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Research Article

A novel kind of beta logarithmic function and their properties

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Abstract

The main objective is to introduce a novel kind of beta function known as the beta logarithmic function using extended beta functions and logarithmic mean. Further, we study its essential properties and investigate various formulas of beta logarithmic functions such as integral representation, summation formula, transform formula and their statistical properties. Based on this concept, we introduce new hypergeometric and confluent hypergeometric functions and study their properties.

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1. Introduction and Preliminaries

The Euler beta function plays a significant role in a special function and related areas. It has a wide range of applications in science especially in engineering mathematics. The extension of the classical Euler beta and gamma functions has been an exciting topic for researchers due to their pivot role in advanced research in the last few years. In 1997, Chaudhary et al. and after that many researcher provided an extension of Euler beta functions (see [2–4, 7, 10, 11],[8]). In this paper, we investigate the new parametrization of the beta function known as beta logarithmic functions and discuss their various properties such as summation formula, beta distribution and some statistical formulas. Further, we discuss the hypergeometric and confluent hypergeometric functions based on these concepts and their essential properties.

Definition 1.1. The logarithmic mean defined as;

$$L(a,b) = \int_0^1 a^{1-t} b^t \, dt = \begin{cases} \frac{a-b}{log(a)-log(b)} & a \neq b, \\ a & \text{a=b.} \end{cases} \tag{1.1}$$

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Definition 1.2. The classical Gauss hypergeometric functions (see [1]) is defined as

$$F(x,y;w;z)=\sum_{n=0}^{\infty}\frac{(x)_n\,(y)_n}{(w)_n}\,\frac{z^n}{n!}, \tag{1.2}$$

and the confluent hypergeometric functions (see [1]) is defined by

$$\Phi(y;w;z)=\sum_{n=0}^{\infty}\frac{(y)_n}{(w)_n}\,\frac{z^n}{n!}, \tag{1.3}$$

where $(\mu)_n$ is called the Pochhammer symbol define as follows:

$$(\mu)_n=\frac{\Gamma(\mu+n)}{\Gamma(\mu)},\ (\mu\in\mathbb{C}). \tag{1.4}$$

Definition 1.3. The Euler beta function $B(x,y)$ (see [1]) is defined by

$$B(x,y)=\int_0^1 t^{x-1}\,(1-t)^{y-1}dt=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}=\frac{(x-1)!\,(y-1)!}{(x+y-1)!},$$

where $\Re(x)>0$, $\Re(y)>0$.

Choudhary et al. [3] introduced an extension of beta functions in 1997, which defined as:

$$B^p(x,y)=\int_0^1 t^{x-1}\,(1-t)^{y-1}\,\exp\left(-\frac{p}{t(1-t)}\right)dt, \tag{1.5}$$

where

$$(\Re(p)>0,\ \Re(x)>0,\ \Re(y)>0).$$

Chaudhary et al. [4] used new extended beta function $B^p(x,y)$ to introduced extended Gauss hypergeometric and confluent hypergeometric function defined respectively as:

$$F_p\,(x,y;w;z)=\sum_{n=0}^{\infty}(x)_n\,\frac{B_p(y+n,w-y)}{B(y,w-y)}\,\frac{z^n}{n!} \tag{1.6}$$

$$(p\geq 0,\ \ |z|<1,\ \ \Re(w)>\Re(y)>0),$$

and

$$\Phi_p\,(y;w;z)=\sum_{n=0}^{\infty}\frac{B_p(y+n,w-y)}{B(y,w-y)}\,\frac{z^n}{n!} \tag{1.7}$$

$$(p\geq 0,\ \ \Re(w)>\Re(y)>0).$$

Now the integral representation for extended Gauss hypergeometric functions and extended confluent hypergeometric functions are defined as:

$$F_p\,(x,y;w;z)=\frac{1}{B(y,w-y)}\int_0^1 t^{y-1}\,(1-t)^{w-y-1}\,(1-zt)^{-x}\,exp\left(-\frac{p}{t(1-t)}\right)dt \tag{1.8}$$

$$(p\geq 0;\quad |\arg(1-z)|<\pi;\ \Re(w)>\Re(y)>0),$$

$$\Phi_p\,(y;w;z)=\frac{1}{B(y,w-y)}\int_0^1 t^{y-1}\,(1-t)^{w-y-1}\,exp\left(zt-\frac{p}{t(1-t)}\right)dt \tag{1.9}$$

$$(p\geq 0;\ \ and\ \ \Re(w)>\Re(y)>0).$$

Recently, Khan et al. [8] introduced an extension of beta function using generalized Mittag-Leffler function as follow:

$$B_{\alpha,\beta}^{p,\eta,\nu}(x,y)=\int_0^1 t^{x-1}\,(1-t)^{y-1}\,E_{\alpha,\beta}\left(-\frac{p}{t^{\eta}(1-t)^{\nu}}\right)dt \tag{1.10}$$

$$(\Re(x) > 0,\, \Re(y) > 0; \quad a, b, \alpha, \beta, \eta, \nu \in \mathbb{R}^+,\, p \geq 0),$$

where $E_{\alpha,\beta}(.)$ is the generalized Mittag-Leffler function (see [14]) define as

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \tag{1.11}$$

where $\alpha, \beta \in \mathbb{R}_0^+$, $z \in \mathbb{C}$.

By using (1.10), we furthor extended the Gauss hypergeometric and confluent hypergeometric functions and their integral representation are defined as

$$F_{\alpha,\beta}^{p,\eta,\nu}(x, y; w; z) = \sum_{n=0}^{\infty} (x)_n \frac{B_{\alpha,\beta}^{p,\eta,\nu}(y+n, w-y)}{B(y, w-y)} \frac{z^n}{n!} \tag{1.12}$$

$$(p \geq 0,\ |z| < 1,\ \alpha, \beta, \eta, \nu \in \mathbb{R}^+,\ \Re(w) > \Re(y) > 0),$$

$$\Phi_{\alpha,\beta}^{p,\eta,\nu}(y; w; z) = \sum_{n=0}^{\infty} \frac{B_{\alpha,\beta}^{p,\eta,\nu}(y+n, w-y)}{B(y, w-y)} \frac{z^n}{n!} \tag{1.13}$$

$$(p \geq 0,\ \alpha, \beta, \eta, \nu \in \mathbb{R}^+,\ \Re(w) > \Re(y) > 0).$$

$$\begin{aligned} F_{\alpha,\beta}^{p,\eta,\nu}(x, y; w; z) =& \frac{1}{B(y, w-y)} \\ &\times \int_0^1 t^{y-1} (1-t)^{w-y-1} (1-zt)^{-x} E_{\alpha,\beta}\left(-\frac{p}{t^\eta (1-t)^\nu}\right) dt \end{aligned} \tag{1.14}$$

$$(p \in \mathbb{R}_0^+, a, b, \alpha, \beta, \eta, \nu \in \mathbb{R}^+ \ \textit{and}\ |\arg(1-z)| < \pi;\ \Re(w) > \Re(y) > 0),$$

and

$$\begin{aligned} \Phi_{\alpha,\beta}^{p,\eta,\nu}(y; w; z) =& \frac{1}{B(y, w-y)} \\ &\times \int_0^1 t^{y-1} (1-t)^{w-y-1} e^{zt} E_{\alpha,\beta}\left(-\frac{p}{t^\eta (1-t)^\nu}\right) dt \end{aligned} \tag{1.15}$$

$$(p \in \mathbb{R}_0^+, a, b, \alpha, \beta, \eta, \nu \in \mathbb{R}^+ \ \textit{and}\ \Re(w) > \Re(y) > 0).$$

Recently, Raïssouli et al. (see[13]) specified a new class of beta functions known as a new parametrized beta function in which connected the beta functions with logarithmic mean and discussed their properties. Motivated by the above result, we introduce the novel kind of beta logarithmic function by connecting the extended beta functions given by Khan et al. (see[8]) and the logarithmic mean and analyzed their essential properties.

2. Main results

Here, we introduce and investigate various kind of properties and representations of a novel kind of beta function known as beta logarithmic function a combined study of beta function (1.10) and logarithmic mean (1.1).

For any fixed $a, b > 0$ the function $t \to a^{1-t} b^t$ is continous in $[0, 1]$ and so it is bounded on $[0, 1]$. It means that there exist $c \geq 0$ and for any $a, b, x, y > 0$, we have

$$\begin{aligned} 0 \quad &\leq \quad a^{1-t}\, b^t\, t^{x-1} (1-t)^{y-1} E_{\alpha,\beta}\left(-\frac{p}{t^\eta (1-t)^\nu}\right) \\ &\leq \quad c\, t^{x-1} (1-t)^{y-1} E_{\alpha,\beta}\left(-\frac{p}{t^\eta (1-t)^\nu}\right) \qquad \forall t \in (0, 1). \end{aligned}$$

Thus, $a^{1-t}\, b^t\, t^{x-1}\,(1-t)^{y-1}\, E_{\alpha,\beta}\left(-\frac{p}{t^\eta(1-t)^\nu}\right)$ is integrable on $(0,1)$. Then we introduce the following defintion:

Definition 2.1. For any $\alpha, \beta, \eta, \nu, a, b \in \mathbb{R}^+$, we define

$$BL^{p,\eta,\nu}_{\alpha,\beta}(a,b;x,y) = \int_0^1 a^{1-t}\, b^t\, t^{x-1}\,(1-t)^{y-1}\, E_{\alpha,\beta}\left(-\frac{p}{t^\eta(1-t)^\nu}\right) dt, \tag{2.1}$$

$$(p \geq 0, \quad \Re(x) > 0, \, \Re(y) > 0).$$

Where $E_{\alpha,\beta}(.)$ is the generalized Mittag-Leffler function defined in (1.11).

Remark 2.2. If we take $a = 1$ and $b = 1$ in (2.1), we get the beta function introduced by Khan et al. (see [8,9]) i.e.

$$BL^{p,\eta,\nu}_{\alpha,\beta}(1,1;x,y) = B^{p,\eta,\nu}_{\alpha,\beta}(x,y).$$

Remark 2.3. If we take $\alpha = \beta = \mu = \nu = a = b = 1$ and $p = 0$ in (2.1), we get an Euler beta functions (1.5) (see [1,5])

$$BL^{0,1,1}_{1,1}(1,1;x,y) = B(x,y).$$

Remark 2.4. If we choose $\alpha = \beta = \mu = \nu = x = y = 1$ and $p = 0$ in (2.1), then we get a logarithmic mean (1.1)

$$BL^{0,1,1}_{1,1}(a,b;1,1) = L(a,b).$$

Proposition 2.5. *For $\alpha, \beta, a, b, x, y > 0$, the following assertions holds true:*

$$BL^{p,\eta,\nu}_{\alpha,\beta}(a,b;x,y) = BL^{p,\eta,\nu}_{\alpha,\beta}(b,a;x,y), \tag{2.2}$$

$$BL^{p,\eta,\nu}_{\alpha,\beta}(a,a;x,y) = a\; B^{p,\eta,\nu}_{\alpha,\beta}(x,y), \tag{2.3}$$

and

$$BL^{p,\eta,\nu}_{\alpha,\beta}(\lambda a,\lambda b;x,y) = \lambda BL^{p,\eta,\nu}_{\alpha,\beta}(a,b;x,y). \tag{2.4}$$

Proof. The result (2.2) can be obtained by changing the variable t by $1-u$ in (2.1). The assertaion (2.3) and (2.4) can be obtained by simple calculation in (2.1). □

Proposition 2.6. *For any $\alpha, \beta, a, b, x, y > 0$, the following assertions holds true:*

$$BL^{p,\eta,\nu}_{\alpha,\beta}(a,b;x+1,y) + BL^{p,\eta,\nu}_{\alpha,\beta}(a,b;x,y+1) = BL^{p,\eta,\nu}_{\alpha,\beta}(a,b;x,y) \tag{2.5}$$

Proof. By using the defintion (2.1) to the left side of (2.5), we get the assertion (2.5). □

Corollary 2.7. *If we set $a = b = 1$ in (2.5), we obtained the known result of Khan et al. (see [8])*

$$B^{p,\eta,\nu}_{\alpha,\beta}(x+1,y) + B^{p,\eta,\nu}_{\alpha,\beta}(x,y+1) = B^{p,\eta,\nu}_{\alpha,\beta}(x,y). \tag{2.6}$$

Proposition 2.8. *For any $\alpha, \beta, a, b, x, y > 0$, the following assertions holds true:*

$$min(a,b)B^{p,\eta,\nu}_{\alpha,\beta}(x,y) \leq BL^{p,\eta,\nu}_{\alpha,\beta}(a,b;x,y) \leq max(a,b)BL^{p,\eta,\nu}_{\alpha,\beta}(x,y). \tag{2.7}$$

Proof. From the inequality,

$$min(a,b) \leq \sqrt{ab} \leq L(a,b) \leq \left(\frac{a+b}{2}\right) \leq max(a,b) \qquad and \qquad B^{p,\eta,\nu}_{\alpha,\beta}(x,y) > 0,$$

we get the following relation

$$min(a,b)B^{p,\eta,\nu}_{\alpha,\beta}(x,y) \leq BL^{p,\eta,\nu}_{\alpha,\beta}(a,b;x,y). \tag{2.8}$$

Using the following well known young's inequality

$$a^{1-t}b^t \leq a(1-t) + bt, \quad \forall t \in [0,1]$$

we obtain

$$\begin{aligned} BL_{\alpha,\beta}^{p,\eta,\nu}(a,b;x,y) \leq & \quad aB_{\alpha,\beta}^{p,\eta,\nu}(x,y+1) + bB_{\alpha,\beta}^{p,\eta,\nu}(x+1,y) \\ \leq & \quad max(a,b)\left(B_{\alpha,\beta}^{p,\eta,\nu}(x,y+1) + B_{\alpha,\beta}^{p,\eta,\nu}(x+1,y)\right). \end{aligned}$$

Using the relation (2.6), we achived the desired result. □

Proposition 2.9. *For any $\alpha, \beta, a, b, x, y > 0$, the following assertions holds true:*

$$BL_{\alpha,\beta}^{p,\eta,\nu}(a,b;x,y) = \sum_{n=0}^{\infty} BL_{\alpha,\beta}^{p,\eta,\nu}(a,b;x+n,y+1). \tag{2.9}$$

Proof. Using the series representation $(1-t)^{-1} = \sum_{n=0}^{\infty} t^n$, for $t \in (0,1)$ with the arguments of uniform convergence of this power series, we have

$$BL_{\alpha,\beta}^{p,\eta,\nu}(a,b;x,y) = \sum_{n=0}^{\infty} \int_0^1 a^{1-t}\, b^t\, t^{x+n-1}\,(1-t)^y\, E_{\alpha,\beta}\left(-\frac{p}{t^\eta(1-t)^\nu}\right) dt.$$

Using the definition (2.1) in the above expression, we achived the desired result. □

Theorem 2.10. *Let $\alpha, \beta, a, b, x, y > 0$, the following representation holds true:*

$$BL_{\alpha,\beta}^{p,\eta,\nu}(a,b;x,y) = \sum_{m,n=0}^{\infty} \frac{B_{\alpha,\beta}^{p,\eta,\nu}(x+n,y+m)}{n!m!}(log(a))^m(log(b))^n. \tag{2.10}$$

Proof. The following power series expansion

$$a^{1-t} = \sum_{m=0}^{\infty} \frac{(log(a))^m}{m!}(1-t)^m, \qquad b^t = \sum_{n=0}^{\infty} \frac{(log(b))^n}{n!} t^n$$

using the above expansion in the result (2.1), we have $BL_{\alpha,\beta}^{p,\eta,\nu}(a,b;x,y) =$

$$\int_0^1 \sum_{m,n=0}^{\infty} \frac{(log(a))^m(log(b))^n}{m!n!}\, t^{x+n-1}\,(1-t)^{y+m-1} E_{\alpha,\beta}\left(-\frac{p}{t^\eta(1-t)^\nu}\right) dt.$$

This together with the fact that the involved power series is uniformly convergent, allows us to interchange the order of the integral with the infinite sum getting (2.10). □

Theorem 2.11. *For any $a, b, \alpha, \beta, \eta, \nu > 0$, the following relation holds true:*

$$BL_{\alpha,\beta}^{p,\eta,\nu}(a,b;x,y) = 2\int_0^{\frac{\pi}{2}} \left(\frac{b}{a}\right)^{cos^2(\theta)} cos^{2x-1}(\theta) sin^{2y-1}(\theta) E_{\alpha,\beta}(-psec^{2\eta}(\theta)csc^{2\nu}(\theta))d\theta. \tag{2.11}$$

Proof. Let $t = cos^2(\theta)$ in (2.1). After simplification, we obtain the desired result (2.11). □

Theorem 2.12. *The Mellin transform of beta logarithmic function as:*

$$\mathsf{M}\left\{BL_{\alpha,\beta}^{p,\eta,\nu}(a,b;x,y);s\right\} = \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\beta-\alpha s)} BL(a,b;x+\eta s, y+\nu s). \tag{2.12}$$

Proof. We know that $\mathsf{M}\left\{BL_{\alpha,\beta}^{p,\eta,\nu}(a,b;x,y);s\right\} =$

$$\begin{aligned} & \int_0^\infty p^{s-1} BL_{\alpha,\beta}^{p,\eta,\nu}(a,b;x,y)dp \\ &= \int_0^\infty p^{s-1} \int_0^1 a^{1-t}\, b^t\, t^{x-1}\,(1-t)^{y-1}\, E_{\alpha,\beta}\left(-\frac{p}{t^\eta(1-t)^\nu}\right) dt\; dp \\ &= \int_0^1 a^{1-t}\, b^t\, t^{x-1}\,(1-t)^{y-1} \left(\int_0^\infty p^{s-1} E_{\alpha,\beta}\left(-\frac{p}{t^\eta(1-t)^\nu}\right) dp\right) dt. \end{aligned}$$

In the above expression substitute $u = \frac{p}{t^{\eta}(1-t)^{\nu}}$ and after simplifacation, we have

$$= \int_0^1 a^{1-t}\, b^t\, t^{x+\eta s-1}\,(1-t)^{y+\nu s-1}\left(\int_0^{\infty} u^{s-1} E_{\alpha,\beta}(-u)du\right)dt, \tag{2.13}$$

we know well known result (see [6], p.102)

$$\int_0^{\infty} u^{s-1} E^{\gamma}_{\alpha,\beta}(-u) = \frac{\Gamma(s)\Gamma(\gamma-s)}{\Gamma(\gamma)\Gamma(\beta-\alpha s)}. \tag{2.14}$$

Using the above expression for $\gamma = 1$ and the definition of beta functions given by Raïssouli et al. (see[13]) in (2.13). Thus, we obtain the required result. □

3. The novel kind of beta logarithmic random variable

We now define the beta logarithmic distribution of (2.1) and obtain its mean, variance and moment generating function.

For $\alpha, \beta, a, b, x, y > 0$, the beta logarithmic distribution is defined as

$$f(t) = \begin{cases} \frac{1}{BL^{p,\eta,\nu}_{\alpha,\beta}(x,y)} a^{1-t} b^t t^{x-1}(1-t)^{y-1}\, E_{\alpha,\beta}\left(-\frac{p}{t^{\eta}(1-t)^{\nu}}\right) & (0 < t < 1), \\ 0 & \text{otherwise}. \end{cases} \tag{3.1}$$

$$(a, b, \alpha, \beta, \eta, \nu \in \mathbb{R}^+, p \geq 0).$$

We have the d^{th} moment of a random variable X as for any real number d.

$$\mathbb{E}(X^d) = \frac{BL^{p,\eta,\nu}_{\alpha,\beta}(a,b;x+d,y)}{BL^{p,\eta,\nu}_{\alpha,\beta}(a,b;x,y)}, \tag{3.2}$$

$$(\ p \geq 0, \alpha, \beta, \eta, \nu \in \mathbb{R}^+, a, b, x, y > 0).$$

When $d = 1$, the mean is obtained as a special case of (3.2) given by

$$\mu = \mathbb{E}(X) = \frac{BL^{p,\eta,\nu}_{\alpha,\beta}(a,b;x+1,y)}{BL^{p,\eta,\nu}_{\alpha,\beta}(a,b;x,y)}. \tag{3.3}$$

The variance of a distribution is discuss as follows:

$$\sigma^2 = \mathbb{E}(X^2) - \{\mathbb{E}(X)\}^2$$

$$\sigma^2 = \frac{BL^{p,\eta,\nu}_{\alpha,\beta}(a,b;x,y)\, BL^{p,\eta,\nu}_{\alpha,\beta}(a,b;x+2,y) - \left\{BL^{p,\eta,\nu}_{\alpha,\beta}(a,b;x+1,y)\right\}^2}{\left\{BL^{p,\eta,\nu}_{\alpha,\beta}(a,b;x,y)\right\}^2}. \tag{3.4}$$

The moment generating function (mgf) of the distribution is defined as

$$M(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\mathbb{E}(X^n) = \frac{1}{BL^{p,\eta,\nu}_{\alpha,\beta}(a,b;x,y)} \sum_{n=0}^{\infty} BL^{p,\eta,\nu}_{\alpha,\beta}(a,b;x+n,y)\frac{t^n}{n!}. \tag{3.5}$$

Here, we recall some known lemma.

Lemma 3.1. *Let Y be a random variable with values that exist inside a finite range $[a, b]$. Then we have for all $\epsilon \in [a, b]$*

$$\left|P(Y \leq \epsilon) - \frac{b - E(Y)}{b-a}\right| \leq \frac{1}{2} + \frac{|\epsilon - \frac{a+b}{2}|}{b-a}. \tag{3.6}$$

Proposition 3.2. *Let X represent a beta logarithmic random variable with parameters $(a, b; x, y)$. Then, for any $d, \epsilon > 0$, the following assumptions are true:*

$$\left| P(X \leq \epsilon) - \frac{BL^{p,\eta,\nu}_{\alpha,\beta}(a, b; x, y+1)}{BL^{p,\eta,\nu}_{\alpha,\beta}(a, b; x, y)} \right| \leq \frac{1}{2} + \left| \epsilon - \frac{1}{2} \right|, \tag{3.7}$$

and

$$P(X^d \geq \epsilon) \leq \frac{BL^{p,\eta,\nu}_{\alpha,\beta}(a, b; x+d, y)}{\epsilon \ BL^{p,\eta,\nu}_{\alpha,\beta}(a, b; x, y)}. \tag{3.8}$$

Proof. With the help of (2.5) and (3.3), we have

$$E(X) = 1 - \frac{BL^{p,\eta,\nu}_{\alpha,\beta}(a, b; x, y+1)}{BL^{p,\eta,\nu}_{\alpha,\beta}(a, b; x, y)}, \tag{3.9}$$

using the above relation in the inequality (3.6), we achived the desired result (3.7). The second inequality (3.8) can be deducted by using the Markov's inequality

$$P(X^d \geq \epsilon) \leq \frac{E(X^d)}{\epsilon},$$

and the definition of $E(X^d)$, we get the desired result (3.8). □

4. Hypergeometric and Confluent hypergeometric functions in terms of logarithmic mean

Many author given the extention and generalization of hypergeometric and confluent hypergeometric functions (see [4,5,12]). Here, we introduce a hypergeometric and confluent hypergeometric functions in terms of beta logarithmic functions

The hypergeometric logarithmic function is defined as

$$FL^{p,\eta,\nu}_{a,b,\alpha,\beta}(x, y; w; z) = \sum_{n=0}^{\infty} (x)_n \frac{BL^{p,\eta,\nu}_{\alpha,\beta}(a, b; y+n, w-y)}{B(y, w-y)} \frac{z^n}{n!}, \tag{4.1}$$

$$(p \geq 0;\ |z| < 1;\ \alpha, \beta, \eta, \nu > 0,\ \Re(w) > \Re(y) > 0,\quad a, b > 0).$$

The confluent hypergeometric logarithmic function is defined as

$$\Phi L^{p,\eta,\nu}_{a,b;\alpha,\beta}(y; w; z) = \sum_{n=0}^{\infty} \frac{BL^{p,\eta,\nu}_{\alpha,\beta}(a, b; y+n, w-y)}{B(y, w-y)} \frac{z^n}{n!}, \tag{4.2}$$

$$(p \geq 0;\ a, b, \alpha, \beta, \eta, \nu > 0,\ \Re(w) > \Re(y) > 0).$$

Remark 4.1. If we choosen $a = b = 1$ in (4.1) and (4.2), then we achived the known result given by Khan et al. (see [8]).

4.1. Integral representation

Theorem 4.2. *The following integral representations for the hypergeometric and confluent hypergeometric logarithmic function holds:*

$$FL^{p,\eta,\nu}_{a,b,\alpha,\beta}(x, y; w; z) = \frac{1}{B(y, w-y)} \times \int_0^1 a^{1-t} b^t \ t^{y-1} (1-t)^{w-y-1} (1-zt)^{-x} E_{\alpha,\beta}\left(-\frac{p}{t^\eta (1-t)^\nu}\right) dt \tag{4.3}$$

$$(p \geq 0;\ a, b, \alpha, \beta, \eta, \nu \in \mathbb{R}^+;\ \textit{and}\ |\arg(1-z)| < \pi;\ \Re(w) > \Re(y) > 0),$$

and

$$\phi L^{p,\eta,\nu}_{a,b,\alpha,\beta}(y;w;z) = \frac{1}{B(y,w-y)} \times \int_0^1 a^{1-t}b^t\, t^{y-1}(1-t)^{w-y-1} e^{zt} E_{\alpha,\beta}\left(-\frac{p}{t^\eta(1-t)^\nu}\right) dt \tag{4.4}$$

$$(p \geq 0; \quad a,b,\alpha,\beta,\eta,\nu \in \mathbb{R}^+; \ \Re(w) > \Re(y) > 0).$$

Proof. By using the definition of beta logarithmic function (2.1) into (4.1) and by rearranging the order of integral and summation which is verified here, we have

$$FL^{p,\eta,\nu}_{a,b,\alpha,\beta}(x,y;w;z) = \frac{1}{B(y,w-y)} \times \int_0^1 a^{1-t}b^t\, t^{y-1}(1-t)^{w-y-1} E_{\alpha,\beta}\left(-\frac{p}{t^\eta(1-t)^\nu}\right) \sum_{n=0}^{\infty} (x)_n \frac{(zt)^n}{n!} dt. \tag{4.5}$$

Applying the binomial theorem in (4.5), we achived the desired result (4.3).
Similarly, we can obtain (4.4). □

4.2. Derivative formula

Theorem 4.3. *The following derivative formulae holds true:*

$$\frac{d^n}{dz^n}\left\{FL^{p,\eta,\nu}_{a,b,\alpha,\beta}(x,y;w;z)\right\} = \frac{(x)_n(y)_n}{(w)_n} FL^{p,\eta,\nu}_{\alpha,\beta}(x+n,y+n;w+n;\ z), \tag{4.6}$$

and

$$\frac{d^n}{dz^n}\left\{\Phi L^{p,\eta,\nu}_{\alpha,\beta}(y;w;z)\right\} = \frac{(y)_n}{(w)_n} \phi L^{p,\eta,\nu}_{\alpha,\beta}(y+n;w+n;\ z), \tag{4.7}$$

where

$$(p \geq 0, \ a,b,\alpha,\beta,\eta,\nu \in \mathbb{R}^+; \ \Re(w) > \Re(y) > 0; \ n \in \mathbb{N}_0).$$

Proof. We know well known relation of Euler beta functions,

$$B(y,w-y) = \frac{w}{y} B(y+1,w-y) \tag{4.8}$$

Differentiating (4.1) with respect to variable z, we get

$$\frac{d}{dz} FL^{p,\eta,\nu}_{a,b,\alpha,\beta}(x,y;w;z) = \sum_{n=1}^{\infty} (x)_n \frac{BL^{p,\eta,\nu}_{\alpha,\beta}(a,b;y+n,w-y)}{B(y,w-y)} \frac{z^{n-1}}{(n-1)!}$$
$$= \sum_{n=0}^{\infty} (x)_{n+1} \frac{BL^{p,\eta,\nu}_{\alpha,\beta}(a,b;y+n+1,w-y)}{B(y,w-y)} \frac{z^n}{n!},$$

using (1.4) and (4.8) in the above expression, we obtain

$$\frac{d}{dz} FL^{p,\eta,\nu}_{a,b,\alpha,\beta}(x,y;w;z) = \frac{xy}{w} \sum_{n=0}^{\infty} (x+1)_n \frac{BL^{p,\eta,\nu}_{\alpha,\beta}(a,b;y+n+1,w-y)}{B(y+1,w-y)} \frac{z^n}{n!}.$$

Continue the same process upto $(n-1)$, yield the required result (4.6).
Similarly, the same process apply on (4.2), we get the desired result (4.7). □

Remark 4.4. If we choose $a = b = 1$ in the expression (4.6) and (4.7), we get the known result introduced by Khan et al. (see[8]).

Theorem 4.5. *The Mellin transform of hypergeometric and confluent hypergeometric logarithmic functions are as follows:*

$$\begin{aligned}\mathsf{M}\left\{FL^{p,\eta,\nu}_{a,b,\alpha,\beta}(x,y;w;z);s\right\}=&\frac{\Gamma(s)\Gamma(1-s)B(y+\eta s,w-y+\nu s)}{\Gamma(\beta-s\alpha)B(y,w-y)}\\&\times FL_{a,b}(x,y+\eta s;w+(\eta+\nu)s;z),\end{aligned}\tag{4.9}$$

and

$$\begin{aligned}\mathsf{M}\left\{\phi L^{p,\eta,\nu}_{a,b,\alpha,\beta}(y;w;z);s\right\}=&\frac{\Gamma(s)\Gamma(1-s)B(y+\eta s,w-y+\nu s)}{\Gamma(\beta-s\alpha)B(y,w-y)}\\&\times\phi L_{a,b}(y+\eta s;w+(\eta+\nu)s;z).\end{aligned}\tag{4.10}$$

Proof. Using the definition of Mellin transform, we have

$$\mathsf{M}\left\{FL^{p,\eta,\nu}_{a,b,\alpha,\beta}(x,y;w;z);s\right\}=\int_0^\infty p^{s-1}FL^{p,\eta,\nu}_{a,b,\alpha,\beta}(x,y;w;z)\,dp$$

Now using the relation (4.3) in the right hand side, we get

$$\begin{aligned}\mathsf{M}\left\{FL^{p,\eta,\nu}_{a,b,\alpha,\beta}(x,y;w;z);s\right\}=&\frac{1}{B(y,w-y)}\int_0^1 a^{1-t}b^t t^{y-1}(1-t)^{w-y-1}(1-zt)^{-x}\\&\times\left(\int_0^\infty p^{s-1}E_{\alpha,\beta}\left(-\frac{p}{t^\eta(1-t)^\nu}\right)dp\right)dt.\end{aligned}$$

Let $u=\frac{p}{t^\eta(1-t)^\nu}$ in the above expression and simplify it, we get $\mathsf{M}\left\{FL^{p,\eta,\nu}_{a,b,\alpha,\beta}(x,y;w;z);s\right\}=$

$$\begin{aligned}&\frac{1}{B(y,w-y)}\int_0^1 a^{1-t}b^t t^{y+\eta s-1}(1-t)^{w-y+\nu s-1}(1-zt)^{-x}\left(\int_0^\infty u^{s-1}E_{\alpha,\beta}(-u)du\right)dt\\&=\frac{B(y+\eta s,w+\nu s)}{B(y,w-y)B(y+\eta s,w+\nu s)}\int_0^1 a^{1-t}b^t t^{y+\eta s-1}(1-t)^{w-y+\nu s-1}(1-zt)^{-x}\\&\times\left(\int_0^\infty u^{s-1}E_{\alpha,\beta}(-u)du\right)dt.\end{aligned}$$

Using the relation (2.14) (for $\gamma=1$) and the integral (4.3) (for $p=0,\alpha=\beta=\eta=\nu=1$) in the above integral, we get the required result (4.9).

Similarly, following the same procedure, we obtain the second result (4.10). □

5. Transformation formulas

Theorem 5.1. *The hypergeometric logarithmic and confluent hypergeometric logarithmic functions have the following formulas:*

$$FL^{p,\eta,\nu}_{a,b,\alpha,\beta}(x,y;w;\,z)=(1-z)^{-x}\,FL^{p,\eta,\nu}_{a,b,\alpha,\beta}\left(x,w-y;w;\,-\frac{z}{1-z}\right),\tag{5.1}$$

$$FL^{p,\eta,\nu}_{a,b,\alpha,\beta}\left(x,y;w;\,1-\frac{1}{z}\right)=z^x\,FL^{p,\eta,\nu}_{a,b,\alpha,\beta}(x,w-y;w;\,1-z),\tag{5.2}$$

$$FL^{p,\eta,\nu}_{a,b,\alpha,\beta}\left(x,y;w;\,\frac{z}{1+z}\right)=(1+z)^x\,FL^{p,\eta,\nu}_{a,b,\alpha,\beta}(x,w-y;w;\,-z),\tag{5.3}$$

$$\Phi L^{p,\eta,\nu}_{a,b,\alpha,\beta}(y,w;\,z)=e^z\,\Phi L^{p,\eta,\nu}_{a,b,\alpha,\beta}(w-y;w;\,-z),\tag{5.4}$$

$$(p\geq 0,a,b,\alpha,\beta,\eta,\nu\in\mathbb{R}^+;\;|z|<1;\,\Re(w)>\Re(y)>0).$$

Proof. Replacing t by $1-t$ in $(1-zt)^{-x}$ and substituting

$$[1-z(1-t)]^{-x}=(1-z)^{-x}\left(1+\frac{z}{1-z}t\right)^{-x}$$

in (4.3), we obtain

$$FL^{p,\eta,\nu}_{a,b,\alpha,\beta}(x,y;w;\,z)=\frac{(1-z)^{-x}}{B(y,w-y)}\times\int_0^1 t^{y-1}(1-t)^{w-y-1}\left(1+\frac{z}{1-z}t\right)^{-x}E_{\alpha,\beta}\left(-\frac{p}{t^{\eta}(1-t)^{\nu}}\right)dt, \tag{5.5}$$

further, we have

$$FL^{p,\eta,\nu}_{a,b,\alpha,\beta}(x,y;w;\,z)=\frac{(1-z)^{-x}}{B(y,w-y)}\times\int_0^1 t^{y-1}(1-t)^{w-y-1}\left(1-\frac{-z}{1-z}t\right)^{-x}E_{\alpha,\beta}\left(-\frac{p}{t^{\eta}(1-t)^{\nu}}\right)dt, \tag{5.6}$$

In view of (4.3), we get the desired result (5.1).
Replacing z by $1-\frac{1}{z}$ and $\frac{z}{1+z}$ in (5.1) yield (5.2) and (5.3) respectively. ☐

Similarly apply the same process in (5.1) by simple calculation, we can establish (5.4).

Theorem 5.2. *The following relation holds true:*

$$FL^{p,\eta,\nu}_{a,b,\alpha,\beta}(x,y;w;\,1)=\frac{BL^{p,\eta,\nu}_{\alpha,\beta}(a,b;x,w-x-y)}{B(y,w-y)} \tag{5.7}$$

$$(p\geq 0,a,b,\alpha,\beta,\eta,\nu\in\mathbb{R}^{+};\Re(w-x-y)>0).$$

Proof. Putting $z=1$ in (4.3) and using the definition (2.1), we obtain desired result (5.7). ☐

6. Generating function

Theorem 6.1. *The generating function for* $FL^{p,\eta,\nu}_{a,b,\alpha,\beta}(x,y;w;\,z)$ *holds the following relation:*

$$\sum_{k=0}^{\infty}(x)_k\,FL^{p,\eta,\nu}_{a,b,\alpha,\beta}(x+k,y;w;z)\,\frac{t^k}{k!}=(1-z)^{-x}\,FL^{p,\eta,\nu}_{a,b,\alpha,\beta}\left(x,y;w;\,\frac{z}{1-t}\right) \tag{6.1}$$

$$(a,b,\alpha,\beta,\eta,\nu\in\mathbb{R}^{+};p\geq 0,|t|<1).$$

Proof. Let the left hand side of (6.1) is $\mathfrak{L}$ then from (4.1), we have

$$\mathfrak{L}=\sum_{k=0}^{\infty}(x)_k\left(\sum_{n=0}^{\infty}\frac{(x+k)_n\,BL^{p,\eta,\nu}_{\alpha,\beta}(a,b;y+n,w-y)}{B(y,w-y)}\frac{z^n}{n!}\right)\frac{t^k}{k!}.$$

Using the well known identity $(a)_n(a+n)_k=(a)_k(a+k)_n$, we obtain

$$\mathfrak{L}=\sum_{n=0}^{\infty}(x)_n\,\frac{BL^{p,\eta,\nu}_{\alpha,\beta}(a,b;y+n,w-y)}{B(y,w-y)}\left(\sum_{k=0}^{\infty}(x+n)_k\frac{t^k}{k!}\right)\frac{z^n}{n!}.$$

Since we know that $\sum_{n=0}^{\infty}(x+n)_n\frac{t^n}{n!}=(1-t)^{-x-n}$, we have

$$\mathfrak{L}=\sum_{n=0}^{\infty}(x)_n\,\frac{BL^{p,\eta,\nu}_{\alpha,\beta}(a,b;y+n,w-y)}{B(y,w-y)}(1-t)^{-x-n}\frac{z^n}{n!}$$

$$\mathfrak{L}=(1-t)^{-x}\sum_{n=0}^{\infty}(x)_n\,\frac{BL^{p,\eta,\nu}_{\alpha,\beta}(a,b;y+n,w-y)}{B(y,w-y)}\left(\frac{z}{1-t}\right)^n\frac{1}{n!}. \tag{6.2}$$

Finally by using (4.1) in (6.2), we get the right side of (6.1). □

7. Conclusion

Inspired by various uses in different fields of sciences and engineering. This article studies the new parametrization of extended beta functions using logarithmic mean and discusses their statistical properties. Some algebraic properties of this new extended function are developed, and also discuss its probabilistic concept as an application. For future direction, they are using these beta functions to investigate Appell's functions of two and three variables, Reimann Liouville fractional derivative operator, and Whittaker functions and their properties.

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