

Design Equations from Geometric Programming

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Abstract

Geometric programming is an optimization tool that permits the development of design relationships. Most researchers do not develop the design relationships, but only solve the specific optimization problem for a set of specific input parameters and a new solution must be developed for any changes.

For some problems with few degrees of difficulty, design relations can be developed which given an insight into the importance of the input constants. An example from a previous paper using the Cobb-Douglas production function is used to illustrate the development of design relationships.

Keywords: *Geometric Programming, Design Equation Development, Cobb-Douglas production function*

1. Introduction

Clarence Zener is credited as being the father of geometric programming with the publishing of the paper "A mathematical aid in optimizing engineering designs" in the Proceedings of the National Academy of Science[1] in 1961. He is better known for the invention of the Zener diode. He later co-authored with Richard Duffin and Elmor Peterson the book "Geometric Programming"[2] in 1967 published by John Wiley. Several books have been written about geometric programming, but few consider or emphasize the development of design equations.

The mathematics of geometric programming are rather complex and presented in more detail in the references presented[3-5]. Geometric programming is similar to linear programming in that it has both a primal and

a dual formulation. The primal problem formulation is somewhat similar to the primal formulation in linear programming, and is often solved by traditional search methods. The dual formulation is harder to formulate, but is much easier to solve. The

design equations can be found by utilizing the primal-dual relationships. The example presented will be with zero degrees of difficulty to illustrate the solution procedure for finding the design equations. It is easier to determine the design equations for cost models than it is for profit models.

The example presented is that of Ibrahim Guney and Ersoy Oz in the paper "An Application of Geometric Programming"[6] in Vol. 2 of the Internal Journal of Electronics, Mechanical and Mathematics

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Engineering. This example concerns the minimization of production costs for a fixed production level using the Cobb-Douglas production function.

The basic formulations of the primal and dual will be shown and then the example will be presented following the steps of the formulations. One of the requirements for geometric programming is that the terms used are posynomials, that is, they are positive polynomials. That prohibits functions such as the $\sin(x)$ and fractional powers that cannot be expanded, such as $(2 + 4x)^{3.3}$.

2. Primal and Dual Formulations

The primal problem is formulated as:

$$Y_m(x) = \sum_{t=1}^{T_m} \sigma_{mt} C_{mt} \prod_{n=1}^N x^{mtn} \quad \text{for } m=0,1,2..M \quad (1)$$

where

$\sigma_{mt} = \pm 1$ (signum function to indicate sign of

term)

$C_{mt} > 0$ positive constant coefficients

$Y_m(x) \leq 1$ for the constraints, $m=1,2,... M$

$Y_0(x) =$ objective function

The dual formulation initially appears more complex, but it results in several linear equations which are easier to solve. The dual objective function is not linear and is solved after the dual variables have been determined from the dual formulation model. The dual objective function is :

$$d(\omega) = \sigma \left[\prod_{m=0}^M \prod_{t=1}^{T_m} (C_{mt} \omega_{m0} / \omega_{mt})^{\sigma_{mt} \omega_{mt}} \right]^{\sigma} \quad (2)$$

for $m = 0,1,2,...M$ and $t = 1,2,...T_m$

where

$\sigma =$ signum function for objective function (1 for minimization and -1 for maximization)

$\sigma_{mt} =$ signum function for dual constraints (± 1)

$C_{mt} > 0$ positive constant coefficients

$\omega_{m0} =$ dual variables from the linear inequality

constraints

$\omega_{mt} =$ dual variables of dual constraints

$\sigma_{mt} =$ signum function for dual constraints

$\omega_{00} = 1$

The dual is formulated from four conditions

First, a normality condition is expressed by:

$$\sum_{t=1}^{T_m} \sigma_{0t} \omega_{0t} = \sigma \quad \text{where } \sigma = \pm 1 \quad (3)$$

and

$\sigma_{0t} =$ signum of dual objective function terms
 $\omega_{0t} =$ dual variables for dual objective function terms

The second conditions are the N orthogonal conditions

$$M \quad T_m$$

$$\sum_{m=0} \sum_{t=1} \sigma_{mt} a_{mnt} \omega_{mt} = 0 \quad (4)$$

where

σ_{mt} = signum of constraint term

a_{mnt} = exponent of design variable term in primal

ω_{mt} = dual variable of dual constraint

The third condition is the T non-negativity conditions that require that the dual variables must not be negative, that is:

$$\omega_{mt} \geq 0 \text{ for } m=0,1,2,..M \text{ and } t=1,2,3,..T_m \quad (5)$$

The fourth condition is the M linear inequality constraints expressed by:

$$\omega_{m0} = \sigma_m \sum_{t=1}^{T_m} \sigma_{mt} \omega_{mt} \geq 0 \quad (6)$$

The complexity of a problem is indicated by the number of degrees of difficulty(D). The higher the degree of difficulty, the more difficult the problem is to solve. The formula for determining the degrees of difficulty is:

$$D = T - (N + 1) \quad (7)$$

where

T = number of terms in the primal formulation

N=number of orthogonal conditions (which is equivalent to the number of primal variables)

Once the dual variables are determined, the primal variables can be determined from the relationships between the primal and dual variables. As in linear programming, the primal and dual objective functions must be equal and thus $Y_0(x)$ and $d(\omega)$ are equal. The two equations relating the primal and dual for determining the primal variables are:

$$C_{0t} \prod_{n=1}^N x_n^{mnt} = \omega_{0t} \sigma d(\omega) \quad (8)$$

and

$$C_{mt} \prod_{n=1}^N x_n^{mnt} = \omega_{mt} / \omega_{m0} \quad (9)$$

for $t=1,2,..T_m$ and $m = 1,2,..M$

3. Cobb-Douglas Cost Minimization Model

The initial formulation is to minimize labor and capital costs to obtain a specific output level. The model by Guney and Oz() is slightly modified and can be stated in its primal form as:

$$Y(x) = r_1 x_1 + r_2 x_2 \quad (10)$$

subject to the Cobb-Douglas production constraint

$$q = A x_1^\alpha x_2^\beta \quad (11)$$

where

x_1 = labor amount

r_1 = labor rate

x_2 = capital amount
 r_2 = capital rate
 q = desired output level
 A = total productivity factor
 α = labor elasticity
 β = capital elasticity

The constraints must be written in the form of inequalities with the right hand side being unity and thus the constraint becomes:

$$(q/A) x_1^{-\alpha} x_2^{-\beta} \leq 1 \quad (12)$$

Thus the primal objective function is given in Eqn. 10 and Eqn 12.is the constraint. The number of degrees of difficulty using Eqn 7 is:

$$D = 3 - (2 + 1) = 0$$

The degrees of freedom must be greater than or equal to zero.

Since all the terms in Eqns. 10 and 12 are positive, the signum values are all positive, that is

$$\begin{aligned}
 \sigma_{00} &= 1 \text{ (objective function is minimization)} \\
 \sigma_{01} &= 1 \\
 \sigma_{02} &= 1 \\
 \sigma_{11} &= 1 \\
 \sigma_{10} &= 1 \text{ (RHS of constraint is positive)}
 \end{aligned}$$

The dual can be formulated using Eqns.3, 4 and 6 as:

$$\text{Eqn 3} \quad \omega_{01} + \omega_{02} = 1 \quad (13)$$

$$\text{Eqn 4 (} x_1 \text{)} \quad \omega_{01} - \alpha \omega_{11} = 0 \quad (14)$$

$$\text{Eqn 4 (} x_2 \text{)} \quad \omega_{01} - \beta \omega_{11} = 0 \quad (15)$$

Solving Eqns 13-15 for the dual variables one obtains:

$$\omega_{01} = (\alpha / (\alpha + \beta)) \quad (16)$$

$$\omega_{02} = (\beta / (\alpha + \beta)) \quad (17)$$

$$\omega_{11} = (1 / (\alpha + \beta)) \quad (18)$$

Now ω_{10} can be determined using Eqn 6 and is

$$\begin{aligned}
 \omega_{10} &= \sigma_{10} \sum \sigma_{mt} \omega_{mt} = 1 * (1 * (1/(\alpha + \beta))) \\
 &= 1/(\alpha + \beta) \quad (19)
 \end{aligned}$$

The dual objective function of Eqn 2 can now be determined and is:

$$\begin{aligned}
 d(\omega) &= 1 * [\{r_1 * 1/(\alpha(\alpha+\beta))\}^{(1*(\alpha/(\alpha+\beta)))} * \\
 &\{r_2 * 1/(\beta(\alpha+\beta))\}^{(1*(\beta/(\alpha+\beta)))} * \{q/A\}^{(1*/(\alpha+\beta))}] \\
 &(20)
 \end{aligned}$$

Note that both ω_{11} and ω_{10} are equal. Now using the primal-dual relationship of Eqn. 8 for the two terms of the objective function, one obtains:

$$r_1 x_1 = (\alpha(\alpha + \beta)) * 1 * d(\omega) \quad (21)$$

$$r_2 x_2 = (\beta(\alpha + \beta)) * 1 * d(\omega) \quad (22)$$

Solving Eqns. 21 and 22 for x_1 one obtains:

$$x_1 = (\alpha/\beta) (r_2 / r_1) x_2 \quad (23)$$

Now using Eqn 22 in Eqn 11 and solving for x_2 :

$$x_2 = (q/A)^{1/(\alpha + \beta)} (\alpha r_2 / \beta r_1)^{-\alpha/(\alpha + \beta)} \quad (24)$$

Using Eqn 24 in Eqn 21 x_1 is found to be

$$x_1 = (q/A)^{1/(\alpha + \beta)} (\alpha r_2 / \beta r_1)^{\beta/(\alpha + \beta)} \quad (25)$$

The primal objective function can now be determined from the primal variables and Eqn 10 becomes

$$Y(x) = r_1 * (q/A)^{1/(\alpha + \beta)} (\alpha r_2 / \beta r_1)^{\beta/(\alpha + \beta)} + r_2 * (q/A)^{1/(\alpha + \beta)} (\alpha r_2 / \beta r_1)^{-\alpha/(\alpha + \beta)} \quad (26)$$

Equations 20 and 26 have quite different appearances, but the numerical values will be the same.

4. Model Results and Validation

The equations developed were used to compare with the data reported by Guney and Oz[6] on the construction sector in Turkey. The input values are given in Table 1 and the output values are in Table 2. The input values of A , α , and β were fixed at 1.0, 0.53, and 0.47 for all four reported cases.

Table 1. Yearly Input Data for Estimate Calculations

Year	Production Index q	Labor Index r_1	Capital Index r_2
2006	118.4	121.88	114.32
2007	124.9	137.80	122.32
2008	115.6	153.85	140.06
2009	96.4	158.53	131.48

The results for x_1 and x_2 , the labor and capital estimates, were in complete agreement with those of Guney and Oz[6] so only one set are included in results given in Table 2. The primal and dual values of the objective function from Equations 20 and 26 are identical as expected and although the objective function was not given in the reference[6], it would most likely have been the same. In the model equations presented, the values for A , α , and β from the Cobb-Douglas production equation could be varied and a sensitivity analysis of these parameters could be evaluated and would not require resolving the adjusted problem.

Table 2. *Out put Values of Model for Estimates.*

Year	Labor Estimate x_1	Capital Estimate x_2	Primal Total Cost (Y)	Dual Total Cost $d(\omega)$
2006	121.56	114.93	27955.11	27955.11
2007	124.96	124.84	32488.98	32488.98
2008	117.03	114.000	33973.08	33973.08
2008	93.41	99.88	27941.39	27941.39

5. Conclusions

The development of design equations for a geometric programming model of from data of the construction sector in Turkey. These design equations (Eqns. 20,24,25,and 26) give the solutions for the model outputs and the model does not need to be resolved. The design equations for the variables x_1 and x_2 can then be used to determine the total cost to meet the desired production level.

The design equations also permit easy analysis of the impact of the Cobb-Douglas elasticity exponents and total productivity factor upon the total cost. The development of design equations takes considerable effort, but the equations permit a more rapid analysis of the impact of the input variables upon the output.

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