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Research Article

Smarandache Curves of Involute-Evolute Curve According to Frenet Frame

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Abstract: In this paper, the invariants of the Smarandache curves, which consist of Frenet vectors of the involute curve, are calculated in terms of the evolute curve.

Keywords: Curvature, evolute curve, Frenet frame, involute curve, Smarandache curves, torsion.

1. Introduction and Preliminaries

A regular curve in Minkowski space-time, whose position vector is composed of Frenet frame vectors on another regular curve, is called a Smarandache curve [17]. Special Smarandache curves have been studied by some authors. Turgut and Yılmaz's article deals with interesting knowledge of special Smarandache curves in the space $\mathbb{E}_1^4$. For example, they obtained another orthonormal frame [17]. In the light of the reference [17], Ali adapted Smarandache curve to regular curves in the $\mathbb{E}^3$ [2]. Ergüt et al. defined the isotropic types of Smarandache curves. Then they examined these kinds of isotropic Smarandache curves according to the Cartan frame in the complex 4-space [6]. By using the Darboux frame, Bektaş and Yüce obtained the results about Smarandache curves [4]. In another study, they studied the spatial quaternionic curve and the relationship between Frenet frames of the involute curve of the spatial quaternionic curve which are expressed by using the angle between the Darboux vector and binormal vector [15]. Şenyurt et al. used special curves as a base to create Smarandache curves, and then studied their geometric properties [12–14]. Al-Dayal and Solouma study some properties of spacelike Smarandache curves regarding Bishop frame of a spacelike curve in Minkowski 3-space [1]. There are many studies about Smarandache curves [9, 11, 16]

Huygens discovered an involute-evolute curve while trying to build a more accurate clock. The involute of a curve is a well-known concept in the Euclidean space [7, 8, 10]. The involute-

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evolute curve has attracted mathematicians' attention. In [5], authors found the relationships between the Frenet frames of the timelike curve and the spacelike involute curve. In another study, Bishop curvatures of the involute-evolute curve were examined and some important results were found [3].

In this paper, the invariants of the Smarandache curves, which consist of Frenet vectors of the involute curve, are calculated in terms of the evolute curve.

The inner product can be given by

$$\langle\ ,\rangle = x_1^2 + x_2^3 + x_3^2,$$

where $(x_1, x_2, x_3) \in \mathbb{E}^3$. Let $\alpha : I \to \mathbb{E}^3$ be a unit speed curve with the moving Frenet frame $\{V_1(s), V_2(s), V_3(s)\}$ the moving Frenet frame. For an arbitrary curve $\alpha \in \mathbb{E}^3$, with first and second curvature k_1 and k_2, respectively, the Frenet formulas are given by [7]

$$\begin{cases} V_1{}'(s) = k_1(s)V_2(s), \\ V_2{}'(s) = -k_1(s)V_1(s) + k_2(s)V_3(s), \\ V_3{}'(s) = -k_2(s)V_2(s). \end{cases}$$

For any unit speed curve $\alpha : I \to \mathbb{E}^3$, the vector W is called Darboux vector defined by

$$W = k_2V_1 + k_1V_3.$$

If we consider the normalization of the Darboux vector $C = \frac{1}{\|W\|}W$, we have

$$\sin\varnothing = \frac{k_2}{\|W\|}, \quad \cos\varnothing = \frac{k_1}{\|W\|}$$

and

$$C = \sin\varnothing V_1 + \cos\varnothing V_3,$$

where $\langle W, V_3\rangle = \varnothing$.

Theorem 1.1 *[10] Let the Frenet frames of α and α^* be $\{V_1(s), V_2(s), V_3(s)\}$ and $\{V_1{}^*(s), V_2{}^*(s), V_3{}^*(s)\}$ respectively. The relations between the Frenet frames are as follows;*

$$\begin{cases} V_1{}^*(s) = V_2(s), \\ V_2{}^*(s) = -\cos\varnothing V_1(s) + \sin\varnothing V_3(s), \\ V_3{}^*(s) = \sin\varnothing V_1(s) + \cos\varnothing V_3(s). \end{cases} \tag{1}$$

Definition 1.2 *[7] Let unit speed regular curve $\alpha : I \to \mathbb{E}^3$ and $\alpha^* : I \to \mathbb{E}^3$ be given. For all $s \in I$, the curve α^* is called the involute of the curve α if the tangent at the point $\alpha(s)$ to the curve α passes through the tangent at the point $\alpha^*(s)$ to the curve α^* and $\langle V_1(s), V_1{}^*(s)\rangle = 0$. The curve α is called the evolute curve.*

Theorem 1.3 *[7] The distance between corresponding points of the evolute-involute curve in* $\mathbb{E}^3$ *is, for all* $s \in I$

$$d\big(\alpha(s), \alpha^*(s)\big) = |c - s|,$$

where c *is a constant.*

Theorem 1.4 *[10] Let* (α, α^*) *be a evolute-involute curves in* $\mathbb{E}^3$*. For the curvatures and the torsions of the evolute-involute curve* (α, α^*) *we have*

$$\begin{cases} {k_1}^* = \dfrac{\sqrt{{k_1}^2 + {k_2}^2}}{(c-s)k_1}, \lambda = c - s, \\ \\ {k_2}^* = \dfrac{k_1 {k_2}' - {k_1}' k_2}{(c-s)k_1({k_1}^2 + {k_2}^2)}. \end{cases}$$

2. Smarandache Curves of Involute-Evolute Curve Couple According to Frenet Frame

In this subsection, special Smarandache curves belonging to involute curves such as ${V_1}^*{V_2}^*$, ${V_2}^*{V_3}^*, {V_1}^*{V_3}^*$ and ${V_1}^*{V_2}^*{V_3}^*$ drawn by Frenet frame are defined. Curvatures and torsions of involute curves are expressed depending upon the evolute curve and some related results are given.

Definition 2.1 *Let* (α, α^*) *be a evolute-involute curves in* $\mathbb{E}^3$*.* ${V_1}^*{V_2}^*$ *- Smarandache curve can be defined by*

$$\beta_1(s) = \frac{1}{\sqrt{2}}({V_1}^* + {V_2}^*).$$

If equation (1) is taken into account, the above expression is

$$\beta_1(s) = \frac{-\cos\varnothing V_1 + V_2 + \sin\varnothing V_3}{\sqrt{2}}. \tag{2}$$

Theorem 2.2 *Frenet vectors of Smarandache curve* β_1 *are given as follows;*

$$T_{\beta_1} = \frac{(\varnothing' \sin\varnothing - k_1)V_1 - \|W\|V_2 + (\varnothing' \cos\varnothing + k_2)V_3}{\sqrt{\varnothing'^2 + 2\|W\|^2}}, \tag{3}$$

$$N_{\beta_1} = \frac{\bar{\omega_1} V_1 + \bar{\omega_2} V_2 + \bar{\omega_3} V_3}{\sqrt{\bar{\omega_1}^2 + \bar{\omega_2}^2 + \bar{\omega_3}^2}},$$

$$B_{\beta_1} = \frac{(-\|W\|\bar{\omega_3} - (\varnothing' \cos\varnothing + k_2)\bar{\omega_2})V_1 + (\bar{\omega_1}(\varnothing' \cos\varnothing + k_2) - \bar{\omega_3}(\varnothing' \sin\varnothing - k_1))V_2}{\sqrt{(\varnothing'^2 + 2\|W\|^2)(\bar{\omega_1} + \bar{\omega_2} + \bar{\omega_3})}}$$

$$+ \frac{(\bar{\omega_2}(\varnothing' \sin\varnothing - k_1) + \bar{\omega_1}\|W\|)V_3}{\sqrt{(\varnothing'^2 + 2\|W\|^2)(\bar{\omega_1} + \bar{\omega_2} + \bar{\omega_3})}}.$$

Here, the coefficients are

$$\begin{cases} \bar{\omega_1} = (\varnothing'' \sin\varnothing + \varnothing'^2 \cos\varnothing - k_1{}' + k_1\|W\|)\sqrt{\varnothing'^2 + 2\|W\|^2} \\ \quad -(\varnothing' \sin\varnothing - k_1)\left(\sqrt{\varnothing'^2 + 2\|W\|^2}\right)', \\ \bar{\omega_2} = (-\|W\|^2 - \|W\|')\sqrt{\varnothing'^2 + 2\|W\|^2} + \|W\|\left(\sqrt{\varnothing'^2 + 2\|W\|^2}\right)', \\ \bar{\omega_3} = (\varnothing'' \cos\varnothing - \varnothing'^2 \sin\varnothing + k_2{}' - k_2\|W\|)\sqrt{\varnothing'^2 + 2\|W\|^2} \\ \quad -(\varnothing' \cos\varnothing + k_2)\left(\sqrt{\varnothing'^2 + 2\|W\|^2}\right)'. \end{cases}$$

Proof The derivative of the equation (2) is

$$\beta_1' = T_{\beta_1}\frac{ds_{\beta_1}}{ds} = \frac{(\varnothing' \sin\varnothing - k_1)V_1 - \|W\|V_2 + (\varnothing' \cos\varnothing + k_2)V_3}{\sqrt{2}}.$$

By taking the norm of the above equation, we can write

$$\frac{ds_{\beta_1}}{ds} = \sqrt{\frac{\varnothing'^2 + 2\|W\|^2}{2}}.$$

If necessary operations are taken, the tangent vector is

$$T_{\beta_1}(s) = \frac{(\varnothing' \sin\varnothing - k_1)V_1 - \|W\|V_2 + (\varnothing' \cos\varnothing + k_2)V_3}{\sqrt{\varnothing'^2 + 2\|W\|^2}}. \tag{4}$$

In the light of the pieces of information, the principal normal and the binormal vectors are respectively given by

$$N_{\beta_1} = \frac{\bar{\omega_1}V_1 + \bar{\omega_2}V_2 + \bar{\omega_3}V_3}{\sqrt{\bar{\omega_1}^2 + \bar{\omega_2}^2 + \bar{\omega_3}^2}},$$

$$B_{\beta_1} = \frac{(-\|W\|\bar{\omega_3} - (\varnothing' \cos\varnothing + k_2)\bar{\omega_2})V_1 + (\bar{\omega_1}(\varnothing' \cos\varnothing + k_2) - \bar{\omega_3}(\varnothing' \sin\varnothing - k_1))V_2}{\sqrt{(\varnothing'^2 + 2\|W\|^2)(\bar{\omega_1} + \bar{\omega_2} + \bar{\omega_3})}}$$

$$+\frac{(\bar{\omega_2}(\varnothing' \sin\varnothing - k_1) + \bar{\omega_1}\|W\|)V_3}{\sqrt{(\varnothing'^2 + 2\|W\|^2)(\bar{\omega_1} + \bar{\omega_2} + \bar{\omega_3})}}.$$

□

Theorem 2.3 *Curvature and torsion belonging to Smarandache curve β_1 are, respectively*

$$k_{1\beta_1} = \frac{\sqrt{2}}{(\varnothing'^2+2\|W\|^2)^{\frac{3}{2}}}\Bigg(\Big((\varnothing''\sin\varnothing+\varnothing'^2\cos\varnothing-k_1'+k_1\|W\|)\sqrt{\varnothing'^2+2\|W\|^2}$$
$$-(\varnothing'\sin\varnothing-k_1)\Big(\sqrt{\varnothing'^2+2\|W\|^2}\Big)'\Big)^2+\Big((-\|W\|^2-\|W\|')\sqrt{\varnothing'^2+2\|W\|^2}$$
$$+\|W\|\Big(\sqrt{\varnothing'^2+2\|W\|^2}\Big)'\Big)^2+\Big((\varnothing''\cos\varnothing-\varnothing'^2\sin\varnothing+k_2'-k_2\|W\|)$$
$$\sqrt{\varnothing'^2+2\|W\|^2}-(\varnothing'\cos\varnothing+k_2).\Big(\sqrt{\varnothing'^2+2\|W\|^2}\Big)'\Big)^2\Bigg)^{\frac{1}{2}},$$

$$k_{2\beta_1} = \frac{\begin{array}{l}\sqrt{2}\Big[\bar{\nu_1}(\varnothing'''\sin\varnothing+3\varnothing'\varnothing''\cos\varnothing-\varnothing'^3\sin\varnothing-k_1''+k_1'\|W\|+2k_1\|W\|' \\ +k_1\|W\|^2)+\bar{\nu_2}(\varnothing'^2\|W\|-k_1k_1'-k_2k_2'+\|W\|^3-2\|W\|\|W\|'+\|W\|'') \\ +\bar{\nu_3}(\varnothing'''\cos\varnothing-3\varnothing'\varnothing''\sin\varnothing-\varnothing'^3\cos\varnothing+k_2''-k_2'\|W\|-2k_2\|W\|'-k_2\|W\|^2)\Big]\end{array}}{\bar{\nu_1}^2+\bar{\nu_2}^2+\bar{\nu_3}^2},$$

where

$$\begin{cases}\bar{\nu_1}=-\|W\|(\varnothing''\cos\varnothing-\varnothing'^2\sin\varnothing+k_2'-k_2\|W\|)+(\|W\|^2+\|W\|')(\varnothing'\cos\varnothing+k_2),\\ \bar{\nu_2}=(\varnothing'\cos\varnothing+k_2)(\varnothing''\sin\varnothing+\varnothing'^2\cos\varnothing-k_1'+k_1\|W\|)-(\varnothing'\sin\varnothing-k_1)(\varnothing''\cos\varnothing \\ \qquad -\varnothing'^2\sin\varnothing+k_2'-k_2\|W\|),\\ \bar{\nu_3}=(\varnothing'\sin\varnothing-k_1)(-\|W\|^2-\|W\|')+\|W\|(\varnothing''\sin\varnothing+\varnothing'^2\cos\varnothing-k_1'+k_1\|W\|).\end{cases}$$

Proof The first curvature is

$$k_{1\beta_1}=\|T'_{\beta_1}\|. \tag{5}$$

Taking the derivative of the equation (4), we obtain

$$T'_{\beta_1}(s)=\sqrt{2}\frac{\bar{\omega_1}V_1+\bar{\omega_2}V_2+\bar{\omega_3}V_3}{(\varnothing'^2+2\|W\|^2)^{\frac{3}{2}}}. \tag{6}$$

If the expression (6) is written in (5), the first curvature is

$$k_{1\beta_1}=\|T'_{\beta_1}\|=\sqrt{2}\frac{\sqrt{\bar{\omega_1}^2+\bar{\omega_2}^2+\bar{\omega_3}^2}}{(\varnothing'^2+2\|W\|^2)^{\frac{3}{2}}}.$$

If the coefficients are written instead, the desired result is obtained.

To calculate the torsion of the curve β_1, we differentiate

$$\beta_1'' = \left(-\cos\varnothing\left(\left(\frac{\|W\|}{(c-s)k_1}\right)' - \frac{\|W\|^2}{(c-s)^2{k_1}^2} - \frac{(k_1{k_2}' - {k_1}'k_2)^2}{(c-s)^2{k_1}^2\|W\|^4}\right)\right.$$

$$+\sin\varnothing\Big(\frac{\|W\|(k_1{k_2}' - {k_1}'k_2)}{(c-s)^2{k_1}^2\|W\|^4} + \big(\frac{(k_1{k_2}' - {k_1}'k_2)}{(c-s)k_1\|W\|^2}\big)'\Big)\Bigg)V_1 + \left(\frac{\|W\|^2}{(c-s)^2{k_1}^2}\right.$$

$$+\left(\frac{\|W\|}{(c-s)k_1}\right)'\Bigg)V_2 + \left(\sin\varnothing\left(\left(\frac{\|W\|}{(c-s)k_1}\right)' - \frac{\|W\|^2}{(c-s)^2{k_1}^2} - \frac{(k_1{k_2}' - {k_1}'k_2)^2}{(c-s)^2{k_1}^2\|W\|^4}\right)\right.$$

$$+\cos\varnothing\left(\frac{\|W\|(k_1{k_2}' - {k_1}'k_2)}{(c-s)^2{k_1}^2\|W\|^4} + \left(\frac{(k_1{k_2}' - {k_1}'k_2)}{(c-s)k_1\|W\|^2}\right)'\right)\Bigg)V_3$$

and thus

$$\beta_1''' = \frac{(-\bar{\eta}_2\cos\varnothing + \bar{\eta}_3\sin\varnothing)V_1 + \bar{\eta}_1V_2 + (\bar{\eta}_2\sin\varnothing + \bar{\eta}_3\cos\varnothing)V_3}{\sqrt{2}},$$

where

$$\begin{cases} \bar{\eta}_1 = \varnothing'''\sin\varnothing + 3\varnothing'\varnothing''\cos\varnothing - {\varnothing'}^3\sin\varnothing - {k_1}'' + {k_1}'\|W\| + 2k_1\|W\|' + k_1\|W\|^2, \\ \bar{\eta}_2 = {\varnothing'}^2\|W\| - k_1{k_1}' - k_2{k_2}' + \|W\|^3 - 2\|W\|\|W\|' + \|W\|'', \\ \bar{\eta}_3 = \varnothing'''\cos\varnothing - 3\varnothing'\varnothing''\sin\varnothing - {\varnothing'}^3\cos\varnothing + {k_2}'' - {k_2}'\|W\| - 2k_2\|W\|' - k_2\|W\|^2. \end{cases}$$

The torsion is then given by

$$k_{2_{\beta_1}} = \frac{det(\beta_1', \beta_1'', \beta_1''')}{\|\beta_1' \wedge \beta_1''\|^2} \ ,$$

$$k_{2_{\beta_1}} = \frac{\begin{array}{l}\sqrt{2}\Big[\bar{\nu}_1(\varnothing'''\sin\varnothing + 3\varnothing'\varnothing''\cos\varnothing - {\varnothing'}^3\sin\varnothing - {k_1}'' + {k_1}'\|W\| + 2k_1\|W\|' \\ +k_1\|W\|^2) + \bar{\nu}_2({\varnothing'}^2\|W\| - k_1{k_1}' - k_2{k_2}' + \|W\|^3 - 2\|W\|\|W\|' \\ +\|W\|'') + \bar{\nu}_3(\varnothing'''\cos\varnothing - 3\varnothing'\varnothing''\sin\varnothing - {\varnothing'}^3\cos\varnothing + {k_2}'' - {k_2}'\|W\| \\ -2k_2\|W\|' - k_2\|W\|^2)\Big]\end{array}}{{\bar{\nu}_1}^2 + {\bar{\nu}_2}^2 + {\bar{\nu}_3}^2}.$$

□

Definition 2.4 *Let (α, α^*) be a evolute-involute curves in $\mathbb{E}^3$. ${V_2}^*{V_3}^*$ - Smarandache curve can be defined by*

$$\beta_2(s) = \frac{1}{\sqrt{2}}({V_2}^* + {V_3}^*).$$

If equation (1) is taken into account, the above expression is

$$\beta_2(s) = \frac{(\sin\varnothing - \cos\varnothing)V_1 + (\sin\varnothing + \cos\varnothing)V_3}{\sqrt{2}}.$$

Theorem 2.5 *The Frenet invariants of the β_2 curve are given as follows;*

$$T_{\beta_2} = \frac{(\varnothing'\cos\varnothing + \varnothing'\sin\varnothing)V_1 - \|W\|V_2 + (\varnothing'\cos\varnothing - \varnothing'\sin\varnothing)V_3}{\sqrt{2\varnothing'^2 + \|W\|^2}},$$

$$N_{\beta_2} = \frac{\bar{\varsigma}_1 V_1 + \bar{\varsigma}_2 V_2 + \bar{\varsigma}_3 V_3}{\sqrt{\bar{\varsigma}_1^2 + \bar{\varsigma}_2^2 + \bar{\varsigma}_3^2}},$$

$$\begin{aligned} B_{\beta_2} = & \frac{-\|W\|\bar{\varsigma}_3 - (\varnothing'\cos\varnothing - \varnothing'\sin\varnothing)\bar{\varsigma}_2}{\sqrt{2\varnothing'^2 + \|W\|^2(\bar{\varsigma}_1^2 + \bar{\varsigma}_2^2 + \bar{\varsigma}_3^2)}}V_1 \\ & + \frac{(\varnothing'\cos\varnothing - \varnothing'\sin\varnothing)\bar{\varsigma}_1 - (\varnothing'\cos\varnothing + \varnothing'\sin\varnothing)\bar{\varsigma}_3}{\sqrt{2\varnothing'^2 + \|W\|^2(\bar{\varsigma}_1^2 + \bar{\varsigma}_2^2 + \bar{\varsigma}_3^2)}}V_2 \\ & + \frac{\|W\|\bar{\varsigma}_1 + (\varnothing'\cos\varnothing + \varnothing'\sin\varnothing)\bar{\varsigma}_2}{\sqrt{2\varnothing'^2 + \|W\|^2(\bar{\varsigma}_1^2 + \bar{\varsigma}_2^2 + \bar{\varsigma}_3^2)}}V_3, \end{aligned}$$

$$\begin{aligned} {k_1}_{\beta_2} = & \frac{\sqrt{2}}{(2\varnothing'^2 + \|W\|^2)^{\frac{3}{2}}}\Bigg(\bigg(\big((\varnothing''\cos\varnothing - {\varnothing'}^2\sin\varnothing + \varnothing''\sin\varnothing + {\varnothing'}^2\cos\varnothing \\ & + k_1\|W\|)\sqrt{2\varnothing'^2 + \|W\|^2} - \left(\sqrt{2\varnothing'^2 + \|W\|^2}\right)'\big)^2(\varnothing'\cos\varnothing \\ & + \varnothing'\sin\varnothing) + \bigg((\|W\|\varnothing' - \|W\|')\sqrt{2\varnothing'^2 + \|W\|^2} + \|W\| \\ & .\left(\sqrt{2\varnothing'^2 + \|W\|^2}\right)'\bigg)^2 + \Big((\cos\varnothing(\varnothing'' - {\varnothing'}^2) - \sin\varnothing(\varnothing'' + {\varnothing'}^2) - k_2\|W\|) \\ & \sqrt{2\varnothing'^2 + \|W\|^2} - (\varnothing'\cos\varnothing - \varnothing'\sin\varnothing)\left(\sqrt{2\varnothing'^2 + \|W\|^2}\right)'\Big)^2\Bigg)^{\frac{1}{2}}, \end{aligned}$$

$$k_{2\beta_2} = \frac{\begin{array}{l}\sqrt{2}\Big[h_1\big(\varnothing''' \cos\varnothing - 3\varnothing'\varnothing'' \sin\varnothing - \varnothing'^3 \cos\varnothing + \varnothing''' \sin\varnothing + 3\varnothing'\varnothing'' \cos\varnothing \\ -\varnothing'^3 \sin\varnothing + k_1{}'\|W\| + 2k_1\|W\|' - k_1\varnothing'\|W\|\big) + h_2\big(2\varnothing''\|W\| + \varnothing'^2\|W\| \\ +\|W\|^3 + \|W\|'\varnothing' - \|W\|''\big) + h_3\big(\varnothing''' \cos\varnothing - 3\varnothing'\varnothing'' \sin\varnothing - \varnothing'^3 \cos\varnothing \\ -\varnothing''' \sin\varnothing - 3\varnothing'\varnothing'' \cos\varnothing + \varnothing'^3 \sin\varnothing - k_2{}'\|W\| - 2k_2\|W\|' + k_2\varnothing'\|W\|\big)\end{array}}{h_1{}^2 + h_2{}^2 + h_3{}^2},$$

where

$$\begin{cases} \bar{\varsigma}_1 = (\varnothing'' \cos\varnothing - \varnothing'^2 \sin\varnothing + \varnothing'' \sin\varnothing + \varnothing'^2 \cos\varnothing + k_1\|W\|)\sqrt{2\varnothing'^2 + \|W\|^2} \\ \quad -(\varnothing' \cos\varnothing + \varnothing' \sin\varnothing)\left(\sqrt{2\varnothing'^2 + \|W\|^2}\right)', \\ \bar{\varsigma}_2 = (\|W\|\varnothing' - \|W\|')\sqrt{2\varnothing'^2 + \|W\|^2} + \|W\|\left(\sqrt{2\varnothing'^2 + \|W\|^2}\right)', \\ \bar{\varsigma}_3 = (\varnothing'' \cos\varnothing - \varnothing'^2 \sin\varnothing - \varnothing'' \sin\varnothing - \varnothing'^2 \cos\varnothing - k_2\|W\|)\sqrt{2\varnothing'^2 + \|W\|^2} \\ \quad -(\varnothing' \cos\varnothing - \varnothing' \sin\varnothing)\left(\sqrt{2\varnothing'^2 + \|W\|^2}\right)' . \end{cases}$$

$$\begin{cases} h_1 = -\|W\|(\varnothing'' \cos\varnothing - \varnothing'^2 \sin\varnothing - \varnothing'' \sin\varnothing - \varnothing'^2 \cos\varnothing - k_2\|W\|) \\ \quad -(\varnothing' \cos\varnothing - \varnothing' \sin\varnothing)(\|W\|\varnothing' - \|W\|'), \\ h_2 = (\varnothing' \cos\varnothing - \varnothing' \sin\varnothing)(\varnothing'' \cos\varnothing - \varnothing'^2 \sin\varnothing + \varnothing'' \sin\varnothing + \varnothing'^2 \cos\varnothing + k_1\|W\|), \\ \quad -(\varnothing' \cos\varnothing + \varnothing' \sin\varnothing)(\varnothing'' \cos\varnothing - \varnothing'^2 \sin\varnothing - \varnothing'' \sin\varnothing - \varnothing'^2 \cos\varnothing - k_2\|W\|) \\ h_3 = \|W\|(\varnothing'' \cos\varnothing - \varnothing'^2 \sin\varnothing + \varnothing'' \sin\varnothing + \varnothing'^2 \cos\varnothing + k_1\|W\|) \\ \quad +(\varnothing' \cos\varnothing + \varnothing' \sin\varnothing)(\|W\|\varnothing' - \|W\|')\cdot \end{cases}$$

Proof The theorem is similar to Theorem 2.2 and Theorem 2.3, therefore we omit its proof. □

Definition 2.6 *Let (α, α^*) be a evolute-involute curves in $\mathbb{E}^3$. $V_1{}^*V_3{}^*$ - Smarandache curve can be defined by*

$$\beta_3(s) = \frac{1}{\sqrt{2}}(V_1{}^* + V_3{}^*).$$

If equation (1) is taken into account, the above expression is

$$\beta_3(s) = \frac{\sin\varnothing V_1 + V_2 + \cos\varnothing V_3}{\sqrt{2}}\cdot$$

Theorem 2.7 *The Frenet invariants of the β_3 curve are given as follows;*

$$T_{\beta_3} = \frac{(\varnothing'\cos\varnothing - k_1)V_1 + (-\varnothing'\sin\varnothing + k_2)V_3}{\sqrt{\varnothing'^2 - 2\varnothing'\|W\| + \|W\|^2}},$$

$$N_{\beta_3} = \frac{\bar{o_1}V_1 + \bar{o_2}V_2 + \bar{o_3}V_3}{\sqrt{\bar{o_1}^2 + \bar{o_1}^2 + \bar{o_3}^2}},$$

$$B_{\beta_3} = \frac{[(\varnothing'\sin\varnothing - k_2)\bar{o_2}]V_1 + [(\varnothing'\sin\varnothing - k_2)\bar{o_1} - (\varnothing'\cos\varnothing - k_1)\bar{o_3}]V_2}{\sqrt{(\bar{o_1}^2 + \bar{o_1}^2 + \bar{o_3}^2)(\varnothing'^2 - 2\varnothing'\|W\| + \|W\|^2)}}$$
$$+\frac{[(\varnothing'\cos\varnothing - k_1)\bar{o_2}]V_3}{\sqrt{(\bar{o_1}^2 + \bar{o_1}^2 + \bar{o_3}^2)(\varnothing'^2 - 2\varnothing'\|W\| + \|W\|^2)}},$$

$$k_{1\beta_3} = \frac{\sqrt{2}}{(\varnothing'^2 - 2\varnothing'\|W\| + \|W\|^2)^{\frac{3}{2}}}\Bigg(\Big((\varnothing''\cos\varnothing - \varnothing'^2\sin\varnothing - k_1')\sqrt{\varnothing'^2 - 2\varnothing'\|W\| + \|W\|^2}$$
$$-(\varnothing'\cos\varnothing - k_1)\big(\sqrt{\varnothing'^2 - 2\varnothing'\|W\| + \|W\|^2}\big)'\Big)^2 + (\varnothing'\|W\| - \|W\|^2)^2$$
$$.(\varnothing'^2 - 2\varnothing'\|W\| + \|W\|^2) + \Big((-\varnothing''\sin\varnothing - \varnothing'^2\cos\varnothing + k_2')\sqrt{\varnothing'^2 - 2\varnothing'\|W\| + \|W\|^2}$$
$$-(-\varnothing'\sin\varnothing + k_2)\big(\sqrt{\varnothing'^2 - 2\varnothing'\|W\| + \|W\|^2}\big)'\Big)^2\Bigg)^{\frac{1}{2}},$$

$$k_{2\beta_3} = \frac{\begin{array}{c}\sqrt{2}\Big[\big(\varnothing'''\cos\varnothing - 3\varnothing'\varnothing''\sin\varnothing - \varnothing'^3\cos\varnothing - k_1'' - k_1\varnothing'\|W\| + k_1\|W\|^2\big)\bar{p_1} \\ +\big(\varnothing''\|W\| - k_1k_1' - k_2k_2' + \varnothing''\|W\| + \varnothing'\|W\|' - 2\|W\|\|W\|'\big)\bar{p_2} \\ +\big(-\varnothing'''\sin\varnothing - 3\varnothing'\varnothing'\cos\varnothing + \varnothing'^3\sin\varnothing + k_2'' + k_2\varnothing'\|W\| - k_2\|W\|^2\big)\bar{p_3}\end{array}}{\bar{p_1}^2 + \bar{p_2}^2 + \bar{p_3}^2},$$

where

$$\begin{cases} \bar{o_1} = (\varnothing''\cos\varnothing - \varnothing'^2\sin\varnothing - k_1')\sqrt{\varnothing'^2 - 2\varnothing'\|W\| + \|W\|^2} \\ \qquad -(\varnothing'\cos\varnothing - k_1)\big(\sqrt{\varnothing'^2 - 2\varnothing'\|W\| + \|W\|^2}\big)', \\ \bar{o_2} = (\varnothing'\|W\| - \|W\|^2)\sqrt{\varnothing'^2 - 2\varnothing'\|W\| + \|W\|^2}, \\ \bar{o_3} = (-\varnothing''\sin\varnothing - \varnothing'^2\cos\varnothing + k_2')\sqrt{\varnothing'^2 - 2\varnothing'\|W\| + \|W\|^2} \\ \qquad -(-\varnothing'\sin\varnothing + k_2)\big(\sqrt{\varnothing'^2 - 2\varnothing'\|W\| + \|W\|^2}\big)'; \end{cases}$$

$$\begin{cases} \bar{p_1} = (\varnothing' \sin\varnothing - k_2)(\varnothing' \|W\| - \|W\|^2), \\ \bar{p_2} = (\varnothing' \sin\varnothing - k_2)(\varnothing'' \cos\varnothing - \varnothing'^2 \sin\varnothing - k_1{}') - (\varnothing' \cos\varnothing - k_1) \\ \qquad .(-\varnothing'' \sin\varnothing - \varnothing'^2 \cos\varnothing + k_2{}'), \\ \bar{p_3} = (\varnothing' \cos\varnothing - k_1)(\varnothing' \|W\| - \|W\|^2). \end{cases}$$

Proof The theorem is similar to Theorem 2.2 and Theorem 2.3, therefore we omit its proof. □

Definition 2.8 *Let (α, α^*) be a evolute-involute curves in $\mathbb{E}^3$. ${V_1}^*{V_2}^*{V_3}^*$ - Smarandache curve can be defined by*

$$\beta_4(s) = \frac{1}{\sqrt{3}}({V_1}^* + {V_2}^* + {V_3}^*).$$

If equation (1) is taken into account, the above expression is

$$\beta_4(s) = \frac{(\sin\varnothing - \cos\varnothing)V_1 + V_2 + (\cos\varnothing + \sin\varnothing)V_3}{\sqrt{3}}.$$

Theorem 2.9 *The Frenet invariants of the β_4 curve are given as follows;*

$$T_{\beta_4} = \frac{\big(\varnothing' \cos\varnothing + \varnothing' \sin\varnothing - k_1\big)V_1 - \|W\|V_2 + \big(\varnothing' \cos\varnothing - \varnothing' \sin\varnothing + k_2\big)V_3}{\sqrt{2(\varnothing'^2 - \varnothing'\|W\| + \|W\|^2)}},$$

$$N_{\beta_4} = \frac{\bar{g_1}V_1 + \bar{g_2}V_2 + \bar{g_3}V_3}{\sqrt{\bar{g_1}^2 + \bar{g_1}^2 + \bar{g_3}^2}},$$

$$\begin{aligned} B_{\beta_4} = & \left(\frac{-\|W\|\bar{g_3} - \big(\varnothing' \cos\varnothing - \varnothing' \sin\varnothing + k_2\big)\bar{g_2}}{\sqrt{2(\varnothing'^2 - \varnothing'\|W\| + \|W\|^2)(\bar{g_1}^2 + \bar{g_2}^2 + \bar{g_3}^2)}}\right)V_1 \\ & + \left(\frac{\big(\varnothing' \cos\varnothing - \varnothing' \sin\varnothing + k_2\big)\bar{g_1} - \big(\varnothing' \cos\varnothing + \varnothing' \sin\varnothing - k_1\big)\bar{g_3}}{\sqrt{2(\varnothing'^2 - \varnothing'\|W\| + \|W\|^2)(\bar{g_1}^2 + \bar{g_2}^2 + \bar{g_3}^2)}}\right)V_2 \\ & + \left(\frac{\|W\|\bar{g_1} + \big(\varnothing' \cos\varnothing + \varnothing' \sin\varnothing - k_1\big)\bar{g_2}}{\sqrt{2(\varnothing'^2 - \varnothing'\|W\| + \|W\|^2)(\bar{g_1}^2 + \bar{g_2}^2 + \bar{g_3}^2)}}\right)V_3, \end{aligned}$$

$$
\begin{aligned}
k_{1\beta_4} \;=\; & \frac{\sqrt{3}}{2((\varnothing'^2-2\varnothing'\|W\|+\|W\|^2)^{\frac{3}{2}})}\Bigg(\Big((\varnothing''\cos\varnothing-\varnothing'^2\sin\varnothing+\varnothing''\cos\varnothing+\varnothing'^2\cos\varnothing-{k_1}'\\
& +k_1\|W\|)\sqrt{\varnothing'^2-\varnothing'\|W\|+\|W\|^2}-\Big(\sqrt{\varnothing'^2-\varnothing'\|W\|+\|W\|^2}\Big)'\Big)^2(\varnothing'\cos\varnothing\\
& +\varnothing'\sin\varnothing-k_1)+\Big((\|W\|\varnothing'-\|W\|^2-\|W\|')\sqrt{\varnothing'^2-\varnothing'\|W\|+\|W\|^2}\\
& +\|W\|\Big(\sqrt{\varnothing'^2-\varnothing'\|W\|+\|W\|^2}\Big)'\Big)^2+\Big((\varnothing''\cos\varnothing-\varnothing'^2\sin\varnothing-\varnothing''\sin\varnothing-\varnothing'^2\cos\varnothing\\
& +{k_2}'-k_2\|W\|)\sqrt{\varnothing'^2-\varnothing'\|W\|+\|W\|^2}-\Big(\sqrt{\varnothing'^2-\varnothing'\|W\|+\|W\|^2}\Big)'\Big)^2\\
& .(\varnothing'\cos\varnothing-\varnothing'\sin\varnothing+k_2)\Bigg)^{\frac{1}{2}},
\end{aligned}
$$

$$
k_{2\beta_4} = \frac{\begin{array}{l}\sqrt{3}\Big[\big(\varnothing'''\cos\varnothing-3\varnothing'\varnothing''\sin\varnothing-\varnothing'^3\cos\varnothing+\varnothing'''\sin\varnothing+3\varnothing'\varnothing''\cos\varnothing-\varnothing'''\sin\varnothing-{k_1}''\\
+{k_1}'\|W\|+2k_1\|W\|'-k_1\varnothing'\|W\|+k_1\|W\|^2\big)\bar{f_1}+\big(\varnothing''\|W\|+\varnothing'^2\|W\|\|W\|^3\\
-k_1{k_1}'-k_2{k_2}'+\varnothing'\|W\|'+\varnothing''\|W\|-2\|W\|\|W\|'-\|W\|''\big)\bar{f_2}+\big(\varnothing'''\cos\varnothing\\
-3\varnothing'\varnothing''\sin\varnothing-\varnothing'^3\cos\varnothing-\varnothing'''\sin\varnothing-3\varnothing'\varnothing''\cos\varnothing-\varnothing'^3\sin\varnothing+{k_2}''-{k_2}'\|W\|\\
-2k_2\|W\|'+k_2\varnothing'\|W\|-k_2\|W\|^2\big)\bar{f_3}\Big]\end{array}}{\bar{f_1}^2+\bar{f_2}^2+\bar{f_3}^2},
$$

where

$$
\begin{cases}
\bar{g_1}=(\varnothing''\cos\varnothing-\varnothing'^2\sin\varnothing+\varnothing''\cos\varnothing+\varnothing'^2\cos\varnothing-{k_1}'+k_1\|W\|)\\
\quad .\sqrt{\varnothing'^2-\varnothing'\|W\|+\|W\|^2}-(\varnothing'\cos\varnothing+\varnothing'\sin\varnothing-k_1)\Big(\sqrt{\varnothing'^2-\varnothing'\|W\|+\|W\|^2}\Big)',\\
\bar{g_2}=(\|W\|\varnothing'-\|W\|^2-\|W\|')\sqrt{\varnothing'^2-\varnothing'\|W\|+\|W\|^2}\\
\quad +\|W\|\Big(\sqrt{\varnothing'^2-\varnothing'\|W\|+\|W\|^2}\Big)',\\
\bar{g_3}=(\varnothing''\cos\varnothing-\varnothing'^2\sin\varnothing-\varnothing''\sin\varnothing-\varnothing'^2\cos\varnothing+{k_2}'-k_2\|W\|)\\
\quad .\sqrt{\varnothing'^2-\varnothing'\|W\|+\|W\|^2}-(\varnothing'\cos\varnothing-\varnothing'\sin\varnothing+k_2)\Big(\sqrt{\varnothing'^2-\varnothing'\|W\|+\|W\|^2}\Big)';
\end{cases}
$$

$$\begin{cases} \bar{f}_1 = -\|W\|(\varnothing'' \cos\varnothing - \varnothing'^2 \sin\varnothing - \varnothing'' \sin\varnothing - \varnothing'^2 \cos\varnothing + k_2{}' - k_2\|W\|) - (\varnothing'' \cos\varnothing \\ \quad -\varnothing'^2 \sin\varnothing - \varnothing'' \sin\varnothing - \varnothing'^2 \cos\varnothing + k_2{}' - k_2\|W\|)(\varnothing' \cos\varnothing - \varnothing' \sin\varnothing + k_2), \\ \bar{f}_2 = (\varnothing'(\cos\varnothing - \sin\varnothing) + k_2)(2\varnothing'' \cos\varnothing - \varnothing'^2(\sin\varnothing - \cos\varnothing) - k_1{}' + k_1\|W\|), \\ \quad -(\varnothing'(\cos\varnothing + \sin\varnothing) - k_1)(\cos\varnothing(\varnothing'' - \varnothing'^2) - \sin\varnothing(\varnothing'' + \varnothing'^2) + k_2{}' - k_2\|W\|) \\ \bar{f}_3 = (\varnothing' \cos\varnothing + \varnothing' \sin\varnothing - k_1)(\|W\|\varnothing' - \|W\|^2 - \|W\|') \\ \quad +\|W\|(\varnothing'' \cos\varnothing - \varnothing'^2 \sin\varnothing + \varnothing'' \cos\varnothing + \varnothing'^2 \cos\varnothing - k_1{}' + k_1\|W\|). \end{cases}$$

Proof The theorem is similar to Theorem 2.2 and Theorem 2.3, therefore we omit its proof. □

3. Examples

Example 3.1 *Let us consider the unit speed helix curve and involute curve:*

$$\alpha(s) = \left(3\cos\left(\frac{s}{5}\right), 3\sin\left(\frac{s}{5}\right), \frac{4s}{5}\right),$$

$$\alpha^*(s) = \left(-\frac{3}{5}\sin\left(\frac{s}{5}\right)c + \frac{3}{5}\sin\left(\frac{s}{5}\right)s + 3\cos\left(\frac{s}{5}\right), \frac{3}{5}\cos\left(\frac{s}{5}\right)c - \frac{3}{5}\cos\left(\frac{s}{5}\right)s + 3\sin\left(\frac{s}{5}\right), \frac{4c}{5}\right).$$

The Smarandache curves, which consist of Frenet vectors of the involute curve, are, respectively, given as follows;

$$\begin{cases} \beta_1(s) = \left(\frac{\sqrt{2}}{2}\sin\left(\frac{s}{5}\right) - \frac{\sqrt{2}}{2}\cos\left(\frac{s}{5}\right), -\frac{\sqrt{2}}{2}\cos\left(\frac{s}{5}\right) - \frac{\sqrt{2}}{2}\sin\left(\frac{s}{5}\right), \frac{24\sqrt{2}}{25}\right), \\ \beta_2(s) = \left(\frac{\sqrt{2}}{2}\sin\left(\frac{s}{5}\right), -\frac{\sqrt{2}}{2}\cos\left(\frac{s}{5}\right), \frac{109\sqrt{2}}{50}\right), \\ \beta_3(s) = \left(-\frac{\sqrt{2}}{2}\cos\left(\frac{s}{5}\right), -\frac{\sqrt{2}}{2}\sin\left(\frac{s}{5}\right), \frac{61\sqrt{2}}{50}\right), \\ \beta_4(s) = \left(\frac{\sqrt{2}}{2}\sin\left(\frac{s}{5}\right) - \frac{\sqrt{2}}{2}\cos\left(\frac{s}{5}\right), -\frac{\sqrt{2}}{2}\cos\left(\frac{s}{5}\right) - \frac{\sqrt{2}}{2}\sin\left(\frac{s}{5}\right), \frac{109\sqrt{2}}{50}\right). \end{cases}$$

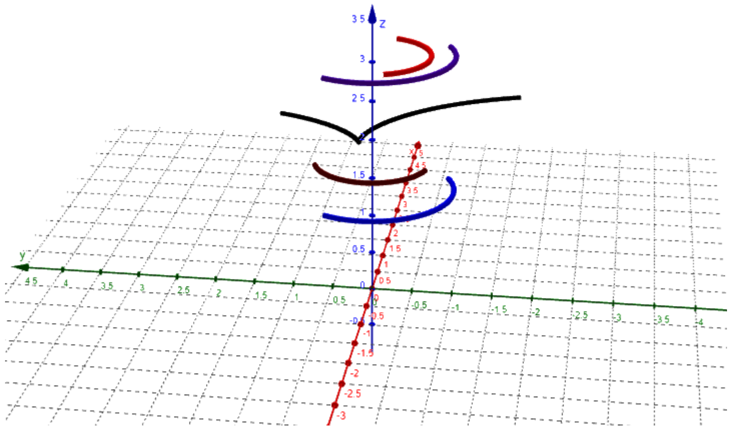

Figure 1: The black curve is the involute curve of the curve α (c=1). The blue, red, brown and purple curves are Smarandache curves, which consist of Frenet vectors of the involute curve, respectively

Example 3.2 *Let us consider the unit speed curve and involute curve:*

$$\alpha(s)=\left(\frac{(1+s)^{\frac{3}{2}}}{3},\frac{(1-s)^{\frac{3}{2}}}{3},\frac{s\sqrt{2}}{2}\right),$$

$$\alpha^*(s)=\left(\frac{c\sqrt{1+s}}{2}-\frac{s\sqrt{1+s}}{2}+\frac{(1+s)^{\frac{3}{2}}}{3},-\frac{c\sqrt{1-s}}{2}+\frac{s\sqrt{1-s}}{2}+\frac{(1-s)^{\frac{3}{2}}}{3},\frac{c\sqrt{2}}{2}\right).$$

The Smarandache curves, which consist of Frenet vectors of the involute curve, are, respectively, given as follows;

$$\begin{cases}\beta_1(s)=\left(\frac{-\sqrt{1+s}}{2}+\frac{\sqrt{1-s}}{2},\frac{\sqrt{1-s}}{2}+\frac{\sqrt{1+s}}{2},0\right),\\ \beta_2(s)=\left(\frac{-\sqrt{1+s}}{2},\frac{\sqrt{1-s}}{2},\frac{\sqrt{2}}{2}\right),\\ \beta_3(s)=\left(\frac{\sqrt{1-s}}{2},\frac{\sqrt{1+s}}{2},\frac{\sqrt{2}}{2}\right),\\ \beta_4(s)=\left(\frac{-\sqrt{1+s}}{2}+\frac{\sqrt{1-s}}{2},\frac{\sqrt{1-s}}{2}+\frac{\sqrt{1+s}}{2},\frac{\sqrt{2}}{2}\right).\end{cases}$$

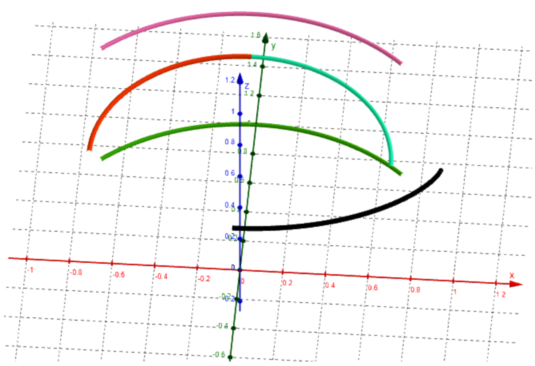

Figure 2: The black curve is the involute curve of the curve α (c=1). The green, orange, turquoise and purple curves are Smarandache curves, which consist of Frenet vectors of the involute curve, respectively

4. Conclusion

We examined the Smarandache curves formed by the Frenet vectors of the involute curve. Then curvatures and torsions of Smarandache curves are calculated. These invariants (Frenet vectors and curvatures) which depend on the evolute curve are explained. Besides, we illustrate the Smarandache curves formed by taking the helix curve.

Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Authors Contributions

Author [Selin Sivas]: Thought and designed the research/problem, contributed to research method or evaluation of data (%55).

Author [Süleyman Şenyurt]: Contributed to research method or evaluation of data (%30).

Author [Abdussamet Çalışkan]: Collected the data, contributed to research examples and figures, wrote the manuscript (%15).

Conflicts of Interest

The authors declare no conflict of interest.

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