q^{23}+8q^{24}+8q^{25}+\cdots,$$


$$\Theta_{\Phi_3}(q)=\Theta_{\Phi_2}(q)\Theta_{\Phi_1}(q)=1+6q^2+12q^4+8q^6+6q^8+24q^{10}+24q^{12}+6q^{13}$$

$$+6q^{14}+24q^{15}+42q^{16}+24q^{17}+78q^{18}+6q^{19}+48q^{20}+48q^{21}+48q^{22}+78q^{23}+80q^{24}+24q^{25}+\cdots,$$

$$\Theta_{\Phi_4}(q)=\Theta_{\Phi_2}(q)\Theta_{\Phi_2}(q)=1+8q^2+24q^{24}+32q^6+24q^8+48q^{10}+96q^{12}+8q^{13}+72q^{14}$$

$$+48q^{15}+80q^{16}+96q^{17}+248q^{18}+72q^{19}+304q^{20}+96q^{21}+208q^{22}+296q^{23}+336q^{24}+304q^{25}+\cdots,$$

$$\Theta_{\Psi_2}(q)=\Theta_{\Psi_1}(q)\Theta_{\Psi_1}(q)=1+4q^4+4q^7+8q^8+8q^{11}+8q^{12}$$

$$+8q^{14}+8q^{15}+8q^{16}+4q^{17}+8q^{18}+8q^{20}+16q^{21}+8q^{22}+8q^{23}+16q^{24}+8q^{25}+\cdots,$$

$$\Theta_{\Psi_3}(q)=\Theta_{\Psi_2}(q)\Theta_{\Psi_1}(q)=1+6q^4+6q^7+18q^8+24q^{11}+32q^{12}+18q^{14}+48q^{15}$$

$$+42q^{16}+6q^{17}+48q^{18}+48q^{19}+48q^{20}+56q^{21}+72q^{22}+48q^{23}+80q^{24}+96q^{25}+\cdots,$$

$$\Theta_{\Psi_4}(q)=\Theta_{\Psi_2}(q)\Theta_{\Psi_2}(q)$$
$$=1+8q^4+8q^7+32q^8+48q^{11}+80q^{12}+32q^{14}+144q^{15}$$

$$+144q^{16}+8q^{17}+144q^{18}+256q^{19}+208q^{20}+128q^{21}+336q^{22}+336q^{23}+320q^{24}+400q^{25}+\cdots,$$

$$\Theta_{F_1\oplus\Phi_1}(q)=\Theta_{F_1}(q)\Theta_{\Phi_1}(q)$$
$$=1+2q+2q^2+4q^3+2q^4+4q^6+2q^8+6q^9+4q^{11}+4q^{12}$$

$$+2q^{13}+6q^{14}+4q^{15}+6q^{16}+16q^{17}+10q^{18}+6q^{19}+12q^{20}+8q^{22}+10q^{23}+8q^{24}+10q^{25}+\cdots,$$

$$\Theta_{F_3\oplus\Phi_1}(q)=\Theta_{F_3}(q)\Theta_{\Phi_1}(q)$$
$$=1+6q+14q^2+20q^3+30q^4+40q^5+36q^6$$

$$48q^7+62q^8+42q^9+72q^{10}+100q^{11}+68q^{12}+122q^{13}+126q^{14}+84q^{16}$$

$$+148q^{17}+182q^{18}+294q^{19}+208q^{20}+256q^{21}+328q^{22}+266q^{23}+312q^{24}+322q^{25}+\cdots,$$

$$\Theta_{F_2\oplus\Phi_2}(q)=\Theta_{F_2}(q)\Theta_{\Phi_2}(q)$$
$$=1+4q+8q^2+16q^3+24q^4+24q^5+32q^6$$

$$+32q^7+24q^8+52q^9+48q^{10}+48q^{11}+96q^{12}+60q^{13}+84q^{14}+136q^{15}+84q^{16}$$

$$+168q^{17}+208q^{18}+180q^{19}+304q^{20}+272q^{21}+232q^{22}+356q^{23}+296q^{24}+356q^{25}+\cdots,$$

$$\Theta_{F_1\oplus\Phi_3}(q)=\Theta_{F_1}(q)\Theta_{\Phi_3}(q)$$
$$=1+2q+6q^2+12q^3+14q^{14}+24q^5+20q^6+16q^7+30q^8$$

$$+14q^9+40q^{10}+60q^{11}+36q^{12}+78q^{13}+66q^{14}+52q^{15}+146q^{16}$$

$$+132q^{17}+150q^{18}+258q^{19}+168q^{20}+240q^{21}+328q^{22}+198q^{23}+392q^{24}+366q^{25}+\cdots,$$

$$\Theta_{F_1\oplus\Psi_1}(q)=\Theta_{F_1}(q)\Theta_{\Psi_1}(q)$$
$$=1+2q+4q^4+4q^5+2q^7+10q^8+6q^9+4q^{11}+4q^{12}$$

$$+6q^{13}+6q^{14}+4q^{15}+8q^{16}+14q^{17}+8q^{18}+6q^{18}+8q^{20}+4q^{21}+4q^{22}+8q^{23}+4q^{24}+6q^{25}+\cdots,$$

$$\Theta_{F_3\oplus\Psi_1}(q)=\Theta_{F_3}(q)\Theta_{\Psi_1}(q)$$
$$=1+6q+12q^2+8q^3+8q^4+36q^5+48q^6+18q^7$$

$$+38q^8+114q^9+112q^{10}+52q^{11}+92q^{12}+180q^{13}+146q^{14}+84q^{15}+132q^{16}$$
$$+234q^{17}+276q^{18}+176q^{19}+176q^{20}+348q^{21}+312q^{22}+168q^{23}+252q^{24}+426q^{25}+\cdots,$$

$$\Theta_{F_2\oplus\Psi_2}(q) = \Theta_{F_2}(q)\Theta_{\Psi_2}(q) = 1 + 4q + 4q^2 + 8q^4 + 24q^5 + 16q^6 + 4q^7 + 44q^8$$

$$+84q^9 + 40q^{10} + 24q^{11} + 120q^{12} + 152q^{13} + 72q^{14} + 88q^{15} + 220q^{16} + 236q^{17}$$

$$+156q^{18} + 176q^{19} + 256q^{20} + 320q^{21} + 280q^{22} + 248q^{23} + 400q^{24} + 500q^{25} + \cdots,$$

$$\Theta_{F_1\oplus\Psi_3}(q) = \Theta_{F_1}(q)\Theta_{\Psi_3}(q) = 1 + 2q + 8q^4 + 12q^5 + 6q^7 + 42q^8 + 38q^9 + 36q^{11}$$

$$+116q^{12} + 76q^{13} + 18q^{14} + 132q^{15} + 216q^{16} + 126q^{17}$$

$$96q^{18} + 240q^{19} + 288q^{20} + 228q^{21} + 280q^{22} + 336q^{23} + 404q^{24} + 454q^{25} + \cdots,$$

$$\Theta_{\Phi_1\oplus\Psi_1}(q) = \Theta_{\Phi_1}(q)\Theta_{\Psi_1}(q) = 1 + 2q^2 + 2q^4 + 4q^6 + 2q^7 + 4q^8 + 4q^9 + 4q^{10} + 4q^{12} + 4q^{13} + 4q^{14}$$

$$+8q^{15} + 14q^{16} + 6q^{17} + 10q^{18} + 6q^{19} + 12q^{20} + 12q^{21} + 12q^{22} + 14q^{23} + 12q^{24} + 8q^{25} + \cdots,$$

$$\Theta_{\Phi_3\oplus\Psi_1}(q) = \Theta_{\Phi_3}(q)\Theta_{\Psi_1}(q) = 1 + 6q^2 + 14q^4 + 20q^6 + 2q^7 + 32q^8 + 12q^9 + 52q^{10}$$

$$+24q^{11} + 60q^{12} + 22q^{13} + 72q^{14} + 36q^{15} + 116q^{16} + 86q^{17} + 174q^{18}$$

$$+114q^{19} + 232q^{20} + 144q^{21} + 292q^{22} + 238q^{23} + 368q^{24} + 336q^{25} + \cdots,$$

$$\Theta_{\Phi_2\oplus\Psi_2}(q) = \Theta_{\Phi_2}(q)\Theta_{\Psi_2}(q) = 1 + 4q^2 + 8q^4 + 16q^6 + 4q^7 + 28q^8 + 16q^9 + 40q^{10}$$

$$+24q^{11} + 56q^{12} + 36q^{13} + 76q^{14} + 64q^{15} + 120q^{16} + 84q^{17} + 164q^{18}$$

$$+116q^{19} + 192q^{20} + 160q^{21} + 288q^{22} + 252q^{23} + 376q^{24} + 368q^{25} + \cdots,$$

$$\Theta_{\Phi_1\oplus\Psi_3}(q) = \Theta_{\Phi_1}(q)\Theta_{\Psi_3}(q) = 1 + 2q^2 + 6q^4 + 12q^6 + 6q^7 + 20q^8 + 12q^9 + 36q^{10}$$

$$+24q^{11} + 44q^{12} + 50q^{13} + 84q^{14} + 60q^{15} + 116q^{16} + 114q^{17} + 146q^{18}$$

$$+110q^{19} + 232q^{20} + 200q^{21} + 252q^{22} + 282q^{23} + 392q^{24} + 328q^{25} + \cdots,$$

$$\Theta_{F_2\oplus\Phi_1\oplus\Psi_1}(q) = \Theta_{F_2}(q)\cdot\Theta_{\Phi_1\oplus\Psi_1}(q) = 1 + 4q + 6q^2 + 8q^3 + 14q^4 + 16q^5 + 20q^6 + 34q^7$$

$$+40q^8 + 48q^9 + 68q^{10} + 80q^{11} + 76q^{12} + 82q^{13} + 108q^{14} + 100q^{15} + 120q^{16}$$

$$+182q^{17} + 190q^{18} + 214q^{19} + 240q^{20} + 256q^{21} + 292q^{22} + 314q^{23} + 336q^{24} + 380q^{25} + \cdots,$$

$$\Theta_{F_1\oplus\Phi_2\oplus\Psi_1}(q)=\Theta_{\Phi_2}(q)\cdot\Theta_{F_1\oplus\Psi_1}(q)=1+2q+4q^2+8q^3+8q^4+12q^5+16q^6+18q^7$$

$$+30q^8+38q^9+48q^{10}+52q^{11}+60q^{12}+64q^{13}+62q^{14}+92q^{15}+104q^{16}$$

$$+130q^{17}+172q^{18}+196q^{19}+208q^{20}+236q^{21}+272q^{22}+292q^{23}+372q^{24}+398q^{25}+\cdots,$$

$$\Theta_{F_1\oplus\Phi_1\oplus\Psi_2}(q)=\Theta_{\Psi_2}(q)\cdot\Theta_{F_1\oplus\Phi_1}(q)=1+2q+2q^2+4q^3+6q^4+8q^5+12q^6$$

$$+20q^7+26q^8+30q^9+48q^{10}+52q^{11}+52q^{12}+74q^{13}+94q^{14}+100q^{15}+116q^{16}+168q^{17}$$

$$+162q^{18}+158q^{19}+216q^{20}+240q^{21}+264q^{22}+314q^{23}+392q^{24}+414q^{25}+\cdots.$$

**Remark 1.**

It is known that

$$L_{\Theta_{F_1}}(s)=2\sum_{a \text{ integral ideal}\in A_{F_1}}N(a)^{-s},\ \text{where}$$

$A_{F_1}=\left[\left[1,\frac{1+i\sqrt{103}}{2}\right]\right]$ is the ideal class corresponding to $F_1$,

$$L_{\Theta_{\Phi_1}}(s)=2_{a \text{ integral ideal}\in A_{\Phi_1}}N(a)^{-s},\ \text{where}$$

$A_{\Phi_1}=\left[\left[2,1+\frac{1+i\sqrt{103}}{2}\right]\right]$ is the ideal class corresponding to $\Phi_1$,

$$L_{\Theta_{\Psi_1}}(s)=2\sum_{a \text{ integral ideal}\in A_{\Psi_1}}N(a)^{-s},\ \text{where}$$

$A_{\Psi_1}=\left[\left[4,-2+\frac{1+i\sqrt{103}}{2}\right]\right]$ is the ideal class corresponding to $\Psi_1$.

In general, if $\rho_k$ is a character of the class group of $\mathbb{Q}(\sqrt{-103})$, i.e.,

$$\rho_k(\Phi_1^m)=e^{2\pi imk/5}\ \text{for}\ m=0,1,2,3,4,k=0,1,2,3,4,$$

**Remark 2.** The Hecke L-functions for these characters will be

$$L_{\rho_k}(s)=L_{\Theta_{F_1}}(s)+\rho_k(\Phi_1)L_{\Theta_{\Phi_1}}(s)+\rho_k(\Phi_1^2)L_{\Theta_{\Psi_1'}}(s)+\rho_k(\Phi_1^3)L_{\Theta_{\Psi_1}}(s)+\rho_k(\Phi_1^4)L_{\Theta_{\Phi_1'}}(s)$$

$$=L_{\Theta_{F_1}}(s)+\big(\rho_k(\Phi_1)+\rho_k(\Phi_1^4)\big)L_{\Theta_{\Phi_1}}(s)+\big(\rho_k(\Phi_1^2)+\rho_k(\Phi_1^3)\big)L_{\Theta_{\Psi_1}}(s)$$

$$L_{\Theta_{F_1}}(s)+(\rho_k(\Phi_1)+\rho_k(\Phi_1)^{-1})L_{\Theta_{\Phi_1}}(s)+(\rho_k(\Phi_1^2)+\rho_k(\Phi_1^2)^{-1})L_{\Theta_{\Psi_1}}(s)$$

which are the L-functions of weight 1 modular forms

$$f_{\rho_k}(q)=\frac{1}{2}\Big(\Theta_{F_1}(q) + \big(\rho_k(\Phi_1)+\rho_k(\Phi_1')\big)\Theta_{\Phi_1}(q) + \big(\rho_k(\Psi_1') + \rho_k(\Psi_1)\big)\Theta_{\Psi_1}(q)\Big)$$

for $k=0,1,2,3,4$.

Because of the unique factorization of the ideals in $\mathbb{Q}(\sqrt{-103})$, these L functions has Euler product, hence the modular forms $f_{\rho_k}$ are Hecke eigenforms.In particular, the L function of the Hecke eigenform of weight 1

$$f_{\rho_0}(q):=\frac{1}{2}\Big(\Theta_{F_1}(q)+2\Theta_{\Phi_1}(q)+2\Theta_{\Psi_1}(q)\Big)$$

$$=\frac{1}{2}((1+2q+2q^4+2q^9+2q^{16}+2q^{25}+\cdots)+$$

$$2(1+2q^2+2q^8+2q^{13}+2q^{14}+2q^{16}+2q^{16}+2q^{18}+2q^{19}+2q^{23}+\cdots)$$

$$+2(1+2q^4+2q^7+2q^8+2q^{14}+2q^{13}+2q^{16}+2q^{17}+\cdots))$$

$$=\frac{5}{2}+q+4q^2+5q^4+4q^7+8q^8+8q^{13}+8q^{14}+9q^{16}+4q^{17}+4q^{18}+\cdots$$

is exactly the Dedekind zeta function of $\mathbb{Q}(\sqrt{-103})$ which is equal to

$$\zeta(s)L(\chi,s),$$

where $\chi(d)=\left(\frac{-103}{d}\right)$ is the natural Dirichlet character associated with the quadratic field $\mathbb{Q}(\sqrt{-103})$[7].Infact, $f_{\rho_0}$ is the Eisenstein series of weight 1. From here, we simply obtain the formula for the sum of the number of representations of an integer n as

$$r(F_1,n)+r(\Phi_1,n)+r(\Phi_1',n)+r(\Psi_1,n)+r(\Psi_1',n)$$

$$=r(F_1,n)+2r(\Phi_1,n)+2r(\Psi_1,n)$$

$$=2\sum_{d|n}\left(\frac{-103}{d}\right),$$

where $\left(\frac{D}{d}\right)$ is the Kronecker symbol.

By [7], it is also known that for $m=0,1,2,3,4$,

$$\sum_{k=0}^{4}\rho_k(\Phi_1^0)L_{\rho_k}(s)=\frac{2\cdot 5}{2}\left(\frac{\sqrt{103}}{2}\right)^{-s}\zeta(2s)\frac{\pi^s}{\Gamma(s)}E\left(-\frac{1}{2}+\frac{i\sqrt{103}}{2},s\right)$$

$$\sum_{k=0}^{4}\rho_k(\Phi_1)L_{\rho_k}(s)=\frac{2\cdot 5}{2}\left(\frac{\sqrt{103}}{2}\right)^{-s}\zeta(2s)\frac{\pi^s}{\Gamma(s)}E\left(-\frac{1}{4}+\frac{i\sqrt{103}}{4},s\right)$$

$$\sum_{k=0}^{4}\rho_k(\Phi_1^2)L_{\rho_k}(s)=\frac{2\cdot 5}{2}\left(\frac{\sqrt{103}}{2}\right)^{-s}\zeta(2s)\frac{\pi^s}{\Gamma(s)}E\left(\frac{3}{8}+\frac{i\sqrt{103}}{8},s\right)$$

$$\sum_{k=0}^{4}\rho_k(\Phi_1^3)L_{\rho_k}(s)=\frac{2\cdot 5}{2}\left(\frac{\sqrt{103}}{2}\right)^{-s}\zeta(2s)\frac{\pi^s}{\Gamma(s)}E\left(-\frac{3}{8}+\frac{i\sqrt{103}}{8},s\right)$$

$$\sum_{k=0}^{4} \rho_k(\Phi_1^4) L_{\rho_k}(s) = \frac{2\cdot 5}{2}\left(\frac{\sqrt{103}}{2}\right)^{-s} \zeta(2s) \frac{\pi^s}{\Gamma(s)} E\left(\frac{1}{4} + \frac{i\sqrt{103}}{4}, s\right),$$

where

$$E(z,s) = \frac{1}{2}\pi^{-s}\Gamma(s) \sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}\setminus\{(0,0)\}} \frac{y^s}{|mz+n|^{2s}} \quad z \in H, s \in \mathbb{C}$$

is the nonholomorphic eisenstein series. $L_{\rho_k}(s)$ has analytic continuation to the whole complex plane as an entire function except for a simple pole at 1 for $\rho_0$ with

$$res_{s=1} L_{\rho_0} = res_{s=1}\zeta_K = \frac{2\pi\cdot 5}{2\sqrt{103}} = \frac{5\pi}{\sqrt{103}}$$

and its completion, i.e.,

$$\Lambda_{\rho_k}(s) = (2\pi)^{-s}\Gamma(s) 103^{s/2} L_{\rho_k}(s)$$

is equal to

$$= \zeta(2s)(\rho_k(\Phi_1^0) E\left(-\frac{1}{2} + \frac{i\sqrt{103}}{2}, s\right) + \rho_k(\Phi_1) E\left(-\frac{1}{4} + \frac{i\sqrt{103}}{4}, s\right) + \rho_k(\Phi_1^2) E\left(\frac{3}{8} + \frac{i\sqrt{103}}{8}, s\right)$$

$$+\rho_k(\Phi_1^3) E\left(-\frac{3}{8} + \frac{i\sqrt{103}}{8}, s\right) + \rho_k(\Phi_1^4) E\left(\frac{1}{4} + \frac{i\sqrt{103}}{4}, s\right))$$

and satisfies the functional equation

$$\Lambda_{\rho_k}(s) = \Lambda_{\rho_k}(1-s) \text{ for all } s \neq 1.$$

## 1. Representation Numbers of n

**Theorem 1.** The difference between the following theta series of the quadratic forms

$$\Theta_{F_4}(q) = \Theta_{F_2}(q)\Theta_{F_2}(q) = 1 + 8q + 24q^2 + 32q^3 + 24q^4 + 48q^5 + 96q^6 + 64q^7 + 24q^8$$

$$104q^9 + 144q^{10} + 96q^{11} + 96q^{12} + 112q^{13} + 192q^{14} + 192q^{15} + 24q^{16} + 144q^{17}$$

$$+312q^{18} + 160q^{19} + 144q^{20} + 256q^{21} + 288q^{22} + 192q^{23} + 96q^{24} + 248q^{25} + \cdots,$$

$$\Theta_{\Phi_4}(q) = \Theta_{\Phi_2}(q)\Theta_{\Phi_2}(q) = 1 + 8q^2 + 24q^{24} + 32q^6 + 24q^8 + 48q^{10} + 96q^{12} + 8q^{13} + 72q^{14}$$

$$+48q^{15} + 80q^{16} + 96q^{17} + 248q^{18} + 72q^{19} + 304q^{20} + 96q^{21} + 208q^{22} + 296q^{23} + 336q^{24} + 304q^{25} + \cdots,$$

$$\Theta_{\Psi_4}(q) = \Theta_{\Psi_2}(q)\Theta_{\Psi_2}(q) = 1 + 8q^4 + 8q^7 + 32q^8 + 48q^{11} + 80q^{12} + 32q^{14} + 144q^{15}$$

$$+144q^{16} + 8q^{17} + 144q^{18} + 256q^{19} + 208q^{20} + 128q^{21} + 336q^{22} + 336q^{23} + 320q^{24} + 400q^{25} + \cdots,$$

$$\Theta_{F_3\oplus\Phi_1}(q) = \Theta_{F_3}(q)\Theta_{\Phi_1}(q) = 1 + 6q + 14q^2 + 20q^3 + 30q^4 + 40q^5 + 36q^6$$

$$48q^7 + 62q^8 + 42q^9 + 72q^{10} + 100q^{11} + 68q^{12} + 122q^{13} + 126q^{14} + 168q^{16}$$

$$+148q^{17}+182q^{18}+294q^{19}+208q^{20}+256q^{21}+328q^{22}+266q^{23}+312q^{24}+322q^{25}+\cdots,$$

$$\Theta_{F_2\oplus\Phi_2}(q)=\Theta_{F_2}(q)\Theta_{\Phi_2}(q)=1+4q+8q^2+16q^3+24q^4+24q^5+32q^6$$

$$+32q^7+24q^8+52q^9+48q^{10}+48q^{11}+96q^{12}+60q^{13}+84q^{14}+136q^{15}+84q^{16}$$

$$+168q^{17}+208q^{18}+180q^{19}+304q^{20}+272q^{21}+232q^{22}+356q^{23}+296q^{24}+356q^{25}+\cdots,$$

$$\Theta_{F_1\oplus\Phi_3}(q)=\Theta_{F_1}(q)\Theta_{\Phi_3}(q)=1+2q+6q^2+12q^3+14q^{14}+24q^5+20q^6+16q^7+30q^8$$

$$+14q^9+40q^{10}+60q^{11}+36q^{12}+78q^{13}+66q^{14}+52q^{15}+140q^{16}$$

$$+132q^{17}+150q^{18}+258q^{19}+168q^{20}+240q^{21}+328q^{22}+198q^{23}+392q^{24}+366q^{25}+\cdots,$$

$$\Theta_{F_3\oplus\Psi_1}(q)=\Theta_{F_3}(q)\Theta_{\Psi_1}(q)=1+6q+12q^2+8q^3+8q^4+36q^5+48q^6+18q^7$$
$$+38q^8+114q^9+112q^{10}+52q^{11}+92q^{12}+180q^{13}+146q^{14}+84q^{15}+132q^{16}$$

$$+234q^{17}+276q^{18}+176q^{19}+176q^{20}+348q^{21}+312q^{22}+168q^{23}+252q^{24}+426q^{25}+\cdots,$$

$$\Theta_{F_2\oplus\Psi_2}(q)=\Theta_{F_2}(q)\Theta_{\Psi_2}(q)=1+4q+4q^2+8q^4+24q^5+16q^6+4q^7+44q^8$$

$$+84q^9+40q^{10}+24q^{11}+120q^{12}+152q^{13}+72q^{14}+88q^{15}+220q^{16}+236q^{17}$$

$$+156q^{18}+176q^{19}+256q^{20}+320q^{21}+280q^{22}+248q^{23}+400q^{24}+500q^{25}+\cdots,$$

$$\Theta_{F_1\oplus\Psi_3}(q)=\Theta_{F_1}(q)\Theta_{\Psi_3}(q)=1+2q+8q^4+12q^5+6q^7+42q^8+38q^9+36q^{11}$$

$$+116q^{12}+76q^{13}+18q^{14}+132q^{15}+216q^{16}+126q^{17}$$

$$96q^{18}+240q^{19}+288q^{20}+228q^{21}+280q^{22}+336q^{23}+404q^{24}+454q^{25}+\cdots,$$

$$\Theta_{\Phi_3\oplus\Psi_1}(q)=\Theta_{\Phi_3}(q)\Theta_{\Psi_1}(q)=1+6q^2+14q^4+20q^6+2q^7+32q^8+12q^9+52q^{10}$$

$$+24q^{11}+60q^{12}+22q^{13}+72q^{14}+36q^{15}+116q^{16}+86q^{17}+174q^{18}$$

$$+114q^{19}+232q^{20}+144q^{21}+292q^{22}+238q^{23}+368q^{24}+336q^{25}+\cdots,$$

$$\Theta_{\Phi_2\oplus\Psi_2}(q)=\Theta_{\Phi_2}(q)\Theta_{\Psi_2}(q)=1+4q^2+8q^4+16q^6+4q^7+28q^8+16q^9+40q^{10}$$

$$+24q^{11}+56q^{12}+36q^{13}+76q^{14}+64q^{15}+120q^{16}+84q^{17}+164q^{18}$$

$$+116q^{19}+192q^{20}+160q^{21}+288q^{22}+252q^{23}+376q^{24}+368q^{25}+\cdots,$$

$$\Theta_{\Phi_1\oplus\Psi_3}(q)=\Theta_{\Phi_1}(q)\Theta_{\Psi_3}(q)=1+2q^2+6q^4+12q^6+6q^7+20q^8+12q^9+36q^{10}$$

$$+24q^{11}+44q^{12}+50q^{13}+84q^{14}+60q^{15}+116q^{16}+114q^{17}+146q^{18}$$

$$+110q^{19}+232q^{20}+200q^{21}+252q^{22}+282q^{23}+392q^{24}+328q^{25}+\cdots,$$

$$\Theta_{F_2\oplus\Phi_1\oplus\Psi_1}(q)=\Theta_{F_2}(q)\cdot\Theta_{\Phi_1\oplus\Psi_1}(q)=1+4q+6q^2+8q^3+14q^4+16q^5+20q^6+34q^7$$

$$+40q^8+48q^9+68q^{10}+80q^{11}+76q^{12}+82q^{13}+108q^{14}+100q^{15}+120q^{16}$$

$$+182q^{17}+190q^{18}+214q^{19}+240q^{20}+256q^{21}+292q^{22}+314q^{23}+336q^{24}+380q^{25}+\cdots,$$

$$\Theta_{F_1\oplus\Phi_2\oplus\Psi_1}(q)=\Theta_{\Phi_2}(q)\cdot\Theta_{F_1\oplus\Psi_1}(q)=1+2q+4q^2+8q^3+8q^4+12q^5+16q^6+18q^7$$

$$+30q^8+38q^9+48q^{10}+52q^{11}+60q^{12}+64q^{13}+62q^{14}+92q^{15}+104q^{16}$$

$$+130q^{17}+172q^{18}+196q^{19}+208q^{20}+236q^{21}+272q^{22}+292q^{23}+372q^{24}+398q^{25}+\cdots,$$

$$\Theta_{F_1\oplus\Phi_1\oplus\Psi_2}(q)=\Theta_{\Psi_2}(q)\cdot\Theta_{F_1\oplus\Phi_1}(q)=1+2q+2q^2+4q^3+6q^4+8q^5+12q^6$$

$$+20q^7+26q^8+30q^9+48q^{10}+52q^{11}+52q^{12}+74q^{13}+94q^{14}+100q^{15}+116q^{16}+168q^{17}$$

$$+162q^{18}+158q^{19}+216q^{20}+240q^{21}+264q^{22}+314q^{23}+392q^{24}+414q^{25}+\cdots,$$

and the Eisenstein series

$$E(q:F_4)=\cdots=E(q:F_1\oplus\Phi_1\oplus\Psi_2)=1+\frac{24}{1061}q+\frac{24\cdot 9}{1061}q^2$$

$$+\frac{24\cdot 28}{1061}q^3+\frac{24\cdot 73}{1061}q^4+\frac{24\cdot 126}{1061}q^5+\frac{24\cdot 252}{1061}q^6$$

$$+\frac{24\cdot 344}{1061}q^7+\frac{24\cdot 585}{1061}q^8+\frac{24\cdot 757}{1061}q^9+\frac{24\cdot 1134}{1061}q^{10}+\frac{24\cdot 1332}{1061}q^{11}$$

$$+\frac{24\cdot 2044}{1061}q^{12}+\frac{24\cdot 2198}{1061}q^{13}+\frac{24\cdot 3096}{1061}q^{14}+\frac{24\cdot 3528}{1061}q^{15}+\frac{24\cdot 4681}{1061}q^{16}$$

$$+\frac{24\cdot 4914}{1061}q^{17}+\frac{24\cdot 6813}{1061}q^{18}+\frac{24\cdot 6860}{1061}q^{19}+\frac{24\cdot 9198}{1061}q^{20}+\frac{24\cdot 9632}{1061}q^{21}$$

$$+\frac{24\cdot 11988}{1061}q^{22}+\frac{24\cdot 12168}{1061}q^{23}+\frac{24\cdot 16380}{1061}q^{24}+\frac{24\cdot 15751}{1061}q^{25}+\cdots$$

are linear combinations of the 20 generalized theta series mentioned in Theorem [1]. The coefficients are given in TABLE_of_COEFFICIENTS103 in[9].

**Proof.** Let's see the situation in the case:

$$\Theta_{F_4}(q)-E(q:F_4)=c_1\Theta_{F_2,\varphi_{11}}(q)+c_2\Theta_{\Phi_2,\varphi_{11}}(q)+c_3\Theta_{\Phi_2,\varphi_{12}}(q)+$$

$$c_4\Theta_{F_1\oplus\Phi_1,\varphi_{11}}(q)+c_5\Theta_{F_1\oplus\Phi_1,\varphi_{12}}(q)+c_6\Theta_{F_1\oplus\Phi_1,\varphi_{33}}(q)+c_7\Theta_{F_1\oplus\Phi_1,\varphi_{34}}(q)$$

$$+c_8\Theta_{\Psi_2,\varphi_{11}}(q)+c_9\Theta_{\Psi_2,\varphi_{12}}(q)+c_{10}\Theta_{F_1\oplus\Psi_1,\varphi_{11}}(q)+c_{11}\Theta_{F_1\oplus\Psi_1,\varphi_{12}}(q)$$

$$+c_{12}\Theta_{F_1\oplus\Psi_1,\varphi_{33}}(q)+c_{13}\Theta_{F_1\oplus\Psi_1,\varphi_{34}}(q)+c_{14}\Theta_{\Phi_1\oplus\Psi_1,\varphi_{11}}(q)+c_{15}\Theta_{\Phi_1\oplus\Psi_1,\varphi_{12}}(q)+c_{16}\Theta_{\Phi_1\oplus\Psi_1,\varphi_{22}}(q)$$

$$+c_{17}\Theta_{\Phi_1\oplus\Psi_1,\varphi_{23}}(q)+c_{18}\Theta_{\Phi_1\oplus\Psi_1,\varphi_{33}}(q)+c_{19}\Theta_{\Phi_1\oplus\Psi_1,\varphi_{34}}(q)+c_{20}\Theta_{\Phi_1\oplus\Psi_1,\varphi_{44}}(q).$$

By equating the coefficients of $q^n$ in both sides for $n=1,2,3,\cdots,25$, we see that there exist solutions of the equations in coefficients.We repeat the same procedure for the other cases. At the end, by solving 25 linear equations in 20 variables we get all the coefficients.We also observe that $c_{20}$ is always zero.

## Conclusion

The representation numbers for the quadratic forms

$$H=F_4,\Phi_4,\Phi_3\oplus\Phi_1',\Phi_2\oplus\Phi_2',\Phi_1\oplus\Phi_3',\Phi_4',\Psi_4,\Psi_3\oplus\Psi_1',\Psi_2\oplus\Psi_2',\Psi_1\oplus\Psi_3',\Psi_4',$$

$$F_3\oplus\Phi_1,F_3\oplus\Phi_1',F_2\oplus\Phi_2,F_2\oplus\Phi_1\oplus\Phi_1',F_2\oplus\Phi_2',F_1\oplus\Phi_3,F_1\oplus\Phi_2\oplus\Phi_1',$$

$$F_1\oplus\Phi_1\oplus\Phi_2',F_1\oplus\Phi_3',F_3\oplus\Psi_1,F_3\oplus\Psi_1',F_2\oplus\Psi_2,F_2\oplus\Psi_1\oplus\Psi_1',F_2\oplus\Psi_2',F_1\oplus\Psi_3,$$

$$F_1\oplus\Psi_2\oplus\Psi_1',F_1\oplus\Psi_1\oplus\Psi_2',F_1\oplus\Psi_3',\Phi_3\oplus\Psi_1,\Phi_2\oplus\Phi_1'\oplus\Psi_1,\Phi_1\oplus\Phi_2'\oplus\Psi_1,$$

$$\Phi_3'\oplus\Psi_1,\Phi_3\oplus\Psi_1',\Phi_2\oplus\Phi_1'\oplus\Psi_1',\Phi_1\oplus\Phi_2'\oplus\Psi_1',\ \Phi_3'\oplus\Psi_1',\Phi_2\oplus\Psi_2,\Phi_2'\oplus\Psi_2,$$

$$\Phi_2\oplus\Psi_2',\Phi_2'\oplus\Psi_2',\Phi_1\oplus\Phi_1'\oplus\Psi_2,\Phi_1\oplus\Phi_1'\oplus\Psi_1\oplus\Psi_1',\Phi_1\oplus\Phi_1'\oplus\Psi_2',\Phi_2\oplus\Psi_1\oplus\Psi_1',$$

$$\Phi_2'\oplus\Psi_1\oplus\Psi_1',\Psi_3\oplus\Phi_1,\Psi_2\oplus\Psi_1'\oplus\Phi_1,\Psi_1\oplus\Psi_2'\oplus\Phi_1,\Psi_3'\oplus\Phi_1,\Psi_3\oplus\Phi_1',\Psi_2\oplus\Psi_1'\oplus\Phi_1',$$

$$\Psi_1\oplus\Psi_2'\oplus\Phi_1',\ \Psi_3'\oplus\Phi_1',F_2\oplus\Phi_1\oplus\Psi_1,\ F_2\oplus\Phi_1'\oplus\Psi_1,\ F_2\oplus\Phi_1\oplus\Psi_1',\ F_2\oplus\Phi_1'\oplus\Psi_1',$$

$$F_1 \oplus \Phi_2 \oplus \Psi_1,\ F_1 \oplus \Phi_2' \oplus \Psi_1,\ F_1 \oplus \Phi_2 \oplus \Psi_1',\ F_1 \oplus \Phi_2' \oplus \Psi_1', F_1 \oplus \Phi_1 \oplus \Phi_1' \oplus \Psi_1,$$

$$F_1 \oplus \Phi_1 \oplus \Phi_1' \oplus \Psi_1', F_1 \oplus \Phi_1 \oplus \Psi_2,\ F_1 \oplus \Phi_1' \oplus \Psi_2,\ F_1 \oplus \Phi_1 \oplus \Psi_2',\ F_1 \oplus \Phi_1' \oplus \Psi_2',$$

$$F_1 \oplus \Phi_1 \oplus \Psi_1 \oplus \Psi_1', F_1 \oplus \Phi_1' \oplus \Psi_1 \oplus \Psi_1',$$

are

$$r(n;H) = \frac{24}{1061}\sigma_3^*(n) + \frac{c_1}{103}\sum_{F_2=n}(103x_1^2 - 26n) + \frac{c_2}{103}\sum_{\Phi_2=n}(103x_1^2 - 13n)$$

$$+\frac{c_3}{2\cdot 103}\sum_{\Phi_2=n}(2\cdot 103x_1x_2 + n) + \frac{c_4}{103}\sum_{F_1\oplus\Phi_1=n}(103x_1^2 - 26n)$$

$$+\frac{c_5}{2\cdot 103}\sum_{F_1\oplus\Phi_1=n}(2\cdot 103x_1x_2 + n) + \frac{c_6}{103}\sum_{F_1\oplus\Phi_1=n}(103x_3^2 - 13n)$$

$$+\frac{c_7}{2\cdot 103}\sum_{F_1\oplus\Phi_1=n}(2\cdot 103x_3x_4 + n) + \frac{c_8}{103}\sum_{\Psi_2=n}(103x_1^2 - 7n)$$

$$+\frac{c_9}{2\cdot 103}\sum_{\Psi_2=n}(2\cdot 103x_1x_2 + n) + +\frac{c_{10}}{103}\sum_{F_1\oplus\Psi_1=n}(103x_1^2 - 26n)$$

$$+\frac{c_{11}}{2\cdot 103}\sum_{F_1\oplus\Psi_1=n}(2\cdot 103x_1x_2 + n) + \frac{c_{12}}{103}\sum_{F_1\oplus\Psi_1=n}(103x_3^2 - 7n)$$

$$+\frac{c_{13}}{206}\sum_{F_1\oplus\Psi_1=n}(206x_3x_4 + n) + \frac{c_{14}}{103}\sum_{\Phi_1\oplus\Psi_1=n}(103x_1^2 - 13n)$$

$$+\frac{c_{15}}{2\cdot 103}\sum_{\Phi_1\oplus\Psi_1=n}(2\cdot 103x_1x_2 + n) + \frac{c_{16}}{103^2}\sum_{\Phi_1\oplus\Psi_1=n}(103^2x_2^2 - 182n)$$

$$+\frac{c_{17}}{103^2}\sum_{\Phi_1\oplus\Psi_1=n}(103^2x_2x_3 + 28n) + \frac{c_{18}}{103}\sum_{\Phi_1\oplus\Psi_1=n}(103x_3^2 - 7n)$$

$$+\frac{c_{19}}{2\cdot 103}\sum_{\Phi_1\oplus\Psi_1=n}(2\cdot 103x_3x_4 + n)$$

the coefficients $\begin{bmatrix} c_1 = \\ c_2 = \\ c_3 = \\ c_4 = \\ c_5 = \\ c_6 = \\ c_7 = \\ c_8 = \\ c_9 = \\ c_{10} = \\ c_{11} = \\ c_{12} = \\ c_{13} = \\ c_{14} = \\ c_{15} = \\ c_{16} = \\ c_{17} = \\ c_{18} = \\ c_{19} = \end{bmatrix}$ corresponding to the quadratic form *H* are given in the TABLE_of_COEFFICIENTS103 in [9]

**Proof.** It follows from the preceding Theorem.

## Conclusion

S_4 (Γ_0 (103))'s basis of a subspace has been found via generalized theta series associated to the direct sums of the quadratic forms $x_1^2 + x_1x_2 + 26x_2^2$, $2x_1^2 \pm x_1x_2 + 13x_2^2$, $4x_1^2 \pm 3x_1x_2 + 7x_2^2$ .

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