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A Novel Operator to Solve Decision-Making Problems Under Trapezoidal Fuzzy Multi Numbers and Its Application

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Abstract — This article investigates solutions to multiple attribute decision-making (MADM) problems in which the attribute values take the form of trapezoidal fuzzy multi-numbers. To do this, this paper proposes a kind of mean aggregation operator called the Bonferroni harmonic mean operator for aggregating trapezoidal fuzzy information. Then, an approach that is a solution algorithm has been developed to find a solution to multi-attribute decision-making problems. Afterwards, an illustrative example has been given to verify the developed approach and to show its usefulness and efficiency. Finally, a comparison table has been presented to compare the proposed method with some existing methods.

Keywords — *Fuzzy multi sets, trapezoidal fuzzy multi numbers, Bonferroni harmonic mean, multiple attribute decision making*

Mathematics Subject Classification (2020) — 03E72, 94D05

1. Introduction

As an extension of the classical sets, fuzzy set theory was introduced by Zadeh [1] in 1965 to model uncertain information. Then, a kind of fuzzy sets were introduced by Yager [2] which is called multi-fuzzy sets (fuzzy bags). It is a different generalization of fuzzy sets and provides complete information in some problems as there are situations where each element has different membership values. Miyamoto [3, 4] and Sebastian and Ramakrishnan [5, 6] expanded and studied the Yager's multi-sets and multi-fuzzy sets in detail. Since some occurrences are with more than one possibility of same or different membership functions, Uluçay et al. [7] developed trapezoidal fuzzy multi-numbers (TFM-numbers) on real number set $\mathbb{R}$ which are extension of both fuzzy numbers and multi-fuzzy sets by allowing the repeated occurrences of any element. Later, various studies have been additionally done by many authors in [8–11].

The Bonferroni mean (BM), primarily proposed by Bonferroni [12] is an aggregation method which is useful to aggregate the crisp data. It can explore the interrelationships among arguments, which have a critical role in multi criteria decision making problems. Therefore, Yager [13] introduced a detailed work of BM and gave some generalizations that enhance its capability. Then, Beliakov et al. [14] made the BM more enhanced by coping with the interrelation of any three aggregated elements instead of any two.

Harmonic mean is a conservative average which give an aggregation locating between the maximum and minimum operators. It is commonly used by scientists as a tool of aggregating data that has tendency to central [15]. In the literature, the harmonic mean is mostly considered as an aggregation

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method of numerical data information including fuzzy informations. For example, Xu [15] proposed the fuzzy harmonic mean operators named fuzzy weighted harmonic mean (FWHM) operator, fuzzy ordered weighted harmonic mean (FOWHM) operator and fuzzy hybrid harmonic mean (FHHM) operator. Furthermore, he applied these operators to multiple attribute group decision making (MAGDM) problems. Wei [16] developed fuzzy induced ordered weighted harmonic mean (FIOWHM) operator and then, he presented the approach to MAGDM based on the FWHM and FIOWHM operators. Sun and Sun [17] introduced the fuzzy Bonferroni harmonic mean (FBHM) operator and the fuzzy ordered Bonferroni harmonic mean (FOBHM) operator. Then, they applied the FOBHM operator to MADM problems. Until now, Bonferroni harmonic mean aggregation operators based on trapezoidal multi fuzzy numbers have not been studied as we know. In order to fill this gap, this article formed and has five sections. In second section, we give definitions of fuzzy sets, multi-fuzzy sets, and trapezoidal fuzzy multi-numbers and some of their basic properties. In third section, we introduce an aggregation method called weighted Bonferroni harmonic mean operator for aggregating the trapezoidal multi fuzzy information. In addition the section reviews its some special cases and some properties. In fourth section, we propose an algorithm to solve multiple attribute decision making problems under the trapezoidal multi fuzzy numbers. Then, in the section, we apply the proposed operator to a multi attribute decision making problem. In fifth section, we give a brief conclusion. The present paper is derived from the first author's master's thesis under the second author's supervision.

2. Preliminary

In this section, we give some basic concepts such as fuzzy set [1], trapezoidal fuzzy multi numbers [7] and etc. In [8–11, 18–23], readers can find further knowledge.

Definition 2.1. [1] A fuzzy set F on X which is a non-empty set is defined as:

$$F = \{\langle x, \mu_F(x)\rangle : x \in X\}$$

where μ_F is a function from X to $[0,1]$.

Definition 2.2. [24] t-norms are associative, monotonic and commutative two valued functions t that map from $[0,1] \times [0,1]$ into $[0,1]$. These properties are formulated with the following conditions:

i. $t(0,0) = 0$ and $t(\mu_{X_1}(x), 1) = t(1, \mu_{X_1}(x)) = \mu_{X_1}(x)$

ii. If $\mu_{X_1}(x) \leq \mu_{X_3}(x)$ and $\mu_{X_2}(x) \leq \mu_{X_4}(x)$, then $t(\mu_{X_1}(x), \mu_{X_2}(x)) \leq t(\mu_{X_3}x), \mu_{X_4}(x))$

iii. $t(\mu_{X_1}(x), \mu_{X_2}(x)) = t(\mu_{X_2}(x), \mu_{X_1}(x))$

iv. $t(\mu_{X_1}(x), t(\mu_{X_2}(x), \mu_{X_3}(x))) = t(t(\mu_{X_1}(x), \mu_{X_2})(x), \mu_{X_3}(x))$

Definition 2.3. [24] s-norm are associative, monotonic and commutative two placed functions s which map from $[0,1] \times [0,1]$ into $[0,1]$. These properties are formulated with the following conditions:

i. $s(1,1) = 1$ *and* $s(\mu_{X_1}(x), 0) = s(0, \mu_{X_1}(x)) = \mu_{X_1}(x)$

ii. if $\mu_{X_1}(x) \leq \mu_{X_3}(x)$ and $\mu_{X_2}(x) \leq \mu_{X_4}(x)$, then $s(\mu_{X_1}(x), \mu_{X_2}(x)) \leq s(\mu_{X_3}(x), \mu_{X_4}(x))$

iii. $s(\mu_{X_1}(x), \mu_{X_2}(x)) = s(\mu_{X_2}(x), \mu_{X_1}(x))$

iv. $s(\mu_{X_1}(x), s(\mu_{X_2}(x), \mu_{X_3}(x))) = s(s(\mu_{X_1}(x), \mu_{X_2})(x), \mu_{X_3}(x))$

For example, $t_2(\mu_{X_1}(x), \mu_{X_2}(x)) = \mu_{X_1}(x)\mu_{X_2}(x)$ is a t-norm and $s_2(\mu_{X_1}(x), \mu_{X_2}(x)) = \mu_{X_1}(x) + \mu_{X_2}(x) - \mu_{X_1}(x)\mu_{X_2}(x)$ is a s-norm.

Definition 2.4. [7] Let $w_N \in [0,1]$, $x_i, y_i, z_i, t_i \in \mathbb{R}$ and $x_i \leq y_i \leq z_i \leq t_i$. A trapezoidal fuzzy number (TF-number) $N = \langle (x_i, y_i, z_i, t_i); w_N \rangle$ is a special fuzzy set on the real number set $\mathbb{R}$. Its membership function is given as:

$$\mu_N(x) = \begin{cases} (x - x_i)w_N/(y_i - x_i), & x_i \leq x < y_i \\ w_N, & y_i \leq x \leq z_i \\ (t_i - x)w_N/(t_i - z_i), & z_i < x \leq t_i \\ 0, & otherwise \end{cases}$$

Definition 2.5. [5] A multi-fuzzy set G on X which is a non-empty set is defined as:

$$G = \{\langle x, \mu_G^1(x), \mu_G^2(x), ..., \mu_G^i(x), ... \rangle : x \in X\}$$

where $\mu_G^i : X \to [0,1]$ for all $i \in \{1, 2, ..., p\}$ and $x \in X$.

Definition 2.6. [7] Let $\eta_N^s \in [0,1]$ $s \in \{1, 2, ..., p\}$ and $x_i, y_i, z_i, t_i \in \mathbb{R}$ such that $x_i \leq y_i \leq z_i \leq t_i$. Then, trapezoidal fuzzy multi-number (TFM-number) shown by $N = \langle (x_i, y_i, z_i, t_i); \eta_N^1, \eta_N^2, ..., \eta_N^P \rangle$ is a special fuzzy multi-set on the real numbers set $\mathbb{R}$ and its membership functions are defined as:

$$\mu_N^s(x) = \begin{cases} (x - x_i)\eta_N^s/(y_i - x_i) & x_i \leq x \leq y_i \\ \eta_N^s & y_i \leq x \leq z_i \\ (t_i - x)\eta_N^s/(t_i - z_i) & z_i \leq x \leq t_i \\ 0 & otherwise \end{cases}$$

From now on the set of all TFM-number on $\mathbb{R}^+$ will be denoted by $\mho(\mathbb{R}^+)$. Moreover, I_p and I_n will be used instead of $\{1, 2, ..., p\}$ and $\{1, 2, ..., n\}$, respectively.

Definition 2.7. [7] Let $N_1 = \langle (x_1, y_1, z_1, t_1); \eta_{N_1}^1, \eta_{N_1}^2, ..., \eta_{N_1}^P \rangle$, $N_2 = \langle (x_2, y_2, z_2, t_2); \eta_{N_2}^1, \eta_{N_2}^2, ..., \eta_{N_2}^P \rangle \in \mho(\mathbb{R}^+)$ and $\gamma \neq 0$, $\gamma \in \mathbb{R}$. Then,

i. $N_1 + N_2 = (x_1 + x_2, y_1 + y_2, z_1 + z_2, t_1 + t_2);$
$\eta_{N_1}^1 + \eta_{N_2}^1 - \eta_{N_1}^1\eta_{N_2}^1, \eta_{N_1}^2 + \eta_{N_2}^2 - \eta_{N_1}^2\eta_{N_2}^2, ..., \eta_{N_1}^P + \eta_{N_2}^P - \eta_{N_1}^P\eta_{N_2}^P)\rangle$

ii. $$N_1 \times N_2 = \begin{cases} \langle (x_1x_2, y_1y_2, z_1z_2, t_1t_2); \eta_{N_1}^1\eta_{N_2}^1, \eta_{N_1}^2\eta_{N_2}^2, ...\eta_{N_1}^P\eta_{N_2}^P)\rangle & (t_1 > 0, t_2 > 0) \\ \langle (x_1t_2, y_1z_2, z_1y_2, t_1x_2); \eta_{N_1}^1\eta_{N_2}^1, \eta_{N_1}^2\eta_{N_2}^2, ...\eta_{N_1}^P\eta_{N_2}^P)\rangle & (t_1 < 0, t_2 > 0) \\ \langle (t_1t_2, z_1z_2, y_1y_2, x_1x_2); \eta_{N_1}^1\eta_{N_2}^1, \eta_{N_1}^2\eta_{N_2}^2, ...\eta_{N_1}^P\eta_{N_2}^P)\rangle & (t_1 < 0, t_2 < 0) \end{cases}$$

iii. $\gamma N_1 = \langle (\gamma x_1, \gamma y_1, \gamma z_1, \gamma t_1); 1 - (1 - \eta_{N_1}^1)^\gamma, 1 - (1 - \eta_{\bar{N}_1}^2)^\gamma, ..., 1 - (1 - \eta_{N_1}^p)^\gamma \rangle (\gamma \geq 0)$

iv. $N_1^\gamma = \langle (x_1^\gamma, y_1^\gamma, z_1^\gamma, t_1^\gamma); (\eta_{N_1}^1)^\gamma, (\eta_{N_1}^2)^\gamma, ..., (\eta_{N_1}^P)^\gamma \rangle (\gamma \geq 0)$

v. Based on negative exponential of a trapezoidal intuitionistic fuzzy number given by Li [25], we can give following property for TFM-numbers:

$$N_1^{-1} = \left\langle \left(\frac{1}{t_1}, \frac{1}{z_1}, \frac{1}{y_1}, \frac{1}{x_1} \right); \eta_{N_1}^1, \eta_{N_1}^2, ..., \eta_{N_1}^P \right\rangle$$

Definition 2.8. [26] Let $N_1 = \langle (x_1, y_1, z_1, t_1); \eta_{N_1}^1, \eta_{N_1}^2, ..., \eta_{N_2}^P \rangle$, $N_2 = \langle (x_2, y_2, z_2, t_2); \eta_{N_2}^1, \eta_{N_2}^2, ..., \eta_{N_2}^P \rangle$ $\in \mho(\mathbb{R}^+)$. Followings are right:

i. If $x_1 < x_2$, $y_1 < y_2$, $z_1 < z_2$, $t_1 < t_2$, $\eta_{N_1}^1 < \eta_{N_2}^1$,$\eta_{N_1}^2 < \eta_{N_2}^2$,...,$\eta_{N_1}^P < \eta_{N_2}^P$ then $N_1 < N_2$

ii. If $x_1 > x_2$, $y_1 > y_2$, $z_1 > z_2$, $t_1 > t_2$, $\eta_{N_1}^1 > \eta_{N_2}^1$,$\eta_{N_1}^2 > \eta_{N_2}^2$,...,$\eta_{N_1}^P > \eta_{N_2}^P$ then $N_1 > N_2$

iii. If $x_1 = x_2$, $y_1 = y_2$, $z_1 = z_2$, $t_1 = t_2$, $\eta_{N_1}^1 = \eta_{N_2}^1$,$\eta_{N_1}^2 = \eta_{N_2}^2$,...,$\eta_{N_1}^P = \eta_{N_2}^P$ then $N_1 = N_2$

Based on score value for trapezoidal hesitant fuzzy numbers given by Deli [26], we propose following definition.

Definition 2.9. Let $N = \langle (x_i, y_i, z_i, t_i); \eta_N^1, \eta_N^2, ..., \eta_N^P \rangle$ be a TFM-number and P is number of η_N^i. Then, score of N denoted $s(N)$ is defined as:

$$s(N) = \frac{t_i^2 + z_i^2 - x_i^2 - y_i^2}{2P} \sum_{s=1}^{P} \eta_N^s$$

Let $N_i = \langle (x_i, y_i, z_i, t_i); \eta_{N_i}^1, \eta_{N_i}^2, ..., \eta_{N_i}^P \rangle$ $(i \in I_2)$ be a TFM-numbers' collection. Then, comparison of N_1 and N_2 is given as:

i. If $s(N_1) > s(N_2)$, then $N_1 > N_2$

ii. If $s(N_1) < s(N_2)$, then $N_1 < N_2$

iii. If $s(N_1) = s(N_2)$, then $N_1 = N_2$

Definition 2.10. [12] Let σ_i $(i \in I_n)$ be a nonnegative numbers' collection and $p, q \in \mathbb{R}$ such that $p, q \geq 0$. Then, Bonferroni mean (BM) of σ_i $(i \in I_n)$ is defined as:

$$BM^{p,q}(\sigma_1, \sigma_2, ..., \sigma_n) = \left(\frac{1}{n(n-1)} \sum_{i,j=1, i \neq j}^{n} \sigma_i^p \sigma_j^q \right)^{\frac{1}{p+q}}$$

For two collections of nonnegative numbers σ_i and υ_i $(i \in I_n)$, the BM has some properties as follows:

i. $BM^{p,q}(0, 0, ..., 0) = 0$

ii. $BM^{p,q}(\sigma, \sigma, ..., \sigma) = \sigma$, if $\sigma_i = \sigma$, for all i

iii. $BM^{p,q}(\sigma_1, \sigma_2, ..., \sigma_n) \geq BM^{p,q}(\upsilon_1, \upsilon_2, ..., \upsilon_n)$,i.e., $BM^{p,q}$ is monotonic, if $\sigma_i \geq \upsilon_i$, for all i

iv. $\min\{\sigma_i\} \leq BM^{p,q}(\sigma_1, \sigma_2, ..., \sigma_n) \leq \max\{\sigma_i\}$

Definition 2.11. [17] Let $p, q \geq 0$ and σ_i $(i \in I_n)$ be a collection of nonnegative numbers and $\upsilon = (\upsilon_1, \upsilon_2, ..., \upsilon_n)^T$ σ_i's weight vector such that $\upsilon_i \geq 0$ $(i \in I_n)$ and $\sum_{i=1}^{n} \upsilon_i = 1$. If

$$WBHM^{p,q}(\sigma_1, \sigma_2, ..., \sigma_n) = \frac{1}{\left(\sum_{i,j=1, i \neq j}^{n} \frac{\upsilon_i \upsilon_j}{\sigma_i^p \sigma_j^q} \right)^{\frac{1}{p+q}}}$$

then $WBHM^{p,q}$ is called the weighted Bonferroni Harmonic Mean (WBHM).

3. Weighted Bonferroni Harmonic Mean Operator on TFM-numbers

In this section we propose TFM weighted Bonferroni harmonic mean based on weighted Bonferroni harmonic mean given by Su et al. [27]. Then, we analyze its properties and review special cases to see how it converts into other operators. In addition, a basic example is presented to see its application to three TFM-numbers.

Definition 3.1. Let $N_i = \langle (x_i, y_i, z_i, t_i); \eta_{N_i}^1, \eta_{N_i}^2, ..., \eta_{N_i}^P \rangle$ $(i \in I_n)$ be a TFM-numbers' collection, $p, q > 0$ and N_i's weight vector is $\upsilon = (\upsilon_1, \upsilon_2, ..., \upsilon_n)^T$. Here, υ_i is N_i's importance degree, satisfying $\upsilon_i \in [0, 1]$, $i \in I_n$ such that $\sum_{i=1}^{n} \upsilon_i = 1$. Then,

$$TFMBHM_{\upsilon}^{p,q}(N_1, N_2, ..., N_n) = \frac{1}{\left(\bigoplus_{i,j=1, i \neq j}^{n} \left(\left(\frac{\upsilon_i}{N_i^p} \right) \otimes \left(\frac{\upsilon_j}{N_j^q} \right) \right) \right)^{\frac{1}{p+q}}}$$

is called TFM weighted Bonferroni harmonic mean operator $(TFMBHM_{\upsilon})$.

Considering operational laws in Definitions 2.7 of TFM-numbers, we can give following theorem:

Theorem 3.2. Let $N_i = \langle (x_i, y_i, z_i, t_i); \eta^1_{N_i}, \eta^2_{N_i}, ..., \eta^P_{N_i} \rangle$ $(i \in I_n)$ be a TFM-numbers' collection, $p, q > 0$ and N_i's weight vector is $v = (v_1, v_2, ..., v_n)^T$. Here, v_i is N_i's importance degree, satisfying $v_i \in [0, 1]$, $i \in I_n$ such that $\sum\limits_{i=1}^{n} v_i = 1$. Then, $TFMBHM^{p,q}_v(A_1, A_2, ..., A_n)$ is a TFM-number and computed as:

$$\begin{aligned} TFMBHM^{p,q}_v(N_1, N_2, ..., N_n) = \Bigg\langle \Bigg(& \frac{1}{\left(\sum\limits_{i,j=1, i\neq j}^{n} \frac{v_i v_j}{x_i^p x_j^q}\right)^{\frac{1}{p+q}}}, \frac{1}{\left(\sum\limits_{i,j=1, i\neq j}^{n} \frac{v_i v_j}{y_i^p y_j^q}\right)^{\frac{1}{p+q}}}, \\ & \frac{1}{\left(\sum\limits_{i,j=1, i\neq j}^{n} \frac{v_i v_j}{z_i^p z_j^q}\right)^{\frac{1}{p+q}}}, \frac{1}{\left(\sum\limits_{i,j=1, i\neq j}^{n} \frac{v_i v_j}{t_i^p t_j^q}\right)^{\frac{1}{p+q}}} \Bigg); \\ & 1 - \prod_{i,j=1, i\neq j}^{n} [1 - (1 - (1 - (\eta^1_{N_i})^p)^{v_i})(1 - (1 - (\eta^1_{N_j})^q)^{v_j})]^{\frac{1}{p+q}}, \\ & 1 - \prod_{i,j=1, i\neq j}^{n} [1 - (1 - (1 - (\eta^2_{N_i})^p)^{v_i})(1 - (1 - (\eta^2_{N_j})^q)^{v_j})]^{\frac{1}{p+q}}, ..., \\ & 1 - \prod_{i,j=1, i\neq j}^{n} [1 - (1 - (1 - (\eta^P_{N_i})^p)^{v_i})(1 - (1 - (\eta^P_{N_j})^q)^{v_j})]^{\frac{1}{p+q}} \Bigg\rangle \end{aligned} \tag{1}$$

Proof. By operation laws in Definition 2.7 of TFM-numbers,

$$\begin{aligned} \left(\frac{v_i}{N_i^p}\right) \otimes \left(\frac{v_j}{N_j^q}\right) = \Bigg\langle & \left(\frac{v_i v_j}{x_i^p x_j^q}, \frac{v_i v_j}{y_i^p y_j^q}, \frac{v_i v_j}{z_i^p z_j^q}, \frac{v_i v_j}{t_i^p t_j^q} \right); \\ & (1 - (1 - (\eta^1_{N_i})^p)^{v_i})(1 - (1 - (\eta^1_{N_j})^q)^{v_j}), \\ & (1 - (1 - (\eta^2_{N_i})^p)^{v_i})(1 - (1 - (\eta^2_{N_j})^q)^{v_j}), ..., \\ & (1 - (1 - (\eta^P_{N_i})^p)^{v_i})(1 - (1 - (\eta^P_{N_j})^q)^{v_j} \Bigg\rangle \end{aligned}$$

First of all, we need to show:

$$\begin{aligned} \bigoplus_{i,j=1, i\neq j}^{n} \left(\left(\frac{v_i}{N_i^p}\right) \otimes \left(\frac{v_j}{N_j^q}\right) \right) = \Bigg\langle & \left(\sum_{i,j=1, i\neq j}^{n} \frac{v_i v_j}{x_i^p x_j^q}, \sum_{i,j=1, i\neq j}^{n} \frac{v_i v_j}{y_i^p y_j^q}, \sum_{i,j=1, i\neq j}^{n} \frac{v_i v_j}{z_i^p z_j^q}, \sum_{i,j=1, i\neq j}^{n} \frac{v_i v_j}{t_i^p t_j^q} \right); \\ & 1 - \prod_{i,j=1, i\neq j}^{n} [1 - (1 - (1 - (\eta^1_{N_i})^p)^{v_i})(1 - (1 - (\eta^1_{N_j})^q)^{v_j})]^{\frac{1}{p+q}}, \\ & 1 - \prod_{i,j=1, i\neq j}^{n} [1 - (1 - (1 - (\eta^2_{N_i})^p)^{v_i})(1 - (1 - (\eta^2_{N_j})^q)^{v_j})]^{\frac{1}{p+q}}, ..., \\ & 1 - \prod_{i,j=1, i\neq j}^{n} [1 - (1 - (1 - (\eta^P_{N_i})^p)^{v_i})(1 - (1 - (\eta^P_{N_j})^q)^{v_j})]^{\frac{1}{p+q}} \Bigg\rangle \end{aligned} \tag{2}$$

If we use mathematical induction on n:
1) when $n = 2$, from Equation (2), we obtain:

$$\bigoplus_{i,j=1, i\neq j}^{2} \left(\left(\frac{v_i}{N_i^p}\right) \otimes \left(\frac{v_j}{N_j^q}\right) \right) = \left(\frac{v_1}{N_1^p} \otimes \frac{v_2}{N_2^q}\right) \oplus \left(\frac{v_2}{N_2^p} \otimes \frac{v_1}{N_1^q}\right)$$

$$
\begin{aligned}
=&\left\langle\left(\frac{v_1v_2}{x_1^px_2^q}+\frac{v_2v_1}{x_2^px_1^q},\frac{v_1v_2}{y_1^py_2^q}+\frac{v_2v_1}{y_2^py_1^q},\frac{v_1v_2}{z_1^pz_2^q}+\frac{v_2v_1}{z_2^pz_1^q},\frac{v_1v_2}{t_1^pt_2^q}+\frac{v_2v_1}{t_2^pt_1^q}\right);\right.\\
&[(1-(1-(\eta^1_{N_1})^p)^{v_1})(1-(1-(\eta^1_{N_2})^q)^{v_2})]\oplus[(1-(1-(\eta^1_{N_2})^p)^{v_2})(1-(1-(\eta^1_{N_1})^q)^{v_1})],\\
&[(1-(1-(\eta^2_{N_1})^p)^{v_1})(1-(1-(\eta^2_{N_2})^q)^{v_2})]\oplus[(1-(1-(\eta^2_{N_2})^p)^{v_2})(1-(1-(\eta^2_{N_1})^q)^{v_1})],...,\\
&\left.[(1-(1-(\eta^P_{N_1})^p)^{v_1})(1-(1-(\eta^P_{N_2})^q)^{v_2})]\oplus[(1-(1-(\eta^P_{N_2})^p)^{v_2})(1-(1-(\eta^P_{N_1})^q)^{v_1})]\right\rangle\\
=&\left\langle\left(\sum_{i,j=1,i\neq j}^{2}\frac{v_iv_j}{x_i^px_j^q},\sum_{i,j=1,i\neq j}^{2}\frac{v_iv_j}{y_i^py_j^q},\sum_{i,j=1,i\neq j}^{2}\frac{v_iv_j}{z_i^pz_j^q},\sum_{i,j=1,i\neq j}^{2}\frac{v_iv_j}{t_i^pt_j^q}\right);\right.\\
&1-\prod_{i,j=1,i\neq j}^{2}[1-(1-(1-(\eta^1_{N_i})^p)^{v_i})(1-(1-(\eta^1_{N_j})^q)^{v_j})]^{\frac{1}{p+q}},\\
&1-\prod_{i,j=1,i\neq j}^{2}[1-(1-(1-(\eta^2_{N_i})^p)^{v_i})(1-(1-(\eta^2_{N_j})^q)^{v_j})]^{\frac{1}{p+q}},...,\\
&\left.1-\prod_{i,j=1,i\neq j}^{2}[1-(1-(1-(\eta^P_{N_i})^p)^{v_i})(1-(1-(\eta^P_{N_j})^q)^{v_j})]^{\frac{1}{p+q}}\right\rangle
\end{aligned}
$$

2) Suppose when $n=k$, the Equation (2) is true, i.e.,

$$
\begin{aligned}
\bigoplus_{i,j=1,i\neq j}^{k}\left(\left(\frac{v_i}{N_i^p}\right)\otimes\left(\frac{v_j}{N_j^q}\right)\right)=&\left\langle\left(\sum_{i,j=1,i\neq j}^{k}\frac{v_iv_j}{x_i^px_j^q},\sum_{i,j=1,i\neq j}^{k}\frac{v_iv_j}{y_i^py_j^q},\sum_{i,j=1,i\neq j}^{k}\frac{v_iv_j}{z_i^pz_j^q},\sum_{i,j=1,i\neq j}^{k}\frac{v_iv_j}{t_i^pt_j^q}\right);\right.\\
&1-\prod_{i,j=1,i\neq j}^{k}[1-(1-(1-(\eta^1_{N_i})^p)^{v_i})(1-(1-(\eta^1_{N_j})^q)^{v_j})]^{\frac{1}{p+q}},\\
&1-\prod_{i,j=1,i\neq j}^{k}[1-(1-(1-(\eta^2_{N_i})^p)^{v_i})(1-(1-(\eta^2_{N_j})^q)^{v_j})]^{\frac{1}{p+q}},...,\\
&\left.1-\prod_{i,j=1,i\neq j}^{k}[1-(1-(1-(\eta^P_{N_i})^p)^{v_i})(1-(1-(\eta^P_{N_j})^q)^{v_j})]^{\frac{1}{p+q}}\right\rangle
\end{aligned}
\tag{3}
$$

3) Now we will show Equation (2) is true for $n=k+1$. If we accept $n=k+1$ in Equation (2):

$$
\begin{aligned}
\bigoplus_{i,j=1,i\neq j}^{k+1}\left(\left(\frac{v_i}{N_i^p}\right)\otimes\left(\frac{v_j}{N_j^q}\right)\right)=&\left(\bigoplus_{i,j=1,i\neq j}^{k}\left(\left(\frac{v_i}{N_i^p}\right)\otimes\left(\frac{v_j}{N_j^q}\right)\right)\right)\oplus\\
&\left(\bigoplus_{i=1}^{k}\left(\left(\frac{v_i}{N_i^p}\right)\otimes\left(\frac{v_{k+1}}{N_{k+1}^q}\right)\right)\right)\oplus\\
&\left(\bigoplus_{j=1}^{k}\left(\left(\frac{v_{k+1}}{N_{k+1}^p}\right)\otimes\left(\frac{v_j}{N_j^q}\right)\right)\right)
\end{aligned}
\tag{4}
$$

$$\bigoplus_{i=1}^{k}\left(\left(\frac{v_i}{N_i^p}\right)\otimes\left(\frac{v_{k+1}}{N_{k+1}^q}\right)\right)=\Big\langle\Big(\sum_{i=1}^{k}\frac{v_iv_{k+1}}{x_i^px_{k+1}^q},\sum_{i=1}^{k}\frac{v_iv_{k+1}}{y_i^py_{k+1}^q},\sum_{i=1}^{k}\frac{v_iv_{k+1}}{z_i^pz_{k+1}^q},\sum_{i=1}^{k}\frac{v_iv_{k+1}}{t_i^pt_{k+1}^q}\Big);$$
$$1-\prod_{i=1}^{k}[1-(1-(1-(\eta_{N_i}^1)^p)^{v_i})(1-(1-(\eta_{N_{k+1}}^1)^q)^{v_{k+1}})]^{\frac{1}{p+q}},$$
$$1-\prod_{i=1}^{k}[1-(1-(1-(\eta_{N_i}^2)^p)^{v_i})(1-(1-(\eta_{N_{k+1}}^2)^q)^{v_{k+1}})]^{\frac{1}{p+q}},...,\tag{5}$$
$$1-\prod_{i=1}^{k}[1-(1-(1-(\eta_{N_i}^P)^p)^{v_i})(1-(1-(\eta_{N_{k+1}}^P)^q)^{v_{k+1}})]^{\frac{1}{p+q}}\Big\rangle$$

and

$$\bigoplus_{j=1}^{k}\left(\left(\frac{v_{k+1}}{N_{k+1}^p}\right)\otimes\left(\frac{v_j}{N_j^q}\right)\right)=\Big\langle\Big(\sum_{j=1}^{k}\frac{v_{k+1}v_j}{x_{k+1}^px_j^q},\sum_{j=1}^{k}\frac{v_{k+1}v_j}{y_{k+1}^py_j^q},\sum_{j=1}^{k}\frac{v_{k+1}v_j}{z_{k+1}^pz_j^q},\sum_{j=1}^{k}\frac{v_{k+1}v_j}{t_{k+1}^pt_j^q}\Big);$$
$$1-\prod_{j=1}^{k}[1-(1-(1-(\eta_{N_{k+1}}^1)^p)^{v_{k+1}})(1-(1-(\eta_{N_j}^1)^q)^{v_j})]^{\frac{1}{p+q}},$$
$$1-\prod_{j=1}^{k}[1-(1-(1-(\eta_{N_{k+1}}^2)^p)^{v_{k+1}})(1-(1-(\eta_{N_j}^2)^q)^{v_j})]^{\frac{1}{p+q}},...,\tag{6}$$
$$1-\prod_{j=1}^{k}[1-(1-(1-(\eta_{N_{k+1}}^P)^p)^{v_{k+1}})(1-(1-(\eta_{N_j}^P)^q)^{v_j})]^{\frac{1}{p+q}}\Big\rangle$$

Finally from Equations (3), (4), (5) and (6), we obtain:

$$TFMBHM_v^{p,q}(N_1,N_2,...,N_n)=\Big\langle\Bigg(\sum_{i,j=1,i\neq j}^{k}\frac{v_iv_j}{x_i^px_j^q},\sum_{i,j=1,i\neq j}^{k}\frac{v_iv_j}{y_i^py_j^q},\sum_{i,j=1,i\neq j}^{k}\frac{v_iv_j}{z_i^pz_j^q},\sum_{i,j=1,i\neq j}^{k}\frac{v_iv_j}{t_i^pt_j^q}\Bigg);$$
$$1-\prod_{i,j=1,i\neq j}^{k}[1-(1-(1-(\eta_{N_i}^1)^p)^{v_i})(1-(1-(\eta_{N_j}^1)^q)^{v_j})]^{\frac{1}{p+q}},$$
$$1-\prod_{i,j=1,i\neq j}^{k}[1-(1-(1-(\eta_{N_i}^2)^p)^{v_i})(1-(1-(\eta_{N_j}^2)^q)^{v_j})]^{\frac{1}{p+q}},...,$$
$$1-\prod_{i,j=1,i\neq j}^{k}[1-(1-(1-(\eta_{N_i}^P)^p)^{v_i})(1-(1-(\eta_{N_j}^P)^q)^{v_j})]^{\frac{1}{p+q}}\Big\rangle$$
$$\otimes\Big\langle\Big(\sum_{i=1}^{k}\frac{v_iv_{k+1}}{x_i^px_{k+1}^q},\sum_{i=1}^{k}\frac{v_iv_{k+1}}{y_i^py_{k+1}^q},\sum_{i=1}^{k}\frac{v_iv_{k+1}}{z_i^pz_{k+1}^q},\sum_{i=1}^{k}\frac{v_iv_{k+1}}{t_i^pt_{k+1}^q}\Big);$$
$$1-\prod_{i=1}^{k}[1-(1-(1-(\eta_{N_i}^1)^p)^{v_i})(1-(1-(\eta_{N_{k+1}}^1)^q)^{v_{k+1}})]^{\frac{1}{p+q}},$$
$$1-\prod_{i=1}^{k}[1-(1-(1-(\eta_{N_i}^2)^p)^{v_i})(1-(1-(\eta_{N_{k+1}}^2)^q)^{v_{k+1}})]^{\frac{1}{p+q}},...,$$
$$1-\prod_{i=1}^{k}[1-(1-(1-(\eta_{N_i}^P)^p)^{v_i})(1-(1-(\eta_{N_{k+1}}^P)^q)^{v_{k+1}})]^{\frac{1}{p+q}}\Big\rangle$$
$$\otimes\Big\langle\Big(\sum_{j=1}^{k}\frac{v_{k+1}v_j}{x_{k+1}^px_j^q},\sum_{j=1}^{k}\frac{v_{k+1}v_j}{y_{k+1}^p.y_j^q},\sum_{j=1}^{k}\frac{v_{k+1}v_j}{z_{k+1}^pz_j^q},\sum_{j=1}^{k}\frac{v_{k+1}v_j}{t_{k+1}^pt_j^q}\Big);$$

$$
1-\prod_{j=1}^{k}[1-(1-(1-(\eta^{1}_{N_{k+1}})^{p})^{v_{k+1}})(1-(1-(\eta^{1}_{N_j})^{q})^{v_j})]^{\frac{1}{p+q}},
$$
$$
1-\prod_{j=1}^{k}[1-(1-(1-(\eta^{2}_{N_{k+1}})^{p})^{v_{k+1}})(1-(1-(\eta^{2}_{N_j})^{q})^{v_j})]^{\frac{1}{p+q}},...,
$$
$$
1-\prod_{j=1}^{k}[1-(1-(1-(\eta^{P}_{N_{k+1}})^{p})^{v_{k+1}})(1-(1-(\eta^{P}_{N_j})^{q})^{v_j})]^{\frac{1}{p+q}}\Bigg\rangle
$$
$$
=\Bigg\langle\Bigg(\sum_{i,j=1,i\neq j}^{k+1}\left(\frac{v_iv_j}{x_i^px_j^q}\right)^{\frac{1}{p+q}},\sum_{i,j=1,i\neq j}^{k+1}\left(\frac{v_iv_j}{y_i^py_j^q}\right)^{\frac{1}{p+q}},\sum_{i,j=1,i\neq j}^{k+1}\left(\frac{v_iv_j}{z_i^pz_j^q}\right)^{\frac{1}{p+q}},\sum_{i,j=1,i\neq j}^{k+1}\left(\frac{v_iv_j}{t_i^pt_j^q}\right)^{\frac{1}{p+q}}\Bigg);
$$
$$
1-\prod_{i,j=1,i\neq j}^{k+1}[1-(1-(1-(\eta^{1}_{N_i})^{p})^{v_i})(1-(1-(\eta^{1}_{N_{k+1}})^{q})^{v_j})]^{\frac{1}{p+q}},
$$
$$
1-\prod_{i,j=1,i\neq j}^{k+1}[1-(1-(1-(\eta^{2}_{N_i})^{p})^{v_i})(1-(1-(\eta^{2}_{N_{k+1}})^{q})^{v_j})]^{\frac{1}{p+q}},...,
$$
$$
1-\prod_{i,j=1,i\neq j}^{k+1}[1-(1-(1-(\eta^{P}_{N_i})^{p})^{v_i})(1-(1-(\eta^{P}_{N_{k+1}})^{q})^{v_j})]^{\frac{1}{p+q}}\Bigg\rangle
$$

□

Note 3.3. Let $p,q>0$ and $N_i=\langle(x_i,y_i,z_i,t_i);\eta^1_{N_i},\eta^2_{N_i},...,\eta^P_{N_i}\rangle$ $(i\in I_n)$ be a TFM-numbers' collection. If $v=\left(\frac{1}{n},\frac{1}{n},...,\frac{1}{n}\right)^T$, then the $TFMBHM_v$ operator converted into the following:

$$
TFMBHM^{p,q}_{\frac{1}{n}}(N_1,N_2,...,N_n)=\frac{1}{\left(\bigoplus_{i,j=1,i\neq j}^{n}\left(\left(\frac{1/n}{N_i^p}\right)\otimes\left(\frac{1/n}{N_j^q}\right)\right)\right)^{\frac{1}{p+q}}}
$$

Proposition 3.4. Let $p,q>0$ and $N_i=\langle(x_i,y_i,z_i,t_i);\eta^1_{N_i},\eta^2_{N_i},...,\eta^P_{N_i}\rangle$, $M_i=\langle(k_i,l_i,m_i,n_i);\eta^1_{M_i},\eta^2_{M_i},...,\eta^P_{M_i}\rangle$ $(i\in I_n)$ be TFM-numbers' two collections. $TFMBHM_v$ operator has following properties:

i. ***(Idempotency)*** If $N_i=N$ for all $(i\in I_n)$, we have

$$
TFMBHM^{p,q}_v(N_1,N_2,...,N_n)=TFMBHM^{p,q}_v(N,N,...,N)=N
$$

ii. ***(Monotonicity)*** Based on Definition 2.8, if $N_i\geq M_i$ for all $i\in I_n$, then $TFMBHM^{p,q}_v$ is monotonic that is,

$$
TFMBHM^{p,q}_v(N_1,N_2,...,N_n)\geq TFMBHM^{p,q}_v(M_1,M_2,...,M_n)
$$

iii. ***(Commutativity)*** If $(\dot{N}_1,\dot{N}_2,...,\dot{N}_n)$ be any permutation of $(N_1,N_2,...,N_n)$ then,

$$
TFMBHM^{p,q}_v(N_1,N_2,...,N_n)=TFMBHM^{p,q}_v(\dot{N}_1,\dot{N}_2,...,\dot{N}_n)
$$

iv. ***(Boundedness)***

$$
N^-\leq TFMBHM^{p,q}_v(N_1,N_2,...,N_n)\leq N^+
$$

where

$$
N^+=\langle(max\{x_i\}_{i\in I_n},max\{y_i\}_{i\in I_n},max\{z_i\}_{i\in I_n},max\{t_i\}_{i\in I_n});\\ max\{\eta^1_{N_i}\}_{i\in I_n},max\{\eta^2_{N_i}\}_{i\in I_n},...,max\{\eta^P_{N_i}\}_{i\in I_n}\rangle
$$

and

$$
N^-=\langle(min\{x_i\}_{i\in I_n},min\{y_i\}_{i\in I_n},min\{z_i\}_{i\in I_n},min\{t_i\}_{i\in I_n});\\ min\{\eta^1_{N_i}\}_{i\in I_n},min\{\eta^2_{N_i}\}_{i\in I_n},...,min\{\eta^P_{N_i}\}_{i\in I_n}\rangle
$$

By considering different values of p and q, a few specific cases of $TFMBHM_v^{p,q}$ are obtained as follows:

Remark 3.5. If $q=0$, $TFMBHM_v^{p,q}$ operator converted into TFM weighted generalized harmonic mean operator:

$$TFMBHM_v^{p,0}(N_1,N_2,...,N_n)=\frac{1}{\left(\bigoplus_{i,j=1,i\neq j}^{n}\left(\frac{v_iv_j}{N_i^p}\right)\right)^{\frac{1}{p}}}=\frac{1}{\left(\bigoplus_{i=1}^{n}\left(\frac{v_i}{N_i^p}\right)\bigoplus_{j=1}^{n}v_j\right)^{\frac{1}{p}}}=\frac{1}{\left(\bigoplus_{i=1}^{n}\left(\frac{v_i}{N_i^p}\right)\right)^{\frac{1}{p}}}$$

which is TFM weighted generalized harmonic mean ($TFMGHM_v$) operator.

Remark 3.6. If $p=1,q=0$, $TFMBHM_v^{p,q}$ operator converted into:

$$TFMBHM_v^{1,0}(N_1,N_2,...,N_n)=\frac{1}{\bigoplus_{i,j=1,i\neq j}^{n}\left(\frac{v_iv_j}{N_i}\right)}=\frac{1}{\left(\bigoplus_{i=1}^{n}\left(\frac{v_i}{N_i}\right)\bigoplus_{j=1}^{n}v_j\right)}=\frac{1}{\bigoplus_{i=1}^{n}\left(\frac{v_i}{N_i}\right)}$$

Remark 3.7. If $p=2,q=0$, $TFMBHM_v^{p,q}$ operator converted into:

$$TFMBHM_v^{2,0}(N_1,N_2,...,N_n)=\frac{1}{\left(\bigoplus_{i,j=1,i\neq j}^{n}\left(\frac{v_iv_j}{N_i^2}\right)\right)^{\frac{1}{2}}}=\frac{1}{\left(\bigoplus_{i=1}^{n}\left(\frac{v_i}{N_i^2}\right)\bigoplus_{j=1}^{n}v_j\right)^{\frac{1}{2}}}=\frac{1}{\left(\bigoplus_{i=1}^{n}\left(\frac{v_i}{N_i^2}\right)\right)^{\frac{1}{2}}}$$

Remark 3.8. If $p=1,q=1$, $TFMBHM_v^{p,q}$ operator converted into:

$$TFMBHM_v^{1,1}(N_1,N_2,...,N_n)=\frac{1}{\left(\bigoplus_{i,j=1,i\neq j}^{n}\left(\frac{v_iv_j}{N_iN_j}\right)\right)^{\frac{1}{2}}}$$

Example 3.9. Suppose we have three TFM-numbers as follows:

$$\begin{aligned}N_1&=\langle(0.1,0.4,0.5,0.6);0.5,0.3,0.4,0.2\rangle\\N_2&=\langle(0.1,0.2,0.5,0.8);0.9,0.6,0.3,0.5\rangle\\N_3&=\langle(0.2,0.3,0.3,0.4);0.7,0.8,0.3,0.4\rangle\end{aligned}$$

Then, based on the operations in Definition 2.7 and Equation (1) for $p,q=1$, we have

$$\begin{aligned}N_1^1\oplus N_2^1&=\langle(0.01,0.08,0.25,0.48);0.45,0.18,0.12,0.10\rangle\\N_2^1\oplus N_1^1&=\langle(0.01,0.08,0.25,0.48);0.45,0.18,0.12,0.10\rangle\\N_1^1\oplus N_3^1&=\langle(0.02,0.12,0.15,0.24);0.35,0.24,0.12,0.08\rangle\\N_3^1\oplus N_1^1&=\langle(0.02,0.12,0.15,0.24);0.35,0.24,0.12,0.08\rangle\\N_2^1\oplus N_3^1&=\langle(0.02,0.06,0.15,0.32);0.63,0.48,0.09,0.20\rangle\\N_3^1\oplus N_2^1&=\langle(0.02,0.06,0.15,0.32);0.63,0.48,0.09,0.20\rangle\end{aligned}$$

and then we obtain:

$$TFMBHM^{1,1}(N_1,N_2,N_3)=\langle(0.0056,0.0289,0.0611,0.1156);0.9912,0.9460,0.7095,0.7492\rangle$$

In a similar way, if $p,q=2$, from Equation (1), we have

$$TFMBHM^{2,2}(N_1,N_2,N_3)=\langle(0.0001,0.0014,0.0060,0.0217);0.9521,0.8440,0.5173,0.5736\rangle$$

if $p=1,q=3$, from Equation (1), we have

$$TFMBHM^{1,3}(N_1,N_2,N_3)=\langle(0.0001,0.0015,0.0063,0.0239);0.9618,0.8664,0.5214,0.5901\rangle$$

if $p=3,q=1$, from Equation (1), we have

$$TFMBHM^{3,1}(N_1,N_2,N_3)=\langle(0.0001,0.0015,0.0063,0.0239);0.9618,0.8664,0.5214,0.5901\rangle$$

4. An Application of Weighted Bonferroni Harmonic Mean Operator on TFM-numbers

This section developes an algorithm based on weighted Bonferroni harmonic mean operator to solve multi criteria decision making problems.

Algorithm

Here, we firstly give an algorithm to solve decision making problems with multi criteria. Then, to see application of the algorithm, we give an example. As for end of the section, we present a table of $TFMBHM_v^{p,q}$ taken by changing (p,q) values.

Table 1. TFM-numbers for linguistic terms

Linguistic terms	**Linguistic values of TFM-numbers**
Absolutely low (AL)	$\langle(0.01, 0.05, 0.10, 0.15); 0.1, 0.2, 0.3, 0.4\rangle$
Very Very Low (VVL)	$\langle(0.05, 0.10, 0.15, 0.20); 0.2, 0.3, 0.4, 0.1\rangle$
Very Low (VL)	$\langle(0.10, 0.15, 0.15, 0.20); 0.2, 0.4, 0.5, 0.3\rangle$
Low (L)	$\langle(0.10, 0.20, 0.20, 0.30); 0.3, 0.4, 0.8, 0.1\rangle$
Fairly low (FL)	$\langle(0.15, 0.20, 0.25, 0.30); 0.4, 0.6, 0.2, 0.5\rangle$
Medium (M)	$\langle(0.25, 0.30, 0.35, 0.40); 0.4, 0.5, 0.6, 0.8\rangle$
Fairly high (FH)	$\langle(0.30, 0.35, 0.40, 0.45); 0.6, 0.1, 0.8, 0.4\rangle$
High (H)	$\langle(0.40, 0.45, 0.50, 0.55); 0.8, 0.9, 0.3, 0.6\rangle$
Very High (VH)	$\langle(0.45, 0.55, 0.65, 0.75); 0.7, 0.8, 0.6, 0.3\rangle$
Very Very High (VVH)	$\langle(0.50, 0.60, 0.70, 0.80); 0.1, 0.7, 0.8, 0.9\rangle$
Absolutely high (AH)	$\langle(0.70, 0.80, 0.90, 1.00); 0.7, 0.8, 0.9, 0.2\rangle$

Step 1: Build the decision matrix based on experts' decision according to Table 1. Let $X = \{x_i | i \in I_m\}$ be set of alternatives and $C = \{c_j | j \in I_n\}$ be set of criteria whose weight vector is $v = (v_1, v_2, ...v_n)^T$ such that $v_i \geq 0$ and $\sum\limits_{i=1}^{n} v_i = 1$.

Preferable of the alternatives x_i based on the criteria c_j expressed by a linguistic terms is computed by following $TFMBHM_v$:

$$N_{ij} = \langle(x_{ij}, y_{ij}, z_{ij}, t_{ij}); \eta^1_{N_{ij}}, \eta^2_{N_{ij}}, ..., \eta^P_{N_{ij}}\rangle \; (i \in I_m)$$

Step 2: Get preferable for x_i based on N_i $(i \in I_m)$ to aggregate the TFM-numbers $N_{i1}, N_{i2}, ..., N_{in}$ as:

$$N_i = TFMBHM_v^{p,q}(N_{i1}, N_{i2}, ..., N_{in})$$

Step 3: Calculate score value $s(N_i)$ whose formula is given in Definition 2.9 for each N_i to rank alternatives.

Step 4: Rank all score value of N_i according to descending order.

4.1. An Illustrative Example

Example 4.1. An entrepreneur wants to set a factory to a suitable place in the country. There are five alternatives to be chosen to set the factory. The entrepreneur takes into account of following criteria in decision making process :

1. Closeness to raw material (c_1)
2. Facility of transportation (c_2)
3. Regional incentive (c_3)
4. Market breadth (c_4)

There are five candidates x_i $(i \in I_5)$. Besides $v = (0.1, 0.3, 0.1, 0.5)^T$ is weight vector of criteria c_j $(j \in I_4)$. Suppose that the preferability of the alternative x_i with regard to the attribute c_j is measured by a TFM $N_{ij} = \langle (x_{ij}, y_{ij}, z_{ij}, t_{ij}); \eta^1_{N_{ij}}, \eta^2_{N_{ij}}, ..., \eta^P_{N_{ij}} \rangle$ $(i \in I_m)$ and then we build TFM decision matrix $(N_{ij})_{5x4}$ according to decision makers' choices based on linguistic Table 1.

Step 1: Decision matrix is constructed according to decision maker's choices:

Table 2. Decision Matrix N_{ij}

	c_1	c_2	c_3	c_4
N_1	$\langle(0.01, 0.05, 0.10, 0.15); 0.1, 0.2, 0.3, 0.4\rangle$	$\langle(0.70, 0.80, 0.90, 1.00); 0.7, 0.8, 0.9, 0.2\rangle$	$\langle(0.05, 0.10, 0.15, 0.200.2, 0.3, 0.4, 0.1\rangle$	$\langle(0.70, 0.80, 0.90, 1.000.7, 0.8, 0.9, 0.2\rangle$
N_2	$\langle(0.45, 0.55, 0.65, 0.75); 0.7, 0.8, 0.6, 0.3\rangle$	$\langle(0.45, 0.55, 0.65, 0.75); 0.7, 0.8, 0.6, 0.3\rangle$	$\langle(0.10, 0.20, 0.20, 0.300.3, 0.4, 0.8, 0.1\rangle$	$\langle(0.01, 0.05, 0.10, 0.150.1, 0.2, 0.3, 0.4\rangle$
N_3	$\langle(0.10, 0.20, 0.20, 0.30); 0.3, 0.4, 0.8, 0.1\rangle$	$\langle(0.30, 0.35, 0.40, 0.45); 0.6, 0.1, 0.8, 0.4\rangle$	$\langle 0.25, 0.30, 0.35, 0.400.4, 0.5, 0.6, 0.8\rangle$	$\langle(0.50, 0.60, 0.70, 0.800.1, 0.7, 0.8, 0.9\rangle$
N_4	$\langle(0.10, 0.15, 0.15, 0.20); 0.2, 0.4, 0.5, 0.3\rangle$	$\langle(0.15, 0.20, 0.25, 0.30); 0.4, 0.6, 0.2, 0.5\rangle$	$\langle(0.40, 0.45, 0.50, 0.550.8, 0.9, 0.3, 0.6\rangle$	$\langle(0.05, 0.10, 0.15, 0.200.2, 0.3, 0.4, 0.1\rangle$
N_5	$\langle(0.30, 0.35, 0.40, 0.45); 0.6, 0.1, 0.8, 0.4\rangle$	$\langle(0.10, 0.15, 0.15, 0.20); 0.2, 0.4, 0.5, 0.3\rangle$	$\langle(0.50, 0.60, 0.70, 0.800.1, 0.7, 0.8, 0.9\rangle$	$\langle(0.45, 0.55, 0.65, 0.750.7, 0.8, 0.6, 0.3\rangle$

Step 2: We aggregate experts' ratings for each alternative, by using TFM-weighted Bonferroni harmonic mean operator. That is,

$$
\begin{aligned}
N_1 = &\ TFMBHM_v^{1,1}(N_{11}, N_{12}, N_{13}, N_{14}) = \frac{1}{\left(\bigoplus\limits_{i,j=1, i\neq j}^{n} \left(\left(\frac{v_i}{N_{1i}^p}\right) \otimes \left(\frac{v_j}{N_{1j}^q}\right)\right)\right)^{\frac{1}{1+1}}} \\
= &\ \langle (0.1212, 0.3091, 0.4629, 0.5941); 0.1583, 0.2536, 0.4058, 0.0175 \rangle
\end{aligned}
$$

$$
\begin{aligned}
\text{where } &\bigoplus_{i,j=1, i\neq j}^{n} \left(\left(\frac{v_i}{N_{1i}^p}\right) \otimes \left(\frac{v_j}{N_{1j}^q}\right)\right) \\
= &\left(\frac{v_1}{N_{11}^p} \otimes \frac{v_2}{N_{12}^q}\right) \oplus \left(\frac{v_1}{N_{11}^p} \otimes \frac{v_3}{N_{13}^q}\right) \oplus \left(\frac{v_1}{N_{11}^p} \otimes \frac{v_4}{N_{14}^q}\right) \oplus \left(\frac{v_2}{N_{12}^p} \otimes \frac{v_1}{N_{11}^q}\right) \oplus \left(\frac{v_2}{N_{12}^p} \otimes \frac{v_3}{N_{13}^q}\right) \oplus \left(\frac{v_2}{N_{12}^p} \otimes \frac{v_4}{N_{14}^q}\right) \\
\oplus &\left(\frac{v_3}{N_{13}^p} \otimes \frac{v_1}{N_{11}^q}\right) \oplus \left(\frac{v_3}{N_{13}^p} \otimes \frac{v_2}{N_{12}^q}\right) \oplus \left(\frac{v_3}{N_{13}^p} \otimes \frac{v_4}{N_{14}^q}\right) \oplus \left(\frac{v_4}{N_{14}^p} \otimes \frac{v_1}{N_{11}^q}\right) \oplus \left(\frac{v_4}{N_{14}^p} \otimes \frac{v_2}{N_{12}^q}\right) \oplus \left(\frac{v_4}{N_{14}^p} \otimes \frac{v_3}{N_{13}^q}\right)
\end{aligned}
$$

$$
\begin{aligned}
N_2 = &\ TFMBHM_v^{1,1}(N_{21}, N_{22}, N_{23}, N_{24}) \\
= &\ \langle (0.0724, 0.1981, 0.2897, 0.4003); 0.0704, 0.1373, 0.1391, 0.0377 \rangle \\
N_3 = &\ TFMBHM_v^{1,1}(N_{31}, N_{32}, N_{33}, N_{34}) \\
= &\ \langle (0.3450, 0.4673, 0.5228, 0.6316); 0.0383, 0.0720, 0.3806, 0.2141 \rangle \\
N_4 = &\ TFMBHM_v^{1,1}(N_{41}, N_{42}, N_{43}, N_{44}) \\
= &\ \langle (0.1191, 0.1930, 0.2490, 0.3169); 0.0592, 0.1439, 0.0457, 0.0414 \rangle \\
N_5 = &\ TFMBHM_v^{1,1}(N_{51}, N_{52}, N_{53}, N_{54}) \\
= &\ \langle (0.2990, 0.3964, 0.4334, 0.5300); 0.0789, 0.1574, 0.2311, 0.0911 \rangle
\end{aligned}
$$

Step 3: Scores of $N_i's$ calculated as:

$$s(N_1) = \frac{0.5941^2 + 0.4629^2 - 0.1212^2 - 0.3091^2}{2.4}(0.1583 + 0.2536 + 0.4058 + 0.0175) = 0.0477,$$

$s(N_2) = 0.0096$, $s(N_3) = 0.0295$, $s(N_4) = 0.0040$, $s(N_5) = 0.0155$

Step 4: Alternatives are ranked:

$$x_1 > x_3 > x_5 > x_2 > x_4$$

Finally, the best alternative to choose is x_1. Moreover, rankings for some alternatives in terms of different $TFMBHM_v^{p;q}$ of the example are given in Table 3.

Table 3. Rankings for some alternatives in terms of different $TFMBHM_v^{p,q}$ of Example 4.1

(p,q)	$s(N_1)$	$s(N_2)$	$s(N_3)$	$s(N_4)$	$s(N_5)$	**Ranking**
$(1.0, 2.0)$	0.0139	0.0023	0.0110	0.0010	0.0042	$x_1 > x_3 > x_5 > x_2 > x_4$
$(3.0, 1.0)$	0.0051	0.0007	0.0052	0.0003	0.0014	$x_3 > x_1 > x_5 > x_2 > x_4$
$(3.0, 2.0)$	0.0028	0.0003	0.0025	0.0001	0.0006	$x_1 > x_3 > x_5 > x_2 > x_4$
$(1.0, 0.5)$	0.0932	0.0234	0.0590	0.0106	0.0339	$x_1 > x_3 > x_5 > x_2 > x_4$
$(2.0, 0.5)$	0.0201	0.0043	0.0190	0.0023	0.0071	$x_1 > x_3 > x_5 > x_2 > x_4$
$(3.0, 0.5)$	0.0068	0.0013	0.0084	0.0008	0.0023	$x_3 > x_1 > x_5 > x_2 > x_4$
$(4.0, 0.5)$	0.0031	0.0006	0.0044	0.0004	0.0010	$x_3 > x_1 > x_5 > x_2 > x_4$

5. Comparison Analysis

Table 4. Some rankings in terms of different methods and proposed method of Example 4.1

Methods	**Operator**	**Ranking**
Proposed method	$TFMBHM_v^{(1,1)}$	$x_1 > x_3 > x_5 > x_2 > x_1$
Method of Deli and Keleş [8]	$S^i(x_i)$	$x_5 > x_3 > x_4 > x_1 > x_2$
Method of Uluçay et al. [7]	$TFMG_v$	$x_5 > x_3 > x_4 > x_1 > x_2$
Method of Şahin et al. [10]	D_v	$x_3 > x_5 > x_1 > x_4 > x_2$
Method of Uluçay [11]	S_v	$x_4 > x_3 > x_1 > x_5 > x_2$

Table 4 compares introduced method with other existing methods such as distance measure operator proposed by Deli & Keleş [8], TFM weighted geometric operator introduced by Uluçay et al. [7], weighted dice vector similarity operator submitted by Şahin et al. [10] and vector similarity operator given by Uluçay [11] based on Example 4.1. If we check over the comparison table given above, we can see result of the proposed aggregation method presents a new perspective to decision making process. So, proposed method can be readily used to solve decision making problems with multiple criteria.

6. Conclusion

In this article, firstly, Bonferroni harmonic mean was introduced. Then, its some special cases were investigated to see how it converted into other operators. Secondly, to see an application of the operator, a basic example was given. Thirdly, an algorithm was proposed to solve decision making problems. Then, a numerical example was given to show performance of the developed approach and to demonstrate its degree of effectiveness and practicality. Finally, a comparison table was presented to compare proposed method with some existing methods.

In future, the operator will be applied to hesitant fuzzy numbers, intuitionistic fuzzy numbers and neutrosophic fuzzy numbers. In addition, we plan to extend our research to AHP method, ANP method, Topsis method, VIKOR method, QUALIFLEX method, ELECTRE I method, ELECTRE II method, ELECTRE III method, defuzzification techniques and so on.

Author Contributions

All authors contributed equally to this work. They all read and approved the last version of the paper.

Conflicts of Interest

All authors declare no conflict of interest.

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