International Electronic Journal of Algebra
Volume 33 (2023) 18-33
DOI: 10.24330/ieja.1152714

PLANAR INDEX AND OUTERPLANAR INDEX OF ZERO-DIVISOR GRAPHS OF COMMUTATIVE RINGS WITHOUT IDENTITY

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Received: 5 May 2021; Revised: 26 January 2022; Accepted: 7 March 2022
Communicated by Abdullah Harmancı

Dedicated to the memory of Professor Edmund R. Puczyłowski

Abstract. Let R be a commutative ring without identity. The zero-divisor graph of R, denoted by $\Gamma(R)$ is a graph with vertex set $Z(R)\setminus\{0\}$ which is the set of all nonzero zero-divisor elements of R, and two distinct vertices x and y are adjacent if and only if $xy = 0$. In this paper, we characterize the rings whose zero-divisor graphs are ring graphs and outerplanar graphs. Further, we establish the planar index, ring index and outerplanar index of the zero-divisor graphs of finite commutative rings without identity.

Mathematics Subject Classification (2020): 05C10, 05C25, 05C75
Keywords: Zero-divisor graph, line graph, iterated line graph, planar index, outerplanar index, ring index

1. Introduction

Throughout this paper, R is a finite commutative ring without identity. Let $Z(R)$ be the set of all zero-divisors and $Z(R)^* = Z(R)\setminus\{0\}$. In [6], Beck defined a simple graph from commutative rings, the vertex set of that graph is formed by all the elements of a commutative ring R and two vertices x and y are adjacent if and only if $xy = 0$. In [3], Anderson and Livingston modified that graph structure and named it the zero-divisor graph $\Gamma(R)$ of R whose vertex set is $Z(R)^*$ and two distinct vertices x and y are adjacent if and only if $xy = 0$ for commutative rings. In [2], Anderson and Weber studied the zero-divisor graph of a commutative ring without identity.

Kuzmina and Maltsev characterized the planar zero-divisor graphs of nilpotent rings and non-nilpotent rings, in [11] and [12], respectively. In [4], Barati gave a full characterization of zero-divisor graphs associated to finite commutative rings with identity with respect to their planar index and outerplanar index.

A ring R is called *local* if it has a unique maximal ideal. If R is a non local commutative ring with identity, then $Z(R)$ need not be an ideal. For every commutative ring without identity, $Z(R) = R$, $Z(R)$ is an ideal. Therefore, if we focus the study of zero divisor graphs of commutative ring without identity, then it reveals the properties of commutative ring without identity. Thus, the zero-divisor graph of commutative rings without identity is a unique structure than commutative rings with identity. Moreover, we obtain the planar index, ring index and outerplanar index of the zero-divisor graphs of finite commutative rings without identity.

2. Preliminaries

Let G be a graph with n vertices and m edges. A *chord* is an edge joining any two non-adjacent vertices in a cycle. A *primitive cycle* is a cycle without chords. The *free rank* of G is the number of primitive cycles of G and it is denoted by frank(G). The *cycle rank* of G is defined as $\text{rank}(G) = m - n + r$ where r is the number of connected components of G. Note that the cycle rank is the dimension of the cycle space of G and it satisfies the inequality $\text{rank}(G) \leq \text{frank}(G)$. The family of graphs satisfying that $\text{rank}(G) = \text{frank}(G)$ is called *ring graphs*.

The line graph of G (denoted by $L(G)$) is a graph whose vertex set consists of the set of all edges of G and two vertices of $L(G)$ are adjacent if the corresponding edges of G are adjacent. The k^{th} iterated line graph of G (denoted by $L^k(G)$) is defined as $L^k(G) = L(L^{k-1}(G))$, for every positive integer k. In particular, $L^0(G) = G$ and $L^1(G) = L(G)$. K_n and P_n denote the complete graph and the path of n vertices, respectively. A set of vertices of the graph G is called an *independent set* if no two vertices in the set are adjacent to each other. The join of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is a graph $G_1 + G_2$ whose vertex set is $V_1 \cup V_2$ and whose edge set contains the edges joining every vertex from V_1 to every vertex in V_2. A vertex v is said to be a cut vertex if removal of the vertex v disconnects the graph G.

For a class of graphs $\mathbb{G}$, the graph G is said to be a *forbidden subgraph* for $\mathbb{G}$ if no member of $\mathbb{G}$ has G as an induced subgraph. We can say that G is a *minimal forbidden subgraph* for $\mathbb{G}$ if it is a forbidden subgraph for $\mathbb{G}$ but none of its proper induced subgraphs are forbidden subgraphs.

For a graph G, the *genus* of G is the minimum positive integer n such that G can be embedded in the surface S_n without edge crossings and it is denoted by $g(G)$. If a graph G can be embedded in the plane without edge crossings, then it is called *planar*, i.e., $g(G) = 0$. If $g(G) \neq 0$, then the graph G is non planar. An outerplanar graph is a graph that can be embedded in the plane such that all vertices lie on the outer face of the drawing; otherwise, the graph is non-outerplanar.

The ring index of a graph G is the smallest integer k such that the k^{th} iterated line graph of G is not a ring graph and it is denoted by $\gamma_r(G)$. The planar index of a graph G is defined as the smallest k such that $L^k(G)$ is non-planar. We denote the planar index of G by $\gamma_p(G)$. The outerplanar index of a graph G is the smallest integer k such that the k^{th} iterated line graph of G is non-outerplanar and it is denoted by $\gamma_o(G)$. If $L^k(G)$ is outerplanar (respectively, ring graph or planar) for all $k \geq 0$, we define $\gamma_o(G) = \infty$ (respectively, $\gamma_r(G) = \infty$ or $\gamma_p(G) = \infty$).

Remark 2.1. In [10], I. Gitler et al. proved the relationship between outerplanar graph, ring graph and planar graph as follows:

$$\textit{outerplanar} \Rightarrow \textit{ring graph} \Rightarrow \textit{planar}$$

(i.e. $\gamma_o(G) \leq \gamma_r(G) \leq \gamma_p(G)$).

In the literature, the notations for the commutative rings without identity are used in many ways. In this paper, we follow the notations used by Anderson and Weber in [2]. With respect to isomorphism, we identify the notations of the commutative rings without identity used in [2] and [11] as follow: $N_{0,2} \cong \mathbb{Z}_2^0$, $N_{0,3} \cong \mathbb{Z}_3^0$, $N_{0,4} \cong \mathbb{Z}_4^0$, $N_{0,5} \cong \mathbb{Z}_5^0$, $N_{2,2} \cong \frac{x\mathbb{Z}_2[x]}{x^3\mathbb{Z}_2[x]}$, $N_{3,3} \cong \frac{x\mathbb{Z}_3[x]}{x^3\mathbb{Z}_3[x]}$, $N_4 \cong \frac{x\mathbb{Z}[x]}{<4x, x^2-2x>}$, $N_9 \cong \frac{x\mathbb{Z}[x]}{<9x, x^2-3x>}$ and $N_{2,4} \cong \frac{x\mathbb{Z}[x]}{<8x, x^2-2x>}$. We denote the ring of integers modulo n by $\mathbb{Z}_n$ and $\mathbb{Z}_q^0$ is the ring with additive group $(\mathbb{Z}_q, +_q)$ and trivial multiplication (i.e. $ab = 0$ for all $a, b \in \mathbb{Z}_q$). The following notations are useful for further reading of this paper.

$$Q_1 = < a, b \mid 4a = 0, 2b = 0, a^2 = b, ab = ba = 2a, b^2 = 0 >;$$

$$Q_2 = < a, b \mid 4a = 0, 2b = 0, a^2 = 0, ab = ba = 2a, b^2 = 0 >;$$

$$Q_3 = < a, b \mid 4a = 0, 2b = 0, a^2 = 2a, ab = ba = 2a, b^2 = 0 >;$$

$$Q_4 = < a, b \mid 4a = 0, 2b = 0, a^2 = 2a, ab = ba = 0, b^2 = 2a >;$$

$$Q_5 = < a, b, c \mid 2a = 2b = 2c = 0, a^2 = b, b^2 = 0, ab = c, c^2 = 0 >;$$

$$Q_6 = < a, b, c \mid 2a = 2b = 2c = 0, a^2 = b^2 = 0, ab = -ba = c, \\ ac = ca = bc = cb = c^2 = 0 >;$$

$$Q_7 = < a, b, c \mid 2a = 2b = 2c = 0, a^2 = c, ab = ba = 0, b^2 = c, \\ ac = ca = bc = cb = c^2 = 0 > .$$

Remark 2.2. The characterization for planar zero-divisor graphs from all finite rings were obtained in [11, Theorem 3.1] and [12, Theorem 1 and 2]. In this characterization, we have exactly 24 (17 from Theorem 3.1 in [11] and 7 from Theorem 2 in [12]) non-isomorphic (up to isomorphism) commutative rings without identity whose zero-divisor graphs are planar.

We have restated the notations and combined the results from Theorem 3.1 in [11] and Theorem 2 in [12] with the restriction that rings are commutative without identity. From these evidence, we get the following theorem.

Theorem 2.3. *Let R be a finite commutative ring without identity and let $\mathbb{F}_{p^n}$ be a finite field with p^n elements where p is a prime. Then $\Gamma(R)$ is planar if and only if R is isomorphic to one of the following rings:*
$\mathbb{Z}_2^0\times\mathbb{Z}_2\times\mathbb{Z}_2$, $\mathbb{Z}_2^0\times\mathbb{F}_{p^n}$, $\mathbb{Z}_3^0\times\mathbb{F}_{p^n}$, $\mathbb{Z}_2^0\times\mathbb{Z}_4$, $\mathbb{Z}_2^0\times\frac{\mathbb{Z}_2[x]}{<x^2>}$, $\mathbb{Z}_2\times\frac{x\mathbb{Z}[x]}{<4x,x^2-2x>}$, $\mathbb{Z}_2\times\frac{x\mathbb{Z}_2[x]}{x^3\mathbb{Z}_2[x]}$, $\mathbb{Z}_2^0\times\mathbb{Z}_2^0$, $\mathbb{Z}_2^0$, $\mathbb{Z}_3^0$, $\mathbb{Z}_4^0$, $\mathbb{Z}_5^0$, $\frac{x\mathbb{Z}_2[x]}{x^3\mathbb{Z}_2[x]}$, $\frac{x\mathbb{Z}_3[x]}{x^3\mathbb{Z}_3[x]}$, $\frac{x\mathbb{Z}[x]}{<4x,x^2-2x>}$, $\frac{x\mathbb{Z}[x]}{<9x,x^2-3x>}$, $\frac{x\mathbb{Z}[x]}{<8x,x^2-2x>}$, Q_i *where* $1\le i\le 7$.

Let q be a prime number. Consider the ring $R=\mathbb{Z}_q^0\times\mathbb{F}_{p^n}$. Note that $Z(R)=R$. Further, the subgraph of $\Gamma(R)$ induced by $(\mathbb{Z}_q^0)^*\times\{0\}$ is K_{q-1} and the subset $R\setminus(\mathbb{Z}_q^0\times\{0\})$ with $(p^n-1)q$ elements induces an independent set in $\Gamma(R)$. Also every element in $(\mathbb{Z}_q^0)^*\times\{0\}$ is adjacent with every element in $R\setminus(\mathbb{Z}_q^0\times\{0\})$ in $\Gamma(R)$. Hence we have the following lemma, which gives the structure of $\Gamma(\mathbb{Z}_q^0\times\mathbb{F}_{p^n})$.

Lemma 2.4. *Let p and q be prime numbers and $R=\mathbb{Z}_q^0\times\mathbb{F}_{p^n}$. Then $\Gamma(R)\cong K_{q-1}+\overline{K_{(p^n-1)q}}$.*

Lemma 2.5. *Let R_1 and R_2 be finite commutative rings. If $\Gamma(R_1)\cong\Gamma(R_2)$, then $\Gamma(S\times R_1)\cong\Gamma(S\times R_2)$ for any commutative ring S.*

Proof. Let $\psi:\Gamma(R_1)\to\Gamma(R_2)$ be a graph isomorphism. Let S be a commutative ring. Consider $\phi:\Gamma(S\times R_1)\to\Gamma(S\times R_2)$ defined by $\phi((a,b))=(a,\psi(b))$. Let (a,b) and (c,d) be two nonzero elements in $S\times R_1$ which are adjacent in $\Gamma(S\times R_1)$. From this $(ac,bd)=(0,0)$ and so $\psi(bd)=\psi(b)\psi(d)=0$. Now $\phi((ac,bd))=(ac,\psi(bd))=(ac,\psi(b)\psi(d))=(0,0)$ and so $(a,\psi(b))(c,\psi(d))=(0,0)$. Therefore, $\phi((a,b))\phi((c,d))=(0,0)$ and so $\phi((a,b))$ and $\phi((c,d))$ are adjacent in $\Gamma(S\times R_2)$. Similarly one can observe that $\phi((a,b))$ and $\phi((c,d))$ are not adjacent in $\Gamma(S\times R_1)$ whenever (a,b) and (c,d) are not adjacent in $\Gamma(S\times R_1)$. Since ψ is bijective, ϕ is bijective and so ϕ is a graph isomorphism. □

The following is useful in the sequel of the paper.

Corollary 2.6. *Assume that R_1 and R_2 are finite commutative rings. If $\Gamma(R_1) \cong \Gamma(R_2)$, then $g(\Gamma(S \times R_1)) = g(\Gamma(S \times R_2))$ for any commutative ring S.*

3. The planar index of zero-divisor graphs

In [8], Ghebleh and Khatirinejad characterized connected graphs with respect to their planar index.

Theorem 3.1. [8, Theorem 10] *Let G be a connected graph. Then:*

(a) $\gamma_p(G) = 0$ *if and only if G is non-planar;*
(b) $\gamma_p(G) = \infty$ *if and only if G is either a path, a cycle, or $K_{1,3}$;*
(c) $\gamma_p(G) = 1$ *if and only if G is planar and either $\Delta(G) \geq 5$ or G has a vertex of degree 4 which is not a cut-vertex;*
(d) $\gamma_p(G) = 2$ *if and only if $L(G)$ is planar and G contains one of the graphs H_i in Figure 1 as a subgraph;*
(e) $\gamma_p(G) = 4$ *if and only if G is one of the graphs X_k or Y_k (Figure 1) for some $k \geq 2$;*
(f) $\gamma_p(G) = 3$ *otherwise.*

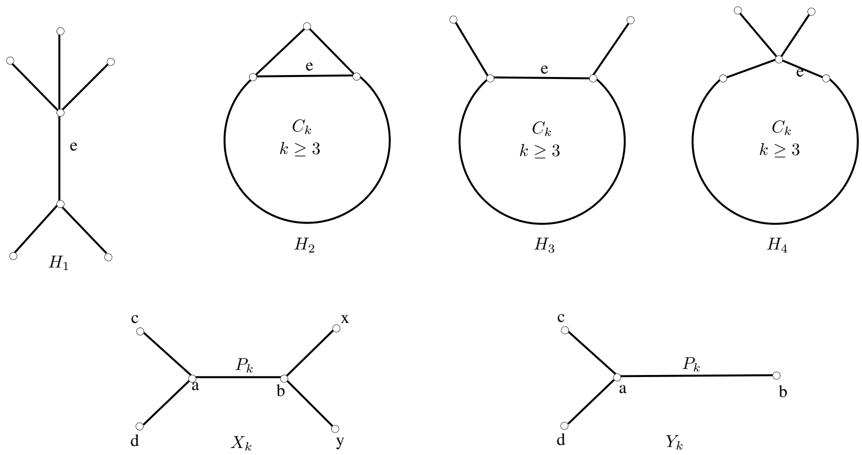

Figure 1

In [11] and [12], Kuzmina studied planarity for all finite rings. Specially, the planarity of zero divisor graphs with non zero identity was studied in [7] and according to these results, the planar index and outerplanar index of these graphs were studied in [4]. In this section, we characterize all zero divisor graphs with respect to the planar index when R is a commutative ring without identity.

Theorem 3.2. *Let R be a finite commutative ring without identity. Then*

(1) $\gamma_p(\Gamma(R)) = \infty$ *if and only if R is isomorphic to one of the following rings:*

(a) $\mathbb{Z}_2^0 \times \mathbb{Z}_2^0$, $\mathbb{Z}_2^0 \times \mathbb{Z}_2$;
(b) $\mathbb{Z}_2^0$, $\mathbb{Z}_3^0$, $\mathbb{Z}_4^0$, $\frac{x\mathbb{Z}[x]}{<4x,x^2-2x>}$, $\frac{x\mathbb{Z}_2[x]}{x^3\mathbb{Z}_2[x]}$;

(2) $\gamma_p(\Gamma(R)) = 1$ *if and only if* R *is isomorphic to one of the following rings:*
(a) $\mathbb{Z}_2^0 \times \mathbb{Z}_2 \times \mathbb{Z}_2$;
(b) $\mathbb{Z}_2^0 \times \mathbb{F}_{p^n}$ *with* $p^n \geq 4$, $\mathbb{Z}_3^0 \times \mathbb{F}_{p^n}$, $\mathbb{Z}_2^0 \times \mathbb{Z}_4$, $\mathbb{Z}_2^0 \times \frac{\mathbb{Z}_2[x]}{<x^2>}$, $\mathbb{Z}_2 \times \frac{x\mathbb{Z}[x]}{<4x,x^2-2x>}$, $\mathbb{Z}_2 \times \frac{x\mathbb{Z}_2[x]}{x^3\mathbb{Z}_2[x]}$;
(c) $\frac{x\mathbb{Z}[x]}{<9x,x^2-3x>}$, $\frac{x\mathbb{Z}_3[x]}{x^3\mathbb{Z}_3[x]}$, $\frac{x\mathbb{Z}[x]}{<8x,x^2-2x>}$, Q_i *where* $1 \leq i \leq 7$;

(3) $\gamma_p(\Gamma(R)) = 2$ *if and only if* R *is isomorphic to* $\mathbb{Z}_5^0$;

(4) $\gamma_p(\Gamma(R)) = 3$ *if and only if* R *is isomorphic to* $\mathbb{Z}_2^0 \times \mathbb{Z}_3$;

(5) $\gamma_p(\Gamma(R)) = 0$ *otherwise.*

Proof. For a non planar graph, the planar index is 0 because of Theorem 3.1. Therefore, we should focused on the case $\Gamma(R)$ is planar. Let R be a finite commutative ring without identity. Then $R \cong R_1 \times R_2 \times \cdots \times R_n$ and R_i's are indecomposable rings for all i such that $1 \leq i \leq n$. By Theorem 2.3, it is enough to consider $n \leq 3$.

Case 1. Suppose $n = 3$. By Theorem 2.3, $\Gamma(R_1 \times R_2 \times R_3)$ is planar if and only if $R \cong \mathbb{Z}_2^0 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. By Figure 2, $\Delta(\Gamma(\mathbb{Z}_2^0 \times \mathbb{Z}_2 \times \mathbb{Z}_2)) = 6$. By Theorem 3.1, we have $\gamma_p(\Gamma(\mathbb{Z}_2^0 \times \mathbb{Z}_2 \times \mathbb{Z}_2)) = 1$.

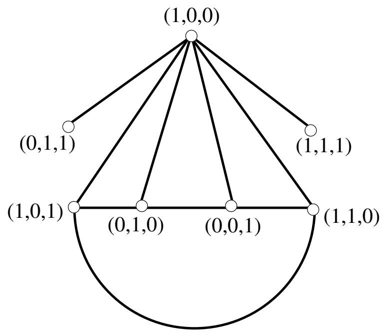

Figure 2. $\Gamma(\mathbb{Z}_2^0 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$

Case 2. Suppose $n = 2$. By Theorem 2.3, $\Gamma(R_1 \times R_2)$ is planar if and only if R is isomorphic to one of the following rings: $\mathbb{Z}_2^0 \times \mathbb{Z}_2^0$, $\mathbb{Z}_2^0 \times \mathbb{F}_{p^n}$, $\mathbb{Z}_3^0 \times \mathbb{F}_{p^n}$, $\mathbb{Z}_2 \times \frac{x\mathbb{Z}[x]}{<4x,x^2-2x>}$, $\mathbb{Z}_2 \times \frac{x\mathbb{Z}_2[x]}{x^3\mathbb{Z}_2[x]}$, $\mathbb{Z}_2^0 \times \mathbb{Z}_4$, $\mathbb{Z}_2^0 \times \frac{\mathbb{Z}_2[x]}{<x^2>}$.

Suppose $R \cong \mathbb{Z}_2^0 \times \mathbb{Z}_2^0$. The products of trivial multiplication yields that $\Gamma(R) \cong K_3$. Now, by Theorem 3.1, we get that $\gamma_p(\Gamma(\mathbb{Z}_2^0 \times \mathbb{Z}_2^0)) = \infty$.

For $R \cong \mathbb{Z}_2^0 \times \mathbb{F}_{p^n}$, by Lemma 2.4, we have $\Gamma(\mathbb{Z}_2^0 \times \mathbb{F}_{p^n}) \cong K_{1,2p^n-2}$. If $p^n \geq 4$, then $\Delta(\Gamma(\mathbb{Z}_2^0 \times \mathbb{F}_{p^n})) \geq 6$. By Theorem 3.1, we have $\gamma_p(\Gamma(\mathbb{Z}_2^0 \times \mathbb{F}_{p^n})) = 1$ where $p^n \geq 4$. If $p^n = 3$, then $\Gamma(\mathbb{Z}_2^0 \times \mathbb{Z}_3) \cong K_{1,4}$. Since the line graph of any star graph is complete, we have $L(\Gamma(\mathbb{Z}_2^0 \times \mathbb{Z}_3)) \cong K_4$ which is planar and H_2 is a subgraph of $L(\Gamma(\mathbb{Z}_2^0 \times \mathbb{Z}_3))$. By Theorem 3.1, $\gamma_p(L(\Gamma(\mathbb{Z}_2^0 \times \mathbb{Z}_3))) = 2$. It implies that $\gamma_p(\Gamma(\mathbb{Z}_2^0 \times \mathbb{Z}_3)) = 3$. Suppose $p^n = 2$. Then $R \cong \mathbb{Z}_2^0 \times \mathbb{Z}_2$. By Lemma 2.4, $\Gamma(R)$ is isomorphic to $K_{1,2}$. Since it is a path, we have $\gamma_p(\Gamma(\mathbb{Z}_2^0 \times \mathbb{Z}_2)) = \infty$.

Suppose $R \cong \mathbb{Z}_3^0 \times \mathbb{F}_{p^n}$. By Lemma 2.4, we have $\Gamma(\mathbb{Z}_3^0 \times \mathbb{F}_{p^n}) \cong K_2 + \overline{K_{3p^n-3}}$. Suppose $p^n \geq 3$. It is easy to see that the graph $\Gamma(\mathbb{Z}_3^0 \times \mathbb{F}_{p^n})$ is planar and $\Delta(\mathbb{Z}_3^0 \times \mathbb{F}_{p^n}) \geq 6$. By Theorem 3.1, $\gamma_p(\Gamma(\mathbb{Z}_3^0 \times \mathbb{F}_{p^n})) = 1$ for $p^n \geq 3$. Suppose $p^n = 2$ and $R \cong \mathbb{Z}_3^0 \times \mathbb{Z}_2$. By Lemma 2.4, $\Gamma(R)$ is isomorphic to $K_2 + \overline{K_3}$. It is a planar graph and it has two vertices of degree 4 which are not cut vertices. By Theorem 3.1, $\gamma_p(\Gamma(\mathbb{Z}_3^0 \times \mathbb{Z}_2)) = 1$.

It is not hard to see that

$$\Gamma(\mathbb{Z}_2^0 \times \mathbb{Z}_2) \cong \Gamma(\tfrac{x\mathbb{Z}[x]}{<4x,x^2-2x>}) \cong \Gamma(\tfrac{x\mathbb{Z}_2[x]}{x^3\mathbb{Z}_2[x]}).$$

By Corollary 2.6, we have that $\Gamma(\mathbb{Z}_2^0 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \cong \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2^0 \times \mathbb{Z}_2) \cong \Gamma\big(\mathbb{Z}_2 \times \frac{x\mathbb{Z}[x]}{<4x,x^2-2x>}\big) \cong \Gamma\big(\mathbb{Z}_2 \times \frac{x\mathbb{Z}_2[x]}{x^3\mathbb{Z}_2[x]}\big)$. We already proved that $\gamma_p(\Gamma(\mathbb{Z}_2^0 \times \mathbb{Z}_2 \times \mathbb{Z}_2)) = 1$. Therefore,

$$\gamma_p(\Gamma\big(\mathbb{Z}_2 \times \tfrac{x\mathbb{Z}[x]}{<4x,x^2-2x>}\big)) = \gamma_p(\Gamma\big(\mathbb{Z}_2 \times \tfrac{x\mathbb{Z}_2[x]}{x^3\mathbb{Z}_2[x]}\big)) = 1.$$

Assume that R is isomorphic to anyone of $\mathbb{Z}_2^0 \times \mathbb{Z}_4$ or $\mathbb{Z}_2^0 \times \frac{\mathbb{Z}_2[x]}{<x^2>}$. Then $\Gamma(R)$ is isomorphic to G_1 represented in Figure 3.

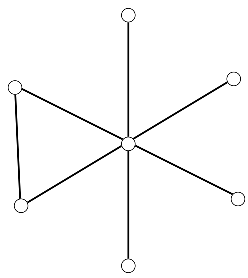

Figure 3. The graph G_1

The degree of the vertex $(1, 0)$ in the graphs $\Gamma\big(\mathbb{Z}_2^0 \times \mathbb{Z}_4\big)$ and $\Gamma\big(\mathbb{Z}_2^0 \times \frac{\mathbb{Z}_2[x]}{<x^2>}\big)$ is 6. By Theorem 3.1, we have

$$\gamma_p(\Gamma(\mathbb{Z}_2^0 \times \mathbb{Z}_4)) = \gamma_p(\Gamma\big(\mathbb{Z}_2^0 \times \tfrac{\mathbb{Z}_2[x]}{<x^2>}\big)) = 1.$$

Case 3. Suppose $n = 1$. Since $\Gamma(R)$ is planar, by Theorem 2.3, R is isomorphic to one of the following rings: $\mathbb{Z}_2^0$, $\mathbb{Z}_3^0$, $\mathbb{Z}_4^0$, $\mathbb{Z}_5^0$, $\frac{x\mathbb{Z}[x]}{<4x,x^2-2x>}$, $\frac{x\mathbb{Z}_2[x]}{x^3\mathbb{Z}_2[x]}$, $\frac{x\mathbb{Z}[x]}{<9x,x^2-3x>}$, $\frac{x\mathbb{Z}_3[x]}{x^3\mathbb{Z}_3[x]}$, $\frac{x\mathbb{Z}[x]}{<8x,x^2-2x>}$, Q_i where $1 \leq i \leq 7$.

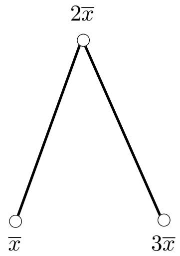

Figure 4(a). $\Gamma(\frac{x\mathbb{Z}[x]}{<4x,x^2-2x>})$

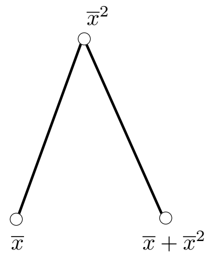

Figure 4(b). $\Gamma(\frac{x\mathbb{Z}_2[x]}{x^3\mathbb{Z}_2[x]})$

Suppose R is isomorphic to either $\mathbb{Z}_2^0$ or $\mathbb{Z}_3^0$ or $\mathbb{Z}_4^0$ or $\frac{x\mathbb{Z}[x]}{<4x,x^2-2x>}$ or $\frac{x\mathbb{Z}_2[x]}{x^3\mathbb{Z}_2[x]}$. The rings $\mathbb{Z}_2^0$, $\mathbb{Z}_3^0$ and $\mathbb{Z}_4^0$ have the zero-divisor graphs K_1, K_2 and K_3 respectively. Moreover, by Figure 4(a) and 4(b), we have that

$$\Gamma(\frac{x\mathbb{Z}[x]}{<4x,x^2-2x>}) \cong \Gamma(\frac{x\mathbb{Z}_2[x]}{x^3\mathbb{Z}_2[x]}) \cong K_{1,2}.$$

So, by Theorem 3.1, we can conclude that $\gamma_p(\Gamma(R)) = \infty$.

If $R \cong \mathbb{Z}_5^0$, then $\Gamma(\mathbb{Z}_5^0) \cong K_4$. By Theorem 3.1, we have $\gamma_p(\Gamma(\mathbb{Z}_5^0)) = 2$.

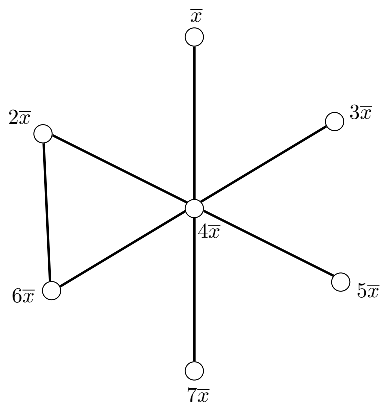

Figure 5. $\Gamma(\frac{x\mathbb{Z}[x]}{<8x,x^2-2x>})$

Suppose that R is isomorphic to either $\frac{x\mathbb{Z}[x]}{<8x,x^2-2x>}$ or Q_i for all i, $1 \leq i \leq 7$. Note that, $\Gamma(Q_1)$, $\Gamma(Q_2)$, $\Gamma(Q_3)$, $\Gamma(Q_4)$, $\Gamma(Q_5)$, $\Gamma(Q_6)$ and $\Gamma(Q_7)$ are illustrated in Figures 1.B, 2.A, 2.B, 3.A, 3.B, 4.A and 4.B of [11], respectively. From these Figures 1.B to 4.B and by Figure 5, one can easily check that $\Delta(\Gamma(R)) = 6$ and $\Gamma(R)$ is planar. By Theorem 3.1, $\gamma_p(\Gamma(R)) = 1$.

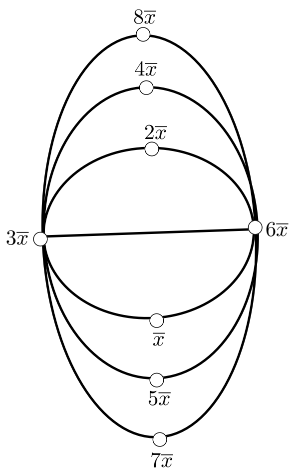

Figure 6(a). $\Gamma(\frac{x\mathbb{Z}[x]}{<9x,x^2-3x>})$

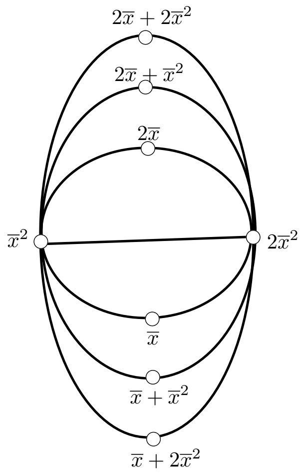

Figure 6(b). $\Gamma(\frac{x\mathbb{Z}_3[x]}{x^3\mathbb{Z}_3[x]})$

Suppose R is isomorphic to either $\frac{x\mathbb{Z}[x]}{<9x,x^2-3x>}$ or $\frac{x\mathbb{Z}_3[x]}{x^3\mathbb{Z}_3[x]}$. By Figures 6(a) and 6(b), $\Gamma(R) \cong K_2 + \overline{K_6}$. Clearly, $\Gamma\big(\frac{x\mathbb{Z}[x]}{<9x,x^2-3x>}\big)$ and $\Gamma\big(\frac{x\mathbb{Z}_3[x]}{x^3\mathbb{Z}_3[x]}\big)$ are planar and $\Delta\big(\Gamma\big(\frac{x\mathbb{Z}[x]}{<9x,x^2-3x>}\big)\big) = \Delta\big(\Gamma\big(\frac{x\mathbb{Z}_3[x]}{x^3\mathbb{Z}_3[x]}\big)\big) = 6$. By Theorem 3.1, we get that $\gamma_p\big(\Gamma\big(\frac{x\mathbb{Z}[x]}{<9x,x^2-3x>}\big)\big) = \gamma_p\big(\Gamma\big(\frac{x\mathbb{Z}_3[x]}{x^3\mathbb{Z}_3[x]}\big)\big) = 1$. □

4. The ring index and outerplanar index of zero-divisor graphs

In this section, we characterize the rings whose zero-divisor graphs are either ring graphs or outerplanar graphs. Further, we give a full characterization of zero-divisor graphs with respect to their ring index and outerplanar index when R is a commutative ring without identity. In [9], Gitler et al. characterized the forbidden induced subgraphs for the family of ring graphs. We need some definitions to use their theorem.

Definition 4.1. (a) A *prism* is a graph consisting of two vertex-disjoint triangles $C_1 = (x_1, x_2, x_3, x_1)$ and $C_2 = (y_1, y_2, y_3, y_1)$, and three paths P_1, P_2 and P_3 pairwise vertex-disjoint, such that each P_i is a path between x_i and y_i for $i = 1, 2, 3$ and the subgraph induced by $V(P_i) \cup V(P_j)$ is a cycle for $1 \leq i < j \leq 3$ (Figure 7a).

(b) A *pyramid* is a graph consisting of a vertex w, a triangle $C = (z_1, z_2, z_3, z_1)$, and three paths P_1, P_2 and P_3 such that P_i is between w and z_i for $i = 1, 2, 3$; $V(P_i) \cap V(P_j) = w$ and the subgraph induced by $V(P_i) \cup V(P_j)$ is a cycle for $1 \leq i < j \leq 3$ and at least one of the P_1, P_2, P_3 has at least two edges (Figure 7b).

(c) A *theta* is a graph consisting of two non adjacent vertices x and y, and three paths P_1, P_2 and P_3 with ends x and y, such that the union of every two of P_1, P_2 and P_3 is an induced cycle (Figure 7c).

(d) A partial wheel is a graph consisting of a cycle C and a vertex z disjoint from C such that z is adjacent to some vertices of C. The cycle C is called the rim of W and z is called the center of W. A partial wheel T with rim C and center z is called a *$\theta-$partial wheel* if $|V(C)| \geq 4$ and there exist two non adjacent vertices in $V(C) \cap N_T(z)$ (Figure 7d).

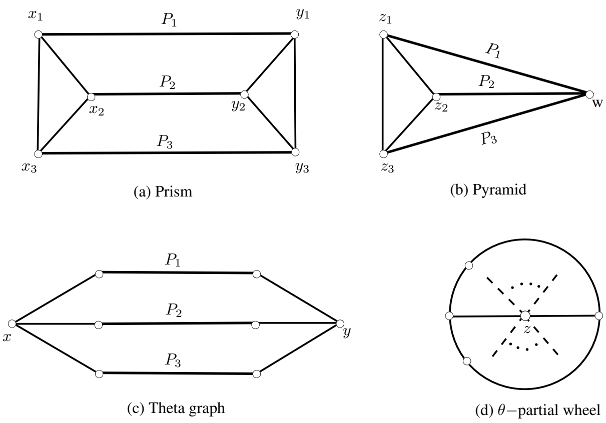

Figure 7

Theorem 4.2. [9, Corollary 4.13] *The minimal forbidden induced subgraphs for ring graphs are: prisms, pyramids, theta graphs, $\theta-$partial wheels and K_4.*

Let $d_1, d_2, \ldots, d_t$ are positive integers with $n \gneq d_1 + d_2 + \cdots + d_t$. We define $I(d_1, d_2, \ldots, d_t)$ as the tree obtained from P_n by adding a leaf to each vertex of P_n that is at in distance of $d_1, d_1 + d_2, \ldots, d_1 + d_2 + \cdots + d_t$ (as in Figure 8). In [5], Barati completely characterized the graphs with respect to their ring index. It can be recalled in the following theorem.

Theorem 4.3. [5, Theorem 1.3] *Let G be a connected graph. Then:*

(a) $\gamma_r(G) = 0$ *if and only if G is not a ring graph if and only if it has an induced subgraph which is prism, pyramid, theta graph, θ-partial wheel or K_4;*

(b) $\gamma_r(G) = \infty$ *if and only if G is either a path, a cycle, or $K_{1,3}$*;

(c) $\gamma_r(G) = 1$ *if and only if G is a ring graph and G has a subgraph homeomorphic to $K_{1,4}$ or $K_1 + P_3$ in* Figure 8;

(d) $\gamma_r(G) = 2$ if and only if $L(G)$ is ring graph and G has a subgraph isomorphic to one of the graphs G_2 or G_3 in Figure 8;

(e) $\gamma_r(G) = 3$ if and only if $G \in I(d_1, d_2, \ldots, d_t)$ where $d_i \geq 2$ for $i = 2, \ldots, t-1$, and $d_1 \geq 1$ (Figure 8).

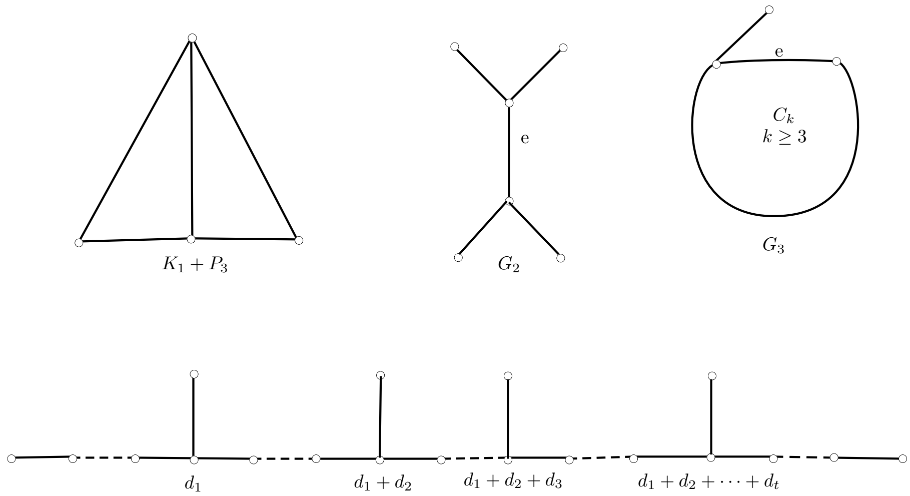

Figure 8

In [13], Lin et al. studied the outerplanarity of the iterated line graphs and they characterized all graphs with respect to their outerplanar index. Their theorem is recalled in the following theorem which is useful for further reading of this paper.

Theorem 4.4. [13, Theorem 3.4] *Let G be a connected graph. Then:*

(a) $\gamma_o(G) = 0$ *if and only if G is non-outerplanar;*

(b) $\gamma_o(G) = \infty$ *if and only if G is either a path, a cycle, or $K_{1,3}$;*

(c) $\gamma_o(G) = 1$ *if and only if G is planar and G has a subgraph homeomorphic to $K_{2,3}$, $K_{1,4}$ or $K_1 + P_3$ in* Figure 8;

(d) $\gamma_o(G) = 2$ if and only if $L(G)$ is planar and G has a subgraph isomorphic to one of the graphs G_2 or G_3 in Figure 8;

(e) $\gamma_o(G) = 3$ if and only if $G \in I(d_1, d_2, \ldots, d_t)$ where $d_i \geq 2$ for $i = 2, \ldots, t-1$, and $d_1 \geq 1$ (Figure 8).

In [1], Afkhami classified all finite commutative rings with identity whose zero-divisor graphs are ring graphs and outerplanar graphs. In the following theorems, we classify all finite commutative rings without identity whose zero-divisor graphs are ring graphs and outerplanar graphs.

Theorem 4.5. *Let R be a finite commutative ring without identity. Then $\Gamma(R)$ is a ring graph if and only if R is isomorphic to one of the following rings:*

$\mathbb{Z}_2^0 \times \mathbb{Z}_2^0$, $\mathbb{Z}_2^0 \times \mathbb{F}_{p^n}$, $\mathbb{Z}_3^0 \times \mathbb{F}_{p^n}$, $\mathbb{Z}_2^0 \times \mathbb{Z}_4$, $\mathbb{Z}_2^0 \times \frac{\mathbb{Z}_2[x]}{<x^2>}$, $\mathbb{Z}_2^0$, $\mathbb{Z}_3^0$, $\mathbb{Z}_4^0$, $\frac{x\mathbb{Z}_2[x]}{x^3\mathbb{Z}_2[x]}$, $\frac{x\mathbb{Z}[x]}{<4x, x^2-2x>}$, $\frac{x\mathbb{Z}_3[x]}{x^3\mathbb{Z}_3[x]}$, $\frac{x\mathbb{Z}[x]}{<9x, x^2-3x>}$, $\frac{x\mathbb{Z}[x]}{<8x, x^2-2x>}$, Q_1, Q_2, Q_5, Q_6.

Proof. Let $R \cong R_1 \times R_2 \times \cdots \times R_n$. We assume that $\Gamma(R)$ is a ring graph. Since every ring graph is planar, by Theorem 2.3, it is enough to consider $n \leq 3$.

Case 1. Assume that $n = 3$ and $R \cong R_1 \times R_2 \times R_3$. So $R \cong \mathbb{Z}_2^0 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Let $S = \{(1,0,1),(0,1,0),(1,1,0),(0,0,1),(1,0,0)\}$. Now, by Figure 2, it is easy to see that the induced subgraph of the graph $\Gamma(\mathbb{Z}_2^0 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ by the set S is isomorphic to a θ-partial wheel. By Theorem 4.2, $\Gamma(\mathbb{Z}_2^0 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ is not a ring graph.

Case 2. Assume that $n = 2$ and R is isomorphic to one of the following rings: $\mathbb{Z}_2^0 \times \mathbb{Z}_2^0$, $\mathbb{Z}_2^0 \times \mathbb{F}_{p^n}$, $\mathbb{Z}_3^0 \times \mathbb{F}_{p^n}$, $\mathbb{Z}_2^0 \times \mathbb{Z}_4$, $\mathbb{Z}_2^0 \times \frac{\mathbb{Z}_2[x]}{<x^2>}$.

Suppose $R \cong \mathbb{Z}_2^0 \times \mathbb{Z}_2^0$. Since the multiplication of R is trivial, $\Gamma(R)$ is isomorphic to K_3. By Theorem 4.2, $\Gamma(R)$ is a ring graph.

By Lemma 2.4, the graph $\Gamma(\mathbb{Z}_2^0 \times \mathbb{F}_{p^n}) \cong K_1 + \overline{K_{2p^n-2}}$ and $\Gamma(\mathbb{Z}_3^0 \times \mathbb{F}_{p^n}) \cong K_2 + \overline{K_{3p^n-3}}$. Since $\Gamma(\mathbb{Z}_2^0 \times \mathbb{F}_{p^n})$ is a star graph, we can deduce that $\Gamma(\mathbb{Z}_2^0 \times \mathbb{F}_{p^n})$ is a ring graph. Also, it is not hard to see that $\mathrm{rank}(\Gamma(\mathbb{Z}_3^0 \times \mathbb{F}_{p^n})) = \mathrm{frank}(\Gamma(\mathbb{Z}_3^0 \times \mathbb{F}_{p^n})) = 3p^n - 3$. So, the graph $\Gamma(\mathbb{Z}_3^0 \times \mathbb{F}_{p^n})$ is a ring graph.

Suppose $R \cong \mathbb{Z}_2^0 \times \mathbb{Z}_4$. Then $\Gamma(R)$ is isomorphic to G_1 in Figure 2 and so $\mathrm{rank}(\Gamma(R)) = \mathrm{frank}(\Gamma(R)) = 1$. Therefore $\Gamma(\mathbb{Z}_2^0 \times \mathbb{Z}_4)$ is a ring graph. Since $\Gamma(\mathbb{Z}_4) \cong \Gamma(\frac{\mathbb{Z}_2[x]}{<x^2>})$, by Corollary 2.6, we get $\Gamma(\mathbb{Z}_2^0 \times \mathbb{Z}_4) \cong \Gamma(\mathbb{Z}_2^0 \times \frac{\mathbb{Z}_2[x]}{<x^2>})$. This implies that $\Gamma(\mathbb{Z}_2^0 \times \frac{\mathbb{Z}_2[x]}{<x^2>})$ is a ring graph.

We know that $\Gamma(\mathbb{Z}_2^0 \times \mathbb{Z}_2) \cong \Gamma(\frac{x\mathbb{Z}[x]}{<4x, x^2-2x>}) \cong \Gamma(\frac{x\mathbb{Z}_2[x]}{x^3\mathbb{Z}_2[x]})$. Now, by Corollary 2.6, $\Gamma(\mathbb{Z}_2^0 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \cong \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2^0 \times \mathbb{Z}_2) \cong \Gamma\big(\mathbb{Z}_2 \times \frac{x\mathbb{Z}[x]}{<4x, x^2-2x>}\big) \cong \Gamma\big(\mathbb{Z}_2 \times \frac{x\mathbb{Z}_2[x]}{x^3\mathbb{Z}_2[x]}\big)$. Since $\Gamma(\mathbb{Z}_2^0 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ is not a ring graph, we can conclude that the graphs $\Gamma(\mathbb{Z}_2 \times \frac{x\mathbb{Z}[x]}{<4x, x^2-2x>})$ and $\Gamma(\mathbb{Z}_2 \times \frac{x\mathbb{Z}_2[x]}{x^3\mathbb{Z}_2[x]})$ are not ring graphs.

Case 3. Assume that $n = 1$ and R is isomorphic to one of the following rings: $\mathbb{Z}_2^0$, $\mathbb{Z}_3^0$, $\mathbb{Z}_4^0$, $\mathbb{Z}_5^0$, $\frac{x\mathbb{Z}_2[x]}{x^3\mathbb{Z}_2[x]}$, $\frac{x\mathbb{Z}[x]}{<4x, x^2-2x>}$, $\frac{x\mathbb{Z}_3[x]}{x^3\mathbb{Z}_3[x]}$, $\frac{x\mathbb{Z}[x]}{<9x, x^2-3x>}$, $\frac{x\mathbb{Z}[x]}{<8x, x^2-2x>}$, Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Q_7.

Since $\Gamma(\mathbb{Z}_n^0) \cong K_{n-1}$, by Theorem 4.2, the graphs $\Gamma(\mathbb{Z}_2^0)$, $\Gamma(\mathbb{Z}_3^0)$, $\Gamma(\mathbb{Z}_4^0)$ are ring graphs and the graph $\Gamma(\mathbb{Z}_5^0)$ is not a ring graph.

If R is isomorphic to either $\frac{x\mathbb{Z}_2[x]}{x^3\mathbb{Z}_2[x]}$ or $\frac{x\mathbb{Z}[x]}{<4x, x^2-2x>}$, then by Figure 4(a) and 4(b), $\Gamma(R)$ is isomorphic to P_3. Therefore $\mathrm{rank}(\Gamma(R)) = 0 = \mathrm{frank}(\Gamma(R))$. So the graphs $\Gamma(\frac{x\mathbb{Z}_2[x]}{x^3\mathbb{Z}_2[x]})$ and $\Gamma(\frac{x\mathbb{Z}[x]}{<4x, x^2-2x>})$ are ring graphs.

Suppose R is isomorphic to either $\frac{x\mathbb{Z}_3[x]}{x^3\mathbb{Z}_3[x]}$ or $\frac{x\mathbb{Z}[x]}{<9x, x^2-3x>}$. By Figure 6(a) and 6(b), $\mathrm{rank}(\Gamma(R)) = 6 = \mathrm{frank}(\Gamma(R))$. Therefore $\Gamma(\frac{x\mathbb{Z}_3[x]}{x^3\mathbb{Z}_3[x]})$ and $\Gamma(\frac{x\mathbb{Z}[x]}{<9x, x^2-3x>})$ are ring graphs.

The zero-divisor graph of the rings $\frac{x\mathbb{Z}[x]}{<8x, x^2-2x>}$, Q_1 and Q_5 are isomorphic to the graph given in Figure 5 and Figures 1.B, 3.B of [11]. Note that rank and frank of this graph is the same and both of them are equal to 1. So, these graphs are ring graphs.

Suppose R is isomorphic to either Q_2 or Q_6. By Figure 2.A and 4.A of [11], we have $\mathrm{rank}(\Gamma(R)) = 3 = \mathrm{frank}(\Gamma(R))$. Hence $\Gamma(Q_2)$ and $\Gamma(Q_6)$ are ring graphs.

Suppose R is isomorphic to either Q_3 or Q_4 or Q_7. By Figure 2.B, 3.A and 4.B of [11], the graphs $\Gamma(Q_3)$, $\Gamma(Q_4)$ and $\Gamma(Q_7)$ are isomorphic. Now, by setting $S = \{\overline{a}, 2\overline{a}, 3\overline{a}, \overline{a}+\overline{b}, 3\overline{a}+\overline{b}\}$, it is easy to see that the induced subgraph by the set S in the graph $\Gamma(Q_3)$ is a θ-partial wheel. So, the graphs $\Gamma(Q_3)$, $\Gamma(Q_4)$ and $\Gamma(Q_7)$ are not ring graphs.

By the above arguments and by Theorem 2.3, the result holds. □

Theorem 4.6. *Let R be a commutative ring without identity. Then $\Gamma(R)$ is an outerplanar graph if and only if R is isomorphic to one of the following:* $\mathbb{Z}_2^0 \times \mathbb{Z}_2^0$, $\mathbb{Z}_2^0 \times \mathbb{F}_{p^n}$, $\mathbb{Z}_2^0 \times \mathbb{Z}_4$, $\mathbb{Z}_2^0 \times \frac{\mathbb{Z}_2[x]}{<x^2>}$ $\mathbb{Z}_2^0$, $\mathbb{Z}_3^0$, $\mathbb{Z}_4^0$, $\frac{x\mathbb{Z}_2[x]}{x^3\mathbb{Z}_2[x]}$, $\frac{x\mathbb{Z}[x]}{<4x, x^2-2x>}$, $\frac{x\mathbb{Z}[x]}{<8x, x^2-2x>}$, Q_1, Q_2, Q_5, Q_6.

Proof. Since every outerplanar graph is a ring graph, it is enough to consider the rings in Theorem 2.3 whose zero-divisor graphs are ring graphs. By similar arguments used in Theorem 4.5, we can verify that the zero-divisor graphs of the rings $\mathbb{Z}_2^0 \times \mathbb{Z}_2^0$, $\mathbb{Z}_2^0 \times \mathbb{F}_{p^n}$, $\mathbb{Z}_2^0 \times \mathbb{Z}_4$, $\mathbb{Z}_2^0 \times \frac{\mathbb{Z}_2[x]}{<x^2>}$ $\mathbb{Z}_2^0$, $\mathbb{Z}_2^0$, $\mathbb{Z}_3^0$, $\mathbb{Z}_4^0$, $\frac{x\mathbb{Z}_2[x]}{x^3\mathbb{Z}_2[x]}$, $\frac{x\mathbb{Z}[x]}{<4x, x^2-2x>}$, $\frac{x\mathbb{Z}[x]}{<8x, x^2-2x>}$, Q_1, Q_2, Q_5 and Q_6 are outerplanar. Also, if R is isomorphic to either $\frac{x\mathbb{Z}_3[x]}{x^3\mathbb{Z}_3[x]}$ or $\frac{x\mathbb{Z}[x]}{<9x, x^2-3x>}$, then by Figures 7(a) and 7(b), $\Gamma(R)$ contains $K_{2,3}$ as a subgraph. Also, since $\Gamma(\mathbb{Z}_3^0 \times \mathbb{F}_{p^n}) \cong K_2 + \overline{K_{3p^n-3}}$, the graph $\Gamma(\mathbb{Z}_3^0 \times \mathbb{F}_{p^n})$ has a copy of the graph $K_{2,3}$, too. So, we can deduce that the graphs $\Gamma(\frac{x\mathbb{Z}_3[x]}{x^3\mathbb{Z}_3[x]})$, $\Gamma(\frac{x\mathbb{Z}[x]}{<9x, x^2-3x>})$ and $\Gamma(\mathbb{Z}_3^0 \times \mathbb{F}_{p^n})$ are not outerplanar graphs. □

In the rest of this section, we study the ring index and outerplanar index of the zero divisor graphs of commutative rings without identity. By Corollary 3.8 and Proposition 3.9 of [5], we conclude that the outerplanar index and ring index are the same when they are equal to 2,3 or ∞. From this classification, we get the following theorem.

Theorem 4.7. *Let R be a finite commutative ring without identity. Then*

(a) $\gamma_r(\Gamma(R)) = \infty$ *if and only if R is isomorphic to one of the following:* $\mathbb{Z}_2^0 \times \mathbb{Z}_2$, $\mathbb{Z}_2^0 \times \mathbb{Z}_2^0$, $\mathbb{Z}_2^0$, $\mathbb{Z}_3^0$, $\mathbb{Z}_4^0$, $\frac{x\mathbb{Z}[x]}{<4x, x^2-2x>}$, $\frac{x\mathbb{Z}_2[x]}{x^3\mathbb{Z}_2[x]}$;

(b) $\gamma_r(\Gamma(R)) = 1$ *if and only if R is isomorphic to one of the following:* $\mathbb{Z}_2^0 \times \mathbb{F}_{p^n}$ *where* $p^n \geq 3$, $\mathbb{Z}_3^0 \times \mathbb{F}_{p^n}$, $\mathbb{Z}_2^0 \times \mathbb{Z}_4$, $\mathbb{Z}_2^0 \times \frac{\mathbb{Z}_2[x]}{<x^2>}$, $\frac{x\mathbb{Z}_3[x]}{x^3\mathbb{Z}_3[x]}$, $\frac{x\mathbb{Z}[x]}{<9x, x^2-3x>}$, $\frac{x\mathbb{Z}[x]}{<8x, x^2-2x>}$, Q_1, Q_2, Q_5, Q_6;

(c) $\gamma_r(\Gamma(R)) = 0$ *otherwise.*

Proof. Let $R \cong R_1 \times R_2 \times \cdots \times R_n$. Since the planar index of a non planar graph is 0, we should focused on the case, $\Gamma(R)$ is planar. For any graph G, by Remark 2.1, $\gamma_r(G) \leq \gamma_p(G)$ together with Theorems 3.2 and Theorem 4.5, would prove assertion (b). So, it is enough to focus on the proof of assertion (a). By Theorem 4.5, we have the following cases.

Case 1. Suppose $n = 2$. Then R is isomorphic to one of the following rings: $\mathbb{Z}_2^0 \times \mathbb{Z}_2^0$, $\mathbb{Z}_2^0 \times \mathbb{Z}_2$.

If $R \cong \mathbb{Z}_2^0 \times \mathbb{Z}_2^0$, then $\Gamma(R) \cong K_3$. By Theorem 4.3, $\gamma_r(\Gamma(R)) = \infty$.

Now, suppose $R \cong \mathbb{Z}_2^0 \times \mathbb{F}_{p^n}$. By Lemma 2.4, $\Gamma(R)$ is isomorphic to $K_1 + \overline{K_{2p^n-2}}$. Therefore if $p^n = 2$, then $\gamma_r(\Gamma(\mathbb{Z}_2^0 \times \mathbb{Z}_2)) = \infty$.

Case 2. Suppose $n = 1$. Then R is isomorphic to one of the following rings: $\mathbb{Z}_2^0$, $\mathbb{Z}_3^0$, $\mathbb{Z}_4^0$, $\frac{x\mathbb{Z}[x]}{<4x,x^2-2x>}$, $\frac{x\mathbb{Z}_2[x]}{x^3\mathbb{Z}_2[x]}$.

We know that if $R \cong \mathbb{Z}_n^0$, then $\Gamma(R)$ is a complete graph with $n-1$ vertices. Then $\Gamma(\mathbb{Z}_2^0)$, $\Gamma(\mathbb{Z}_3^0)$ and $\Gamma(\mathbb{Z}_4^0)$ are isomorphic to either a path or a cycle, and so $\gamma_r(\Gamma(\mathbb{Z}_n^0)) = \infty$ where $n = 2, 3, 4$.

The graph $\Gamma(\frac{x\mathbb{Z}[x]}{<4x,x^2-2x>})$ and $\Gamma(\frac{x\mathbb{Z}[x]}{x^3\mathbb{Z}_2[x]})$ are represented in Figures 4(a) and 4(b). By Theorem 4.3, $\gamma_r(\Gamma(\frac{x\mathbb{Z}[x]}{<4x,x^2-2x>})) = \gamma_r(\Gamma(\frac{x\mathbb{Z}[x]}{x^3\mathbb{Z}_2[x]})) = \infty$. □

In [4], Barati classified the outerplanar index of the zero divisor graphs of finite commutative rings with identity. In the following theorem, we establish the same idea for the zero divisor graphs of finite commutative rings without identity. In fact, we give a full characterization of the zero divisor graphs with respect to their outerplanar index when R is a finite commutative ring without identity.

Theorem 4.8. *Let R be a finite commutative ring without identity. Then*

(a) $\gamma_o(\Gamma(R)) = \infty$ *if and only if R is isomorphic to one of the following:* $\mathbb{Z}_2^0 \times \mathbb{Z}_2$, $\mathbb{Z}_2^0 \times \mathbb{Z}_2^0$, $\mathbb{Z}_2^0$, $\mathbb{Z}_3^0$, $\mathbb{Z}_4^0$, $\frac{x\mathbb{Z}[x]}{<4x,x^2-2x>}$, $\frac{x\mathbb{Z}_2[x]}{x^3\mathbb{Z}_2[x]}$;

(b) $\gamma_o(\Gamma(R)) = 1$ *if and only if R is isomorphic to one of the following:* $\mathbb{Z}_2^0 \times \mathbb{F}_{p^n}$ *where* $p^n \geq 3$, $\mathbb{Z}_2^0 \times \mathbb{Z}_4$, $\mathbb{Z}_2^0 \times \frac{\mathbb{Z}_2[x]}{<x^2>}$, $\frac{x\mathbb{Z}[x]}{<8x,x^2-2x>}$, Q_1, Q_2, Q_5, Q_6;

(c) $\gamma_o(\Gamma(R)) = 0$ *otherwise.*

Proof. For any given graph G, by Remark 2.1 together with Theorems 4.4, 4.6 and 4.7, one can easily verify the assertion (b). By Theorems 4.3 and 4.4, for any graph G, if $\gamma_r(G) = \infty$, then $\gamma_o(G) = \infty$ and by Theorem 4.7, the assertion (a) holds. □

5. Conclusion

In the literature, there are only some few research articles focusing on finite rings without assuming the multiplicative identity. This paper provides the characterization of commutative rings without identity whose zero-divisor graphs are ring graphs and outerplanar graphs. Also, we obtained the planar index, ring index and outerplanar index of the zero-divisor graphs of finite commutative rings without identity. The future work is to address the problem of obtaining various topological indices (like Steiner index, Wiener index, etc.,) for zero-divisor graphs from commutative ring without identity.

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