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Generalizations of Different Type Inequalities for s-Convex, Quasi-Convex and P-Function

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Abstract

The main purpose of this article is to present the Bullen, Midpoint, Trapezoid and Simpson type inequalities, respectively, for different classes of convexity, with the help of identities existing in the literature.

Keywords: *s-convex function, quasi-convex function, P-convex function, Hermite-Hadamard inequalities, Simpson inequality.*
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1. Introduction

Convexity theory is an astonishing and compelling methodology for contemplating the enormous and beautiful issues that arise in many different fields of the pure and applied sciences. Numerous new structures have been presented and explored, including convex sets and related functions. This theory has a rich history and has been the focus and motivation of outstanding mathematical research for more than a century. Also, convexity theory has a critical place in the advancement of the idea of inequality. Inequalities have an interesting mathematical model due to their important applications in traditional calculus, fractional calculus, quantum calculus, interval-valued, stochastic, time-scale calculus, fractal sets, etc.
There are many types of convexity in the literature. The three types of convexity that will be used in this article are as follows.
The concept of s-convex function was introduced in Breckner's paper [3] and a number of properties and connections with s-convexity in the first sense are discussed in the paper [14].

Definition 1.1. *A function $f:[0,\infty)\to\mathbb{R}$ is said to be s-convex in the second sense if*

$$f(\lambda x+(1-\lambda)y)\leq\lambda^{s}f(x)+(1-\lambda)^{s}f(y).$$

for all $x,y\in[0,\infty)$, $t\in[0,1]$ and for some fixed $s\in(0,1]$.

Definition 1.2. *[15] Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ for all $\lambda\in[0,1]$ and all $x,y\in I$, if the following inequality*

$f(\lambda x+(1-\lambda)y)\leq\max\{f(x),f(y)\}$

holds, then f is called a quasi-convex function on I.

Definition 1.3. *[5] A function $f:I\subseteq\mathbb{R}\to\mathbb{R}$ is P-function or that f belongs to the class of $P(I)$, if it is nonnegative and for all $x,y\in I$ and $\lambda\in[0,1]$, satisfies the following inequality:*

$f(\lambda x+(1-\lambda)y)\leq f(x)+f(y).$

Recently, a large number of researchers, including mathematicians, engineers and scientists, have devoted themselves to studying the inequalities and properties associated with convexity in certain different directions. Many integral inequalities have been developed so far by different researchers in the due course of time. In the literature, we have many types of inequalities that involve convex functions, such as Bullen inequality [4], Hermite-Hadamard-Fejér inequality [11], Simpson type inequality [20], and Ostrowski type inequalities [19]. Likewise, there are a lot of well-known integral inequalities but the most notable one is the Hermite-Hadamard integral inequality.

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Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an integrable convex function with $a < b$. Then, the Hermite-Hadamard inequality is expressed as follows: (see [12]):

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}.$$

In [6], Dragomir and Fitzpatrick proved a variant of Hermite-Hadamard inequality which holds for the s-convex functions.

Theorem 1.4. *Suppose that $f : [0,\infty) \to [0,\infty)$ is an s-convex function in the second sense, where $s \in (0,1]$ and let $a,b \in [0,\infty)$, $a < b$. If $f \in L[a,b]$, then the following inequalities hold*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{s+1}. \tag{1.1}$$

There has also been research focusing on the Simpson-type inequality. In particular, Alomari *et al.* [1] studied Simpson's inequality for s-convex functions using differentiable functions. Many researchers have studied Simpson-type inequalities in the literature (see, [2, 7, 10, 13, 16]).

Bullen [4] obtained the well-known Bullen-type inequalities. Bullen-type inequalities for generalized convex functions were obtained by Sarıkaya and Budak [18]. The local fractional version of Bullen-type inequality was presented in [9]. Du *et al.* [8] obtained Bullen-type inequalities using fractional integrals.

In the last few decades, many mathematicians and research scholars have focused their great contributions and attention to the study of this inequality. The aim of this paper, is to establish some new Hermite-Hadamard type inequalities and Simpson-type inequalities for s-convex function, quasi-convex function and P-convex function, respectively.

2. Generalized Bullen Type Inequalities

Theorem 2.1. *Suppose that $f : [0,\infty) \to [0,\infty)$ is an s-convex function in the second sense, where $s \in (0,1]$ and let $a,b \in [0,\infty)$, $a < b$. If $f \in L[a,b]$, then the following inequalities hold*

$$\begin{aligned} 2^{s-1}\left(f\left(\frac{a+x}{2}\right)+f\left(\frac{b+x}{2}\right)\right) &\le \frac{1}{x-a}\int_a^x f(t)dt + \frac{1}{b-x}\int_x^b f(t)dt \\ &\le \frac{f(a)+f(b)+2f(x)}{s+1} \end{aligned} \tag{2.1}$$

for $x \in (a,b)$.

Proof. Since f is a s-convex function in the second sense on $[a,x] \subset [a,b]$, by using the inequalities (1.1) we get

$$2^{s-1} f\left(\frac{a+x}{2}\right) \le \frac{1}{x-a}\int_a^x f(x)dx \le \frac{f(a)+f(x)}{s+1}. \tag{2.2}$$

By similar way for $[x,b] \subset [a,b]$, it follows that

$$2^{s-1} f\left(\frac{b+x}{2}\right) \le \frac{1}{b-a}\int_x^b f(x)dx \le \frac{f(b)+f(x)}{s+1}. \tag{2.3}$$

Consequently, by adding (2.2) and (2.3), we have

$$\begin{aligned} 2^{s-1}\left(f\left(\frac{a+x}{2}\right)+f\left(\frac{b+x}{2}\right)\right) &\le \frac{1}{x-a}\int_a^x f(t)dt + \frac{1}{b-x}\int_x^b f(t)dt \\ &\le \frac{f(a)+f(b)+2f(x)}{s+1} \end{aligned}$$

which completes the proof of the inequality (2.1). □

Remark 2.2. *If we choose the $s = 1$ in the Theorem 2.1, then the inequality (2.1) reduces to the inequality of Theorem 3 in [17].*

3. Trapezoid Type Inequalities

Lemma 3.1. *[17] Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $f' \in L[a,b]$, then the following equality holds:*

$$\begin{aligned} f(x) + \frac{f(a)+f(b)}{2} - \left[\frac{1}{x-a}\int_a^x f(t)dt + \frac{1}{b-x}\int_x^b f(t)dt\right] &= \frac{x-a}{2}\int_0^1 (1-2\lambda)f'(\lambda a+(1-\lambda)x)d\lambda \\ &\quad + \frac{b-x}{2}\int_0^1 (1-2\lambda)f'(\lambda x+(1-\lambda)b)d\lambda \end{aligned}$$

for $x \in (a,b)$.

Theorem 3.2. *Let $f : [a,b] \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $|f'|$ is s-convex in the second sense on $[a,b]$, then*

$$\begin{aligned} &\left| f(x) + \frac{f(a)+f(b)}{2} - \left[\frac{1}{x-a}\int_a^x f(t)dt + \frac{1}{b-x}\int_x^b f(t)dt \right] \right| \\ \leq\ & \left(\frac{1}{2^{s+1}(s+1)(s+2)} + \frac{s}{2(s+1)(s+2)} \right)(b-a)|f'(x)| \\ &+ \left(\frac{1}{2^{s+1}(s+1)(s+2)} + \frac{s}{2(s+1)(s+2)} \right)(x-a)|f'(a)| + (b-x)|f'(b)| \end{aligned} \tag{3.1}$$

for $x \in (a,b)$.

Proof. From Lemma 3.1, by using the properties of modulus and $|f'|$ is s-convex in the second sense on $[a,b]$, we have

$$\begin{aligned} &\left| f(x) + \frac{f(a)+f(b)}{2} - \left[\frac{1}{x-a}\int_a^x f(t)dt + \frac{1}{b-x}\int_x^b f(t)dt \right] \right| \\ \leq\ & \frac{x-a}{2}\int_0^1 |1-2\lambda||f'(\lambda a + (1-\lambda)x)d\lambda| + \frac{b-x}{2}\int_0^1 |1-2\lambda||f'(\lambda x + (1-\lambda)b)d\lambda| \\ \leq\ & \frac{x-a}{2}\int_0^1 |1-2\lambda| \left[\lambda^s|f'(a)| + (1-\lambda)^s|f'(x)|\right] d\lambda \\ &+ \frac{b-x}{2}\int_0^1 |1-2\lambda| \left[\lambda^s|f'(x)| + (1-\lambda)^s|f'(b)|\right] d\lambda \\ =\ & \frac{x-a}{2}\int_0^{\frac{1}{2}} (1-2\lambda) \left[\lambda^s|f'(a)| + (1-\lambda)^s|f'(x)|\right] d\lambda \\ &+ \frac{x-a}{2}\int_{\frac{1}{2}}^1 (2\lambda-1) \left[\lambda^s|f'(a)| + (1-\lambda)^s|f'(x)|\right] d\lambda \\ &+ \frac{b-x}{2}\int_0^{\frac{1}{2}} (1-2\lambda) \left[\lambda^s|f'(x)| + (1-\lambda)^s|f'(b)|\right] d\lambda \\ &+ \frac{b-x}{2}\int_{\frac{1}{2}}^1 (2\lambda-1) \left[\lambda^s|f'(x)| + (1-\lambda)^s|f'(b)|\right] d\lambda \\ =\ & \left(\frac{1}{2^{s+1}(s+1)(s+2)} + \frac{s}{2(s+1)(s+2)} \right)(b-a)|f'(x)| \\ &+ \left(\frac{1}{2^{s+1}(s+1)(s+2)} + \frac{s}{2(s+1)(s+2)} \right)(x-a)|f'(a)| + (b-x)|f'(b)| \end{aligned}$$

which completes the proof of the inequality (3.1). □

Remark 3.3. *If we choose $s = 1$ in the Theorem 3.2, then the inequality (3.1) reduces to the inequality of Theorem 4 in [17].*

Theorem 3.4. *Let $f : [a,b] \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $|f'|^q$ is s-convex in the second sense on $[a,b]$ for some $q > 1$, then*

$$\begin{aligned} &\left| f(x) + \frac{f(a)+f(b)}{2} - \left[\frac{1}{x-a}\int_a^x f(t)dt + \frac{1}{b-x}\int_x^b f(t)dt \right] \right| \\ \leq\ & \frac{x-a}{2(p+1)^{\frac{1}{p}}} \left(\frac{|f'(a)|^q + |f'(x)|^q}{s+1} \right)^{\frac{1}{q}} + \frac{b-x}{2(p+1)^{\frac{1}{p}}} \left(\frac{|f'(b)|^q + |f'(x)|^q}{s+1} \right)^{\frac{1}{q}} \end{aligned} \tag{3.2}$$

where $x \in (a,b)$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 3.1, by using Hölder inequality and $|f'|^q$ is s-convex in the second sense on $[a,b]$, we have

$$\begin{aligned}
&\left|f(x)+\frac{f(a)+f(b)}{2}-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right| \\
\leq\ & \frac{x-a}{2}\int_0^1 |1-2\lambda||f'(\lambda a+(1-\lambda)x)d\lambda|+\frac{b-x}{2}\int_0^1 |1-2\lambda||f'(\lambda x+(1-\lambda)b)d\lambda| \\
\leq\ & \frac{x-a}{2}\left(\int_0^1 |1-2\lambda|^p d\lambda\right)^{\frac{1}{p}}\left(\int_0^1 |f'(\lambda a+(1-\lambda)x)d\lambda|\right)^{\frac{1}{q}} \\
&+\frac{b-x}{2}\left(\int_0^1 |1-2\lambda|^p d\lambda\right)^{\frac{1}{p}}\left(\int_0^1 |f'(\lambda x+(1-\lambda)b)d\lambda|\right)^{\frac{1}{q}} \\
\leq\ & \frac{x-a}{2(p+1)^{\frac{1}{p}}}\left(\int_0^1 (\lambda^s|f'(a)|^q+(1-\lambda)^s|f'(x)|^q)d\lambda\right)^{\frac{1}{q}} \\
&+\frac{b-x}{2(p+1)^{\frac{1}{p}}}\left(\int_0^1 (\lambda^s|f'(x)|^q+(1-\lambda)^s|f'(b)|^q)d\lambda\right)^{\frac{1}{q}} \\
=\ & \frac{x-a}{2(p+1)^{\frac{1}{p}}}\left(\frac{|f'(a)|^q+|f'(x)|^q}{s+1}\right)^{\frac{1}{q}}+\frac{b-x}{2(p+1)^{\frac{1}{p}}}\left(\frac{|f'(b)|^q+|f'(x)|^q}{s+1}\right)^{\frac{1}{q}}
\end{aligned}$$

which completes the proof of the inequality (3.2). □

Remark 3.5. *If we choose $s=1$ in the Theorem 3.4, then the inequality (3.2) reduces to the inequality of Theorem 5 in [17].*

Theorem 3.6. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|$ is quasi-convex on $[a,b]$, then*

$$\begin{aligned}
\left|f(x)+\frac{f(a)+f(b)}{2}-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right| &\leq \frac{x-a}{4}\max\{|f'(a)|,|f'(x)|\}+\frac{b-x}{4}\max\{|f'(x)|,|f'(b)|\} \\
&\leq \frac{b-a}{4}\max\{|f'(x)|,|f'(a)|,|f'(b)|\}
\end{aligned}$$

for $x\in(a,b)$.

Proof. From Lemma 3.1, by using the properties of modulus and $|f'|$ is quasi-convex we have

$$\begin{aligned}
&\left|f(x)+\frac{f(a)+f(b)}{2}-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right| \\
\leq\ & \frac{x-a}{2}\int_0^1 |1-2\lambda||f'(\lambda a+(1-\lambda)x)d\lambda|+\frac{b-x}{2}\int_0^1 |1-2\lambda||f'(\lambda x+(1-\lambda)b)d\lambda| \\
\leq\ & \frac{x-a}{2}\int_0^1 |1-2\lambda|\max\{|f'(a)|,|f'(x)|\}d\lambda+\frac{b-x}{2}\int_0^1 |1-2\lambda|\max\{|f'(x)|,|f'(b)|\}d\lambda \\
=\ & \frac{x-a}{4}\max\{|f'(a)|,|f'(x)|\}+\frac{b-x}{4}\max\{|f'(x)|,|f'(b)|\}.
\end{aligned}$$

So, the proof is completed. □

Theorem 3.7. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|^q$ is quasi-convex on $[a,b]$ for some $q>1$, then*

$$\begin{aligned}
&\left|f(x)+\frac{f(a)+f(b)}{2}-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right| \\
\leq\ & \frac{x-a}{2(p+1)^{\frac{1}{p}}}\left(\max\{|f'(a)|^q,|f'(x)|^q\}\right)^{\frac{1}{q}}+\frac{b-x}{2(p+1)^{\frac{1}{p}}}\left(\max\{|f'(x)|^q,|f'(b)|^q\}\right)^{\frac{1}{q}}
\end{aligned}$$

where $x\in(a,b)$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 3.1, by using Hölder inequality and $|f'|^q$ is quasi-convex on $[a,b]$, we have

$$\begin{aligned}
&\left|f(x)+\frac{f(a)+f(b)}{2}-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right| \\
\leq\ & \frac{x-a}{2}\int_0^1 |1-2\lambda||f'(\lambda a+(1-\lambda)x)d\lambda|+\frac{b-x}{2}\int_0^1 |1-2\lambda||f'(\lambda x+(1-\lambda)b)d\lambda| \\
\leq\ & \frac{x-a}{2}\left(\int_0^1 |1-2\lambda|^p d\lambda\right)^{\frac{1}{p}}\left(|f'(\lambda a+(1-\lambda)x)d\lambda|\right)^{\frac{1}{q}} \\
&+\frac{b-x}{2}\left(\int_0^1 |1-2\lambda|^p d\lambda\right)^{\frac{1}{p}}\left(|f'(\lambda x+(1-\lambda)b)d\lambda|\right)^{\frac{1}{q}} \\
\leq\ & \frac{x-a}{2(p+1)^{\frac{1}{p}}}\left(\int_0^1 \max\{|f'(a)|^q,|f'(x)|^q\}d\lambda\right)^{\frac{1}{q}} \\
&+\frac{b-x}{2(p+1)^{\frac{1}{p}}}\left(\int_0^1 \max\{|f'(x)|^q,|f'(b)|^q\}d\lambda\right)^{\frac{1}{q}} \\
=\ & \frac{x-a}{2(p+1)^{\frac{1}{p}}}\left(\max\{|f'(a)|^q,|f'(x)|^q\}\right)^{\frac{1}{q}}+\frac{b-x}{2(p+1)^{\frac{1}{p}}}\left(\max\{|f'(x)|^q,|f'(b)|^q\}\right)^{\frac{1}{q}}.
\end{aligned}$$

So, the proof is completed. □

Theorem 3.8. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|$ is P-convex on $[a,b]$, then*

$$\left|f(x)+\frac{f(a)+f(b)}{2}-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right| \leq \frac{(x-a)[|f'(a)|+|f'(x)|]}{4}+\frac{(b-x)[|f'(x)|+|f'(b)|]}{4}$$

for $x\in(a,b)$.

Proof. From Lemma 3.1, by using the properties of modulus and $|f'|$ is P-convex on $[a,b]$, we have

$$\begin{aligned}
&\left|f(x)+\frac{f(a)+f(b)}{2}-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right| \\
\leq\ & \frac{x-a}{2}\int_0^1 |1-2\lambda||f'(\lambda a+(1-\lambda)x)d\lambda|+\frac{b-x}{2}\int_0^1 |1-2\lambda||f'(\lambda x+(1-\lambda)b)d\lambda| \\
\leq\ & \frac{x-a}{2}\int_0^1 |1-2\lambda|\left[|f'(a)|+|f'(x)|\right]d\lambda+\frac{b-x}{2}\int_0^1 |1-2\lambda|\left[|f'(x)|+|f'(b)|\right]d\lambda \\
=\ & \frac{(x-a)[|f'(a)|+|f'(x)|]}{4}+\frac{(b-x)[|f'(x)|+|f'(b)|]}{4}.
\end{aligned}$$

So, the proof is completed. □

Theorem 3.9. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|^q$ is P-convex on $[a,b]$ for some $q>1$, then*

$$\left|f(x)+\frac{f(a)+f(b)}{2}-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right| \leq \frac{x-a}{2(p+1)^{\frac{1}{p}}}\left(|f'(a)|^q+|f'(x)|^q\right)^{\frac{1}{q}}+\frac{b-x}{2(p+1)^{\frac{1}{p}}}\left(|f'(b)|^q+|f'(x)|^q\right)^{\frac{1}{q}}$$

where $x\in(a,b)$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 3.1, by using Hölder inequality and $|f'|^q$ is P-convex on $[a,b]$, we have

$$\begin{aligned}
&\left|f(x)+\frac{f(a)+f(b)}{2}-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right| \\
\leq\ & \frac{x-a}{2}\int_0^1 |1-2\lambda||f'(\lambda a+(1-\lambda)x)d\lambda|+\frac{b-x}{2}\int_0^1 |1-2\lambda||f'(\lambda x+(1-\lambda)b)d\lambda| \\
\leq\ & \frac{x-a}{2}\left(\int_0^1 |1-2\lambda|^p d\lambda\right)^{\frac{1}{p}}\left(\int_0^1 |f'(\lambda a+(1-\lambda)x)d\lambda|\right)^{\frac{1}{q}} \\
&+\frac{b-x}{2}\left(\int_0^1 |1-2\lambda|^p d\lambda\right)^{\frac{1}{p}}\left(\int_0^1 |f'(\lambda x+(1-\lambda)b)d\lambda|\right)^{\frac{1}{q}} \\
\leq\ & \frac{x-a}{2(p+1)^{\frac{1}{p}}}\left(\int_0^1 (|f'(a)|^q+|f'(x)|^q)d\lambda\right)^{\frac{1}{q}} \\
&+\frac{b-x}{2(p+1)^{\frac{1}{p}}}\left(\int_0^1 (|f'(x)|^q+|f'(b)|^q)d\lambda\right)^{\frac{1}{q}} \\
=\ & \frac{x-a}{2(p+1)^{\frac{1}{p}}}\left(|f'(a)|^q+|f'(x)|^q\right)^{\frac{1}{q}}+\frac{b-x}{2(p+1)^{\frac{1}{p}}}\left(|f'(b)|^q+|f'(x)|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

So, the proof is completed. □

4. Midpoint Type Inequalities

Lemma 4.1. *[17] Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. $f'\in L[a,b]$, then the following equality holds:*

$$f\left(\frac{a+x}{2}\right)+f\left(\frac{b+x}{2}\right)-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right] = (x-a)\int_0^{\frac{1}{2}}\lambda[f'(\lambda x+(1-\lambda)a)-f'(\lambda a+(1-\lambda)x)]d\lambda$$
$$+(b-x)\int_{\frac{1}{2}}^1(1-\lambda)[f'(\lambda x+(1-\lambda)b)-f'(\lambda b+(1-\lambda)x)]d\lambda$$

for $x\in(a,b)$.

Theorem 4.2. *Let $f:[a,b]\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|$ is s-convex in the second sense on $[a,b]$, then*

$$\left|f\left(\frac{a+x}{2}\right)+f\left(\frac{b+x}{2}\right)-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right| \tag{4.1}$$
$$= \frac{2^{s+1}-1}{2^{s+1}(s+1)(s+2)}(b-a)|f'(x)|+\frac{2^{s+1}-1}{2^{s+1}(s+1)(s+2)}\left[(x-a)|f'(a)|+(b-x)|f'(b)|\right]$$

for $x\in(a,b)$.

Proof. From Lemma 4.1, by using the properties of modulus and $|f'|$ is s-convex in the second sense on $[a,b]$, we have

$$\left|f\left(\frac{a+x}{2}\right)+f\left(\frac{b+x}{2}\right)-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right|$$
$$\leq (x-a)\int_0^{\frac{1}{2}}\lambda[|f'(\lambda x+(1-\lambda)a)|+|f'(\lambda a+(1-\lambda)x)|]d\lambda$$
$$+(b-x)\int_{\frac{1}{2}}^1(1-\lambda)[|f'(\lambda x+(1-\lambda)b)|+|f'(\lambda b+(1-\lambda)x)|]d\lambda$$
$$\leq (x-a)\int_0^{\frac{1}{2}}\lambda[\lambda^s|f'(x)|+(1-\lambda)^s|f'(a)|+\lambda^s|f'(a)|+(1-\lambda)^s|f'(x)|]d\lambda$$
$$+(b-x)\int_{\frac{1}{2}}^1(1-\lambda)[\lambda^s|f'(x)|+(1-\lambda)^s|f'(b)|+\lambda^s|f'(b)|+(1-\lambda)^s|f'(x)|]d\lambda$$
$$= \frac{2^{s+1}-1}{2^{s+1}(s+1)(s+2)}(b-a)|f'(x)|+\frac{2^{s+1}-1}{2^{s+1}(s+1)(s+2)}\left[(x-a)|f'(a)|+(b-x)|f'(b)|\right]$$

which completes the proof of the inequality (4.1). □

Remark 4.3. *If we choose $s=1$ in the Theorem 4.2, then the inequality (4.1) reduces to the inequality of Theorem 6 in [17].*

Theorem 4.4. *Let $f:[a,b]\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|^q$ is s-convex in the second sense on $[a,b]$ for some $q>1$, then*

$$\left|f\left(\frac{a+x}{2}\right)+f\left(\frac{b+x}{2}\right)-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right| \tag{4.2}$$
$$\leq \frac{x-a}{(1+p)^{\frac{1}{p}}2^{1+\frac{1}{p}}}\left[\left(\frac{|f'(x)|^q+|f'(a)|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+|f'(x)|^q}{s+1}\right)^{\frac{1}{q}}\right]$$
$$+\frac{b-x}{(1+p)^{\frac{1}{p}}2^{1+\frac{1}{p}}}\left[\left(\frac{|f'(x)|^q+|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{|f'(b)|^q+|f'(x)|^q}{s+1}\right)^{\frac{1}{q}}\right]$$

where $x\in(a,b)$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 4.1, by using Hölder inequality and $|f'|^q$ is s-convex in the second sense on $[a,b]$, we have

$$\left|f\left(\frac{a+x}{2}\right)+f\left(\frac{b+x}{2}\right)-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right|$$
$$\leq (x-a)\left(\int_0^{\frac{1}{2}}\lambda^p d\lambda\right)^{\frac{1}{p}}\left[\left(\int_0^{\frac{1}{2}}|f'(\lambda x+(1-\lambda)a)|^q\right)^{\frac{1}{q}}+\left(\int_0^{\frac{1}{2}}|f'(\lambda a+(1-\lambda)x)|^q\right)^{\frac{1}{q}}\right]$$
$$+(b-x)\left(\int_{\frac{1}{2}}^1\lambda^p d\lambda\right)^{\frac{1}{p}}\left[\left(\int_{\frac{1}{2}}^1|f'(\lambda x+(1-\lambda)b)|^q\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1|f'(\lambda b+(1-\lambda)x)|^q\right)^{\frac{1}{q}}\right]$$
$$\leq\frac{x-a}{(1+p)^{\frac{1}{p}}2^{1+\frac{1}{p}}}\left[\left(\int_0^{\frac{1}{2}}[\lambda^s|f'(x)|^q+(1-\lambda)^s|f'(a)|^q]d\lambda\right)^{\frac{1}{q}}+\left(\int_0^{\frac{1}{2}}[\lambda^s|f'(a)|^q+(1-\lambda)^s)|f'(x)|^q]d\lambda\right)^{\frac{1}{q}}\right]$$
$$+\frac{b-x}{(1+p)^{\frac{1}{p}}2^{1+\frac{1}{p}}}\left[\left(\int_{\frac{1}{2}}^1[\lambda^s|f'(x)|^q+(1-\lambda)^s|f'(b)|^q]d\lambda\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1[\lambda^s|f'(b)|^q+(1-\lambda)^s)|f'(x)|^q]d\lambda\right)^{\frac{1}{q}}\right]$$
$$=\frac{x-a}{(1+p)^{\frac{1}{p}}2^{1+\frac{1}{p}}}\left[\left(\frac{|f'(x)|^q+|f'(a)|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+|f'(x)|^q}{s+1}\right)^{\frac{1}{q}}\right]$$
$$+\frac{b-x}{(1+p)^{\frac{1}{p}}2^{1+\frac{1}{p}}}\left[\left(\frac{|f'(x)|^q+|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{|f'(b)|^q+|f'(x)|^q}{s+1}\right)^{\frac{1}{q}}\right].$$

So, the proof is completed. □

Remark 4.5. *If we choose $s=1$ in the Theorem 4.4, then the inequality (4.2) reduces to the inequality of Theorem 7 in [17].*

Theorem 4.6. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|$ is quasi-convex on $[a,b]$, then*

$$\left|f\left(\frac{a+x}{2}\right)+f\left(\frac{b+x}{2}\right)-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right| = \frac{x-a}{8}\left(\max\{|f'(x)|,|f'(a)|\}+\max\{|f'(a)|,|f'(x)|\}\right)$$
$$+\frac{b-x}{8}\left(\max\{|f'(x)|,|f'(b)|\}+\max\{|f'(b)|,|f'(x)|\}\right)$$

for $x\in(a,b)$.

Proof. From Lemma 4.1, by using the properties of modulus and $|f'|$ is quasi-convex on $[a,b]$, we have

$$\left|f\left(\frac{a+x}{2}\right)+f\left(\frac{b+x}{2}\right)-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right|$$
$$\leq (x-a)\int_0^{\frac{1}{2}}\lambda[|f'(\lambda x+(1-\lambda)a)|+|f'(\lambda a+(1-\lambda)x)|]d\lambda$$
$$+(b-x)\int_{\frac{1}{2}}^1(1-\lambda)[|f'(\lambda x+(1-\lambda)b)|+|f'(\lambda b+(1-\lambda)x)|]d\lambda$$
$$\leq (x-a)\int_0^{\frac{1}{2}}\lambda[\max\{|f'(x)|,|f'(a)|\}+\max\{|f'(a)|,|f'(x)|\}]d\lambda$$
$$+(b-x)\int_{\frac{1}{2}}^1(1-\lambda)[\max\{|f'(x)|,|f'(b)|\}+\max\{|f'(b)|,|f'(x)|\}]d\lambda$$
$$=\frac{x-a}{8}\left(\max\{|f'(x)|,|f'(a)|\}+\max\{|f'(a)|,|f'(x)|\}\right)$$
$$+\frac{b-x}{8}\left(\max\{|f'(x)|,|f'(b)|\}+\max\{|f'(b)|,|f'(x)|\}\right).$$

So, the proof is completed. □

Theorem 4.7. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|^q$ is quasi-convex on $[a,b]$ for some $q>1$, then*

$$\left|f\left(\frac{a+x}{2}\right)+f\left(\frac{b+x}{2}\right)-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right|$$
$$\leq \frac{x-a}{4(1+p)^{\frac{1}{p}}}\left[\left(\max\{|f'(x)|^q,|f'(a)|^q\}\right)^{\frac{1}{q}}+\left(\max\{|f'(a)|^q,|f'(x)|^q\}\right)^{\frac{1}{q}}\right]$$
$$+\frac{b-x}{4(1+p)^{\frac{1}{p}}}\left[\left(\max\{|f'(x)|^q,|f'(b)|^q\}\right)^{\frac{1}{q}}+\left(\max\{|f'(b)|^q,|f'(x)|^q\}\right)^{\frac{1}{q}}\right]$$

where $x\in(a,b)$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 4.1, by using Hölder inequality and $|f'|^q$ is quasi-convex on $[a,b]$, we have

$$\begin{aligned}
&\left|f\left(\frac{a+x}{2}\right)+f\left(\frac{b+x}{2}\right)-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right| \\
\leq\ & (x-a)\left(\int_0^{\frac{1}{2}}\lambda^p d\lambda\right)^{\frac{1}{p}}\left[\left(\int_0^{\frac{1}{2}}|f'(\lambda x+(1-\lambda)a)|^q\right)^{\frac{1}{q}}+\left(\int_0^{\frac{1}{2}}|f'(\lambda a+(1-\lambda)x)|^q\right)^{\frac{1}{q}}\right] \\
&+(b-x)\left(\int_{\frac{1}{2}}^1\lambda^p d\lambda\right)^{\frac{1}{p}}\left[\left(\int_{\frac{1}{2}}^1|f'(\lambda x+(1-\lambda)b)|^q\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1|f'(\lambda b+(1-\lambda)x)|^q\right)^{\frac{1}{q}}\right] \\
\leq\ & \frac{x-a}{(1+p)^{\frac{1}{p}}2^{1+\frac{1}{p}}}\left[\left(\int_0^{\frac{1}{2}}\max\{|f'(x)|^q,|f'(a)|^q\}d\lambda\right)^{\frac{1}{q}}+\left(\int_0^{\frac{1}{2}}\max\{|f'(a)|,|f'(x)|\}d\lambda\right)^{\frac{1}{q}}\right] \\
&+\frac{b-x}{(1+p)^{\frac{1}{p}}2^{1+\frac{1}{p}}}\left[\left(\int_{\frac{1}{2}}^1\max\{|f'(x)|^q,|f'(b)|^q\}d\lambda\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1\max\{|f'(b)|^q,|f'(x)|^q\}d\lambda\right)^{\frac{1}{q}}\right] \\
=\ & \frac{x-a}{4(1+p)^{\frac{1}{p}}}\left[\left(\max\{|f'(x)|^q,|f'(a)|^q\}\right)^{\frac{1}{q}}+\left(\max\{|f'(a)|^q,|f'(x)|^q\}\right)^{\frac{1}{q}}\right] \\
&+\frac{b-x}{4(1+p)^{\frac{1}{p}}}\left[\left(\max\{|f'(x)|^q,|f'(b)|^q\}\right)^{\frac{1}{q}}+\left(\max\{|f'(b)|^q,|f'(x)|^q\}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

So, the proof is completed. □

Theorem 4.8. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|$ is P-convex on $[a,b]$, then*

$$\begin{aligned}
&\left|f\left(\frac{a+x}{2}\right)+f\left(\frac{b+x}{2}\right)-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right| \\
\leq\ & \frac{(x-a)[|f'(x)|+|f'(a)|]}{4}+\frac{(b-x)[|f'(x)|+|f'(b)|]}{4}
\end{aligned}$$

for $x\in(a,b)$.

Proof. From Lemma 4.1, by using the properties of modulus and $|f'|$ is P-convex on $[a,b]$, we have

$$\begin{aligned}
&\left|f\left(\frac{a+x}{2}\right)+f\left(\frac{b+x}{2}\right)-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right| \\
\leq\ & (x-a)\int_0^{\frac{1}{2}}\lambda[|f'(\lambda x+(1-\lambda)a)|+|f'(\lambda a+(1-\lambda)x)|]d\lambda \\
&+(b-x)\int_{\frac{1}{2}}^1(1-\lambda)[|f'(\lambda x+(1-\lambda)b)|+|f'(\lambda b+(1-\lambda)x)|]d\lambda \\
\leq\ & 2(x-a)\int_0^{\frac{1}{2}}\lambda[|f'(x)|+|f'(a)|]d\lambda+2(b-x)\int_{\frac{1}{2}}^1(1-\lambda)[|f'(x)|+|f'(b)|]d\lambda \\
=\ & \frac{(x-a)[|f'(x)|+|f'(a)|]}{4}+\frac{(b-x)[|f'(x)|+|f'(b)|]}{4}
\end{aligned}$$

So, the proof is completed. □

Theorem 4.9. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|^q$ is P-convex on $[a,b]$, then*

$$\begin{aligned}
&\left|f\left(\frac{a+x}{2}\right)+f\left(\frac{b+x}{2}\right)-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right| \\
\leq\ & \frac{x-a}{(1+p)^{\frac{1}{p}}2^{1+\frac{1}{p}}}\left[\left(\frac{|f'(x)|^q+|f'(a)|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+|f'(x)|^q}{2}\right)^{\frac{1}{q}}\right] \\
&+\frac{b-x}{(1+p)^{\frac{1}{p}}2^{1+\frac{1}{p}}}\left[\left(\frac{|f'(x)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{|f'(b)|^q+|f'(x)|^q}{2}\right)^{\frac{1}{q}}\right]
\end{aligned}$$

where $x\in(a,b)$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 4.1, by using Hölder inequality and $|f'|^q$ is P-convex on $[a,b]$, we have

$$
\begin{aligned}
& \left| f\left(\frac{a+x}{2}\right) + f\left(\frac{b+x}{2}\right) - \left[\frac{1}{x-a}\int_a^x f(t)dt + \frac{1}{b-x}\int_x^b f(t)dt\right]\right| \\
\leq\ & (x-a)\left(\int_0^{\frac{1}{2}} \lambda^p d\lambda\right)^{\frac{1}{p}} \left[\left(\int_0^{\frac{1}{2}} |f'(\lambda x+(1-\lambda)a)|^q\right)^{\frac{1}{q}} + \left(\int_0^{\frac{1}{2}} |f'(\lambda a+(1-\lambda)x)|^q\right)^{\frac{1}{q}}\right] \\
& +(b-x)\left(\int_{\frac{1}{2}}^1 \lambda^p d\lambda\right)^{\frac{1}{p}} \left[\left(\int_{\frac{1}{2}}^1 |f'(\lambda x+(1-\lambda)b)|^q\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 |f'(\lambda b+(1-\lambda)x)|^q\right)^{\frac{1}{q}}\right] \\
\leq\ & \frac{x-a}{(1+p)^{\frac{1}{p}} 2^{1+\frac{1}{p}}} \left[\left(\int_0^{\frac{1}{2}} [|f'(x)|^q + |f'(a)|^q] d\lambda\right)^{\frac{1}{q}} + \left(\int_0^{\frac{1}{2}} [|f'(a)|^q + |f'(x)|^q] d\lambda\right)^{\frac{1}{q}}\right] \\
& + \frac{b-x}{(1+p)^{\frac{1}{p}} 2^{1+\frac{1}{p}}} \left[\left(\int_{\frac{1}{2}}^1 [|f'(x)|^q + |f'(b)|^q] d\lambda\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 [|f'(b)|^q + |f'(x)|^q] d\lambda\right)^{\frac{1}{q}}\right] \\
=\ & \frac{x-a}{(1+p)^{\frac{1}{p}} 2^{1+\frac{1}{p}}} \left[\left(\frac{|f'(x)|^q + |f'(a)|^q}{2}\right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^q + |f'(x)|^q}{2}\right)^{\frac{1}{q}}\right] \\
& + \frac{b-x}{(1+p)^{\frac{1}{p}} 2^{1+\frac{1}{p}}} \left[\left(\frac{|f'(x)|^q + |f'(b)|^q}{2}\right)^{\frac{1}{q}} + \left(\frac{|f'(b)|^q + |f'(x)|^q}{2}\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

So, the proof is completed. □

5. Simpson Type Inequalities

Lemma 5.1. *[17] Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. $f'\in L[a,b]$, then the following equality holds:*

$$
\begin{aligned}
& \frac{1}{3}\left[2f\left(\frac{a+x}{2}\right) + 2f\left(\frac{b+x}{2}\right) + f(x) + \frac{f(a)+f(b)}{2}\right] \\
& -\left[\frac{1}{x-a}\int_a^x f(t)dt + \frac{1}{b-x}\int_x^b f(t)dt\right] \\
=\ & (x-a)\int_0^{\frac{1}{2}} \left(\lambda - \frac{1}{6}\right)[f'(\lambda x+(1-\lambda)a) - f'(\lambda a+(1-\lambda)x)]d\lambda \\
& +(b-x)\int_{\frac{1}{2}}^1 \left(\frac{5}{6} - \lambda\right)[f'(\lambda x+(1-\lambda)b) - f'(\lambda b+(1-\lambda)x)]d\lambda
\end{aligned}
$$

for $x\in(a,b)$.

Theorem 5.2. *Let $f:[a,b]\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|$ is s-convex in the second sense on $[a,b]$, then*

$$
\begin{aligned}
& \left|\frac{1}{3}\left[2f\left(\frac{a+x}{2}\right) + 2f\left(\frac{b+x}{2}\right) + f(x) + \frac{f(a)+f(b)}{2}\right]\right. \\
& \left. -\left[\frac{1}{x-a}\int_a^x f(t)dt + \frac{1}{b-x}\int_x^b f(t)dt\right]\right| \\
\leq\ & \left(\frac{2^{-s-1}(1+5^{s+2})}{3^{s+2}(s+1)(s+2)} - \frac{1}{2^{s+1}(s+1)(s+2)} + \frac{s-4}{6(s+1)(s+2)}\right)|f'(x)| \\
& +\left(\frac{2^{-s-1}(1+5^{s+2})}{3^{s+2}(s+1)(s+2)} - \frac{1}{2^{s+1}(s+1)(s+2)} + \frac{s-4}{6(s+1)(s+2)}\right)[(x-a)|f'(a)| + (b-x)|f'(b)|]
\end{aligned}
\tag{5.1}
$$

for $x\in(a,b)$.

Proof. From Lemma 5.1, by using the properties of modulus and s-convexity of $|f'|$, we have

$$\begin{aligned}
&\left|\frac{1}{3}\left[2f\left(\frac{a+x}{2}\right)+2f\left(\frac{b+x}{2}\right)+f(x)+\frac{f(a)+f(b)}{2}\right]\right.\\
&\left.-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right|\\
\leq&(x-a)\int_0^{\frac{1}{2}}\left|\lambda-\frac{1}{6}\right|[|f'(\lambda x+(1-\lambda)a)|+|f'(\lambda a+(1-\lambda)x)|]d\lambda\\
&+(b-x)\int_{\frac{1}{2}}^1\left|\frac{5}{6}-\lambda\right|[|f'(\lambda x+(1-\lambda)b)|+|f'(\lambda b+(1-\lambda)x)|]d\lambda\\
\leq&(x-a)\int_0^{\frac{1}{2}}\left|\lambda-\frac{1}{6}\right|[\lambda^s|f'(x)|+(1-\lambda)^s|f'(a)|+\lambda^s|f'(a)|+(1-\lambda)^s|f'(x)|]d\lambda\\
&+(b-x)\int_{\frac{1}{2}}^1\left|\frac{5}{6}-\lambda\right|[\lambda^s|f'(x)|+(1-\lambda)^s|f'(b)|+\lambda^s|f'(b)|+(1-\lambda)^s|f'(x)|]d\lambda\\
=&\left(\frac{2^{-s-1}(1+5^{s+2})}{3^{s+2}(s+1)(s+2)}-\frac{1}{2^{s+1}(s+1)(s+2)}+\frac{s-4}{6(s+1)(s+2)}\right)|f'(x)|\\
&+\left(\frac{2^{-s-1}(1+5^{s+2})}{3^{s+2}(s+1)(s+2)}-\frac{1}{2^{s+1}(s+1)(s+2)}+\frac{s-4}{6(s+1)(s+2)}\right)[(x-a)|f'(a)|+(b-x)|f'(b)|]
\end{aligned}$$

which completes the proof of the inequality (5.1). □

Remark 5.3. *If we choose $s=1$ in the Theorem 5.2, then the inequality (5.1) reduces to the inequality of Theorem 8 in [17].*

Theorem 5.4. *Let $f:[a,b]\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|^q$ is s-convex in the second sense on $[a,b]$ for some $q>1$, then*

$$\begin{aligned}
&\left|\frac{1}{3}\left[2f\left(\frac{a+x}{2}\right)+2f\left(\frac{b+x}{2}\right)+f(x)+\frac{f(a)+f(b)}{2}\right]\right.\\
&\left.-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right|\\
\leq&\frac{x-a}{(1+p)^{\frac{1}{p}}}\left[\frac{1}{3^{p+1}}+\frac{1}{6^{p+1}}\right]^{\frac{1}{p}}\left[\left(\frac{|f'(x)|^q+|f'(a)|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+|f'(x)|^q}{s+1}\right)^{\frac{1}{q}}\right]\\
&+\frac{b-x}{(1+p)^{\frac{1}{p}}}\left[\frac{1}{3^{p+1}}+\frac{1}{6^{p+1}}\right]^{\frac{1}{p}}\left[\left(\frac{|f'(x)|^q+|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{|f'(b)|^q+|f'(x)|^q}{s+1}\right)^{\frac{1}{q}}\right]
\end{aligned}\tag{5.2}$$

where $x\in(a,b)$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 5.1, by using Hölder inequality and $|f'|^q$ is s-convex in the second sense on $[a,b]$, we have

$$\begin{aligned}
&\left|\frac{1}{3}\left[2f\left(\frac{a+x}{2}\right)+2f\left(\frac{b+x}{2}\right)+f(x)+\frac{f(a)+f(b)}{2}\right]\right.\\
&\left.-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right|\\
\leq&(x-a)\left(\int_0^{\frac{1}{2}}\left|\lambda-\frac{1}{6}\right|^p d\lambda\right)^{\frac{1}{p}}\left[\left(\int_0^{\frac{1}{2}}|f'(\lambda x+(1-\lambda)a)|^q\right)^{\frac{1}{q}}+\left(\int_0^{\frac{1}{2}}|f'(\lambda a+(1-\lambda)x)|^q\right)^{\frac{1}{q}}\right]\\
&+(b-x)\left(\int_{\frac{1}{2}}^1\left|\frac{5}{6}-\lambda\right|^p d\lambda\right)^{\frac{1}{p}}\left[\left(\int_{\frac{1}{2}}^1|f'(\lambda x+(1-\lambda)b)|^q\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1|f'(\lambda b+(1-\lambda)x)|^q\right)^{\frac{1}{q}}\right]\\
\leq&\frac{1}{(1+p)^{\frac{1}{p}}}\left[\frac{1}{3^{p+1}}+\frac{1}{6^{p+1}}\right]^{\frac{1}{p}}\\
&\times\left\{(x-a)\left[\left(\int_0^{\frac{1}{2}}[\lambda^s|f'(x)|^q+(1-\lambda)^s|f'(a)|^q]d\lambda\right)^{\frac{1}{q}}+\left(\int_0^{\frac{1}{2}}[\lambda^s|f'(a)|^q+(1-\lambda)^s)|f'(x)|^q]d\lambda\right)^{\frac{1}{q}}\right]\right.\\
&\left.+(b-x)\left[\left(\int_{\frac{1}{2}}^1[\lambda^s|f'(x)|^q+(1-\lambda)^s|f'(b)|^q]d\lambda\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1[\lambda^s|f'(b)|^q+(1-\lambda)^s)|f'(x)|^q]d\lambda\right)^{\frac{1}{q}}\right]\right\}\\
=&\frac{x-a}{(1+p)^{\frac{1}{p}}}\left[\frac{1}{3^{p+1}}+\frac{1}{6^{p+1}}\right]^{\frac{1}{p}}\left[\left(\frac{|f'(x)|^q+|f'(a)|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+|f'(x)|^q}{s+1}\right)^{\frac{1}{q}}\right]\\
&+\frac{b-x}{(1+p)^{\frac{1}{p}}}\left[\frac{1}{3^{p+1}}+\frac{1}{6^{p+1}}\right]^{\frac{1}{p}}\left[\left(\frac{|f'(x)|^q+|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{|f'(b)|^q+|f'(x)|^q}{s+1}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

So, the proof is completed. □

Remark 5.5. *If we choose $s=1$ in the Theorem 5.4, then the inequality (5.2) reduces to the inequality of Theorem 9 in [17].*

Theorem 5.6. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|$ is quasi-convex on $[a,b]$, then*

$$\begin{aligned}&\left|\frac{1}{3}\left[2f\left(\frac{a+x}{2}\right)+2f\left(\frac{b+x}{2}\right)+f(x)+\frac{f(a)+f(b)}{2}\right]\right.\\&\left.-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right|\\\le\ &\frac{5(x-a)}{72}\left(\max\{|f'(x)|,|f'(a)|\}+\max\{|f'(a)|,|f'(x)|\}\right)\\&+\frac{5(b-x)}{72}\left(\max\{|f'(x)|,|f'(b)|\}+\max\{|f'(b)|,|f'(x)|\}\right)\end{aligned}$$

for $x\in(a,b)$.

Proof. From Lemma 5.1, by using the properties of modulus and quasi-convexity of $|f'|$, we have

$$\begin{aligned}&\left|\frac{1}{3}\left[2f\left(\frac{a+x}{2}\right)+2f\left(\frac{b+x}{2}\right)+f(x)+\frac{f(a)+f(b)}{2}\right]\right.\\&\left.-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right|\\\le\ &(x-a)\int_0^{\frac{1}{2}}\left|\lambda-\frac{1}{6}\right|[|f'(\lambda x+(1-\lambda)a)|+|f'(\lambda a+(1-\lambda)x)|]d\lambda\\&+(b-x)\int_{\frac{1}{2}}^1\left|\frac{5}{6}-\lambda\right|[|f'(\lambda x+(1-\lambda)b)|+|f'(\lambda b+(1-\lambda)x)|]d\lambda\\\le\ &(x-a)\int_0^{\frac{1}{2}}\left|\lambda-\frac{1}{6}\right|[\max\{|f'(x)|,|f'(a)|\}+\max\{|f'(a)|,|f'(x)|\}]d\lambda\\&+(b-x)\int_{\frac{1}{2}}^1\left|\frac{5}{6}-\lambda\right|[\max\{|f'(x)|,|f'(b|\}+\max\{|f'(b)|,|f'(x)|\}]d\lambda\\=\ &\frac{5(x-a)}{72}\left(\max\{|f'(x)|,|f'(a)|\}+\max\{|f'(a)|,|f'(x)|\}\right)\\&+\frac{5(b-x)}{72}\left(\max\{|f'(x)|,|f'(b)|\}+\max\{|f'(b)|,|f'(x)|\}\right).\end{aligned}$$

So, the proof is completed. □

Theorem 5.7. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|^q$ is quasi-convex on $[a,b]$ for some $q>1$, then*

$$\begin{aligned}&\left|\frac{1}{3}\left[2f\left(\frac{a+x}{2}\right)+2f\left(\frac{b+x}{2}\right)+f(x)+\frac{f(a)+f(b)}{2}\right]\right.\\&\left.-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right|\\\le\ &\frac{x-a}{4(1+p)^{\frac{1}{p}}}\left[\frac{2}{3^{p+1}}\right]^{\frac{1}{p}}\left[\left(\max\{|f'(x)|^q,|f'(a)|^q\}\right)^{\frac{1}{q}}+\left(\max\{|f'(a)|^q,|f'(x)|^q\}\right)^{\frac{1}{q}}\right]\\&+\frac{b-x}{4(1+p)^{\frac{1}{p}}}\left[\frac{2}{3^{p+1}}\right]^{\frac{1}{p}}\left[\left(\max\{|f'(x)|^q,|f'(b)|^q\}\right)^{\frac{1}{q}}+\left(\max\{|f'(b)|^q,|f'(x)|^q\}\right)^{\frac{1}{q}}\right]\end{aligned}$$

where $x\in(a,b)$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 5.1, by using Hölder inequality and $|f'|^q$ is quasi-convex on $[a,b]$, we have

$$\begin{aligned}
&\left|\frac{1}{3}\left[2f\left(\frac{a+x}{2}\right)+2f\left(\frac{b+x}{2}\right)+f(x)+\frac{f(a)+f(b)}{2}\right]\right.\\
&\left.-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right|\\
\leq\ & (x-a)\left(\int_0^{\frac{1}{2}}\left|\lambda-\frac{1}{6}\right|^p d\lambda\right)^{\frac{1}{p}}\left[\left(\int_0^{\frac{1}{2}}|f'(\lambda x+(1-\lambda)a)|^q\right)^{\frac{1}{q}}+\left(\int_0^{\frac{1}{2}}|f'(\lambda a+(1-\lambda)x)|^q\right)^{\frac{1}{q}}\right]\\
&+(b-x)\left(\int_{\frac{1}{2}}^1\left|\frac{5}{6}-\lambda\right|^p d\lambda\right)^{\frac{1}{p}}\left[\left(\int_{\frac{1}{2}}^1|f'(\lambda x+(1-\lambda)b)|^q\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1|f'(\lambda b+(1-\lambda)x)|^q\right)^{\frac{1}{q}}\right]\\
\leq\ & \frac{1}{(1+p)^{\frac{1}{p}}}\left[\frac{1}{3^{p+1}}+\frac{1}{6^{p+1}}\right]^{\frac{1}{p}}\\
&\times\left\{(x-a)\left[\left(\int_0^{\frac{1}{2}}\max\{|f'(x)|^q,|f'(a)|^q\}d\lambda\right)^{\frac{1}{q}}+\left(\int_0^{\frac{1}{2}}\max\{|f'(a)|^q,|f'(x)|^q\}d\lambda\right)^{\frac{1}{q}}\right]\right.\\
&\left.+(b-x)\left[\left(\int_{\frac{1}{2}}^1\max\{|f'(x)|^q,|f'(b)|^q\}d\lambda\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1\max\{|f'(b)|^q,|f'(x)|^q\}d\lambda\right)^{\frac{1}{q}}\right]\right\}\\
=\ & \frac{x-a}{4(1+p)^{\frac{1}{p}}}\left[\frac{2}{3^{p+1}}\right]^{\frac{1}{p}}\left[\left(\max\{|f'(x)|^q,|f'(a)|^q\}\right)^{\frac{1}{q}}+\left(\max\{|f'(a)|^q,|f'(x)|^q\}\right)^{\frac{1}{q}}\right]\\
&+\frac{b-x}{4(1+p)^{\frac{1}{p}}}\left[\frac{2}{3^{p+1}}\right]^{\frac{1}{p}}\left[\left(\max\{|f'(x)|^q,|f'(b)|^q\}\right)^{\frac{1}{q}}+\left(\max\{|f'(b)|^q,|f'(x)|^q\}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

So, the proof is completed. □

Theorem 5.8. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|$ is P-convex on $[a,b]$, then*

$$\begin{aligned}
&\left|\frac{1}{3}\left[2f\left(\frac{a+x}{2}\right)+2f\left(\frac{b+x}{2}\right)+f(x)+\frac{f(a)+f(b)}{2}\right]\right.\\
&\left.-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right|\\
\leq\ & \frac{5(x-a)[|f'(x)|+|f'(a)|}{36}+\frac{5(b-x)[|f'(b)|+|f'(x)|}{36}
\end{aligned}$$

for $x\in(a,b)$.

Proof. From Lemma 5.1, by using the properties of modulus and the fact that $|f'|$ is P-convex on $[a,b]$, we have

$$\begin{aligned}
&\left|\frac{1}{3}\left[2f\left(\frac{a+x}{2}\right)+2f\left(\frac{b+x}{2}\right)+f(x)+\frac{f(a)+f(b)}{2}\right]\right.\\
&\left.-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right|\\
\leq\ & (x-a)\int_0^{\frac{1}{2}}\left|\lambda-\frac{1}{6}\right|[|f'(\lambda x+(1-\lambda)a)|+|f'(\lambda a+(1-\lambda)x)|]d\lambda\\
&+(b-x)\int_{\frac{1}{2}}^1\left|\frac{5}{6}-\lambda\right|[|f'(\lambda x+(1-\lambda)b)|+|f'(\lambda b+(1-\lambda)x)|]d\lambda\\
\leq\ & 2(x-a)\int_0^{\frac{1}{2}}\left|\lambda-\frac{1}{6}\right|[|f'(x)|+|f'(a)|]d\lambda+2(b-x)\int_{\frac{1}{2}}^1\left|\frac{5}{6}-\lambda\right|[|f'(x)|+|f'(b)|]d\lambda\\
=\ & \frac{5(x-a)[|f'(x)|+|f'(a)|}{36}+\frac{5(b-x)[|f'(b)|+|f'(x)|}{36}.
\end{aligned}$$

So, the proof is completed. □

Theorem 5.9. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|^q$ is P-convex on $[a,b]$ for some $q>1$, then*

$$\begin{aligned}
&\left|\frac{1}{3}\left[2f\left(\frac{a+x}{2}\right)+2f\left(\frac{b+x}{2}\right)+f(x)+\frac{f(a)+f(b)}{2}\right]\right.\\
&\left.-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right|\\
\leq\ & \frac{x-a}{(1+p)^{\frac{1}{p}}}\left[\frac{1}{3^{p+1}}+\frac{1}{6^{p+1}}\right]^{\frac{1}{p}}\left[\left(\frac{|f'(x)|^q+|f'(a)|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+|f'(x)|^q}{2}\right)^{\frac{1}{q}}\right]\\
&+\frac{b-x}{(1+p)^{\frac{1}{p}}}\left[\frac{1}{3^{p+1}}+\frac{1}{6^{p+1}}\right]^{\frac{1}{p}}\left[\left(\frac{|f'(x)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{|f'(b)|^q+|f'(x)|^q}{2}\right)^{\frac{1}{q}}\right]
\end{aligned}$$

where $x \in (a,b)$ *and* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 5.1, by using Hölder inequality and $|f'|^q$ is P-convex on $[a,b]$, we have

$$\begin{aligned}
&\left|\frac{1}{3}\left[2f\left(\frac{a+x}{2}\right)+2f\left(\frac{b+x}{2}\right)+f(x)+\frac{f(a)+f(b)}{2}\right]-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right| \\
\leq\ & (x-a)\left(\int_0^{\frac{1}{2}}\left|\lambda-\frac{1}{6}\right|^p d\lambda\right)^{\frac{1}{p}}\left[\left(\int_0^{\frac{1}{2}}|f'(\lambda x+(1-\lambda)a)|^q\right)^{\frac{1}{q}}+\left(\int_0^{\frac{1}{2}}|f'(\lambda a+(1-\lambda)x)|^q\right)^{\frac{1}{q}}\right] \\
&+(b-x)\left(\int_{\frac{1}{2}}^1\left|\frac{5}{6}-\lambda\right|^p d\lambda\right)^{\frac{1}{p}}\left[\left(\int_{\frac{1}{2}}^1|f'(\lambda x+(1-\lambda)b)|^q\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1|f'(\lambda b+(1-\lambda)x)|^q\right)^{\frac{1}{q}}\right] \\
\leq\ & \frac{1}{(1+p)^{\frac{1}{p}}}\left[\frac{1}{3^{p+1}}+\frac{1}{6^{p+1}}\right]^{\frac{1}{p}} \\
&\times\left\{(x-a)\left[\left(\int_0^{\frac{1}{2}}[|f'(x)|^q+|f'(a)|^q]d\lambda\right)^{\frac{1}{q}}+\left(\int_0^{\frac{1}{2}}[|f'(a)|^q+|f'(x)|^q]d\lambda\right)^{\frac{1}{q}}\right]\right. \\
&\left.+\ (b-x)\left[\left(\int_{\frac{1}{2}}^1[|f'(x)|^q+|f'(b)|^q]d\lambda\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1[|f'(b)|^q+|f'(x)|^q]d\lambda\right)^{\frac{1}{q}}\right]\right\} \\
=\ & \frac{x-a}{(1+p)^{\frac{1}{p}}}\left[\frac{1}{3^{p+1}}+\frac{1}{6^{p+1}}\right]^{\frac{1}{p}}\left[\left(\frac{|f'(x)|^q+|f'(a)|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+|f'(x)|^q}{2}\right)^{\frac{1}{q}}\right] \\
&+\frac{b-x}{(1+p)^{\frac{1}{p}}}\left[\frac{1}{3^{p+1}}+\frac{1}{6^{p+1}}\right]^{\frac{1}{p}}\left[\left(\frac{|f'(x)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{|f'(b)|^q+|f'(x)|^q}{2}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

So, the proof is completed. □

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References

[1] M. Alomari, M. Darus, S.S. Dragomir, *New inequalities of Simpson's type for s-convex functions with applications*, RGMIA Res. Rep. Coll., 12(4) (2009).
[2] M.A. Ali, H. Kara, J. Tariboon, S. Asawasamrit, H. Budak, F. Hezenci, *Some new Simpson's formula type inequalities for twice differentiable convex functions via generalized fractional operators*, Symmetry, 13(12) (2021), Art. 2249.
[3] W.W. Breckner, *Stetigkeitsaussagen fr eine Klasse verallgemeinerter konvexer funktionen in topologischen linearen Raumen*, Pupl. Inst. Math., 23 (1978), 1320.
[4] P.S. Bullen, *Error estimates for some elementary quadrature rules*, Publikacije Elektrotehnickog fakulteta. Serija Matematika i fizika, (602/633) (1978), 97-103.
[5] S.S. Dragomir, J. Pečarić and L.E. Person, *Some inequalities of Hadamard Type*, Soochow J. Math., 21(3) (1995), 335-341.
[6] S.S. Dragomir, S. Fitzpatrick, *The Hadamard's inequality for s-convex functions in the second sense*, Demonstratio Math., 32(4) (1999), 687-696.
[7] T. Du, Y. Li, Z. Yang, *A generalization of Simpson's inequality via differentiable mapping using extended (s, m)-convex functions*, Appl. Math. Comput., 293 (2017), 358-369.
[8] T. Du, C. Luo, Z. Cao, *On the Bullen-type inequalities via generalized fractional integrals and their applications*, Fractals, 29(7) (2021), Article ID 2150188, 20 pages.
[9] S. Erden, M.Z. Sarıkaya, *Generalized Bullen-type inequalities for local fractional integrals and its applications*, Palest. J. Math., 9(2) (2020), 945-956.
[10] F. Ertugral, M.Z. Sarıkaya, *Simpson type integral inequalities for generalized fractional integral* Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matem áticas, 113(4) (2019), 3115-3124.
[11] L. Fejer, *Uber die fourierreihen, ii*, Math. Naturwise. Anz Ungar. Akad., Wiss, 24 (1906), 369-390.
[12] J. Hadamard, *Étude sur les propriétés des fonctions entiéres en particulier d'une fonction considérée par Riemann,* J. Math. Pures Appl., 58 (1893), 171-215.

[13] F. Hezenci,H. Budak, H. Kara,*New version of Fractional Simpson type inequalities for twice differentiable functions*, Adv. Differ. Equ., 2021(460) (2021), 1-10.
[14] H. Hudzik, L. Maligranda, *Some remarks on s-convex functions*, Aequationes Math. 48 (1994), 100-111.
[15] D.A. Ion, *Some estimates on the Hermite-Hadamard inequality through quasi-convex functions,* Annals of University of Craiova, Math. Comp. Sci. Ser., 34 (2007), 82-87.
[16] I. Iscan, *Hermite-Hadamard and Simpson-like type inequalities for differentiable harmonically convex functions*, J. Math. (2014)Article ID 346305, 10 pages.
[17] M.Z. Sarıkaya, *On the some generalization of inequalities associated with Bullen, Simpson, Midpoint and Trapezoid type,* ResearchGate, https://www.researchgate.net/publication/358884784.
[18] M.Z. Sarıkaya, H. Budak, *Some integral inequalities for local fractional integrals*, Int. J. Anal. Appl., 14(1) (2017), 9-19.
[19] E. Set, *New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals,* Computers & Mathematics with Applications, 63(7) (2012), 1147-1154.
[20] E. Set, A.O. Akdemir, M.E. Özdemir, *Simpson type integral inequalities for convex functions via Riemann-Liouville integrals,* Filomat, 31(14) (2017), 4415-4420.