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Computational Relationship of the Surface Area and Stiffness of the Spring Constant on Fractional Bagley-Torvik Equation

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Abstract: In this paper, we formulate an efficient algorithm based on a new iterative method for the numerical solution of the Bagley-Torvik equation. The fractional differential equation arises in many areas of applied mathematics including viscoelasticity problems and applied mechanics of the oscillation process. We construct the fractional derivatives via the Caputo-type fractional operator to formulate a three-step algorithm using the MAPLE 18 software package. We further investigate the relationships between the surface area and stiffness of the spring constants of the Bagley-Torvik equation on three case problems and numerical results are presented to demonstrate the efficiency of the proposed algorithm.

Key words: Fractional Bagley-Torvik equation, new iterative method, Caputo derivative, Riemann-Liouville fractional integral operator, MAPLE 18 software package

Kesirli Bagley-Torvik Denklemi Üzerindeki Yay Sabitinin Yüzey Alanı ve Sertliğinin Hesaplamalı İlişkisi

Öz: Bu makalede, Bagley-Torvik denkleminin sayısal çözümü için yeni iterativ yönteme dayalı etkili bir algoritma formüle ediyoruz. Kesirli diferansiyel denklemler, uygulamalı mekanik ve salınım sürecinin viskoelastisite problemleri dahil olmak üzere, uygulamalı matematiğin birçok alanında ortaya çıkar. MAPLE 18 yazılım paketini kullanarak üç adımlı bir algoritma formüle etmek için Caputo tipi kesirli operatör aracılığıyla kesirli türevler üretiyoruz. Üç durumlu bir problemde Bagley-Torvik denkleminin yay sabitlerinin yüzey alanı ve katılığı arasındaki ilişkileri araştırdık ve önerilen algoritmanın etkinliğini göstermek için sayısal sonuçlar sunuldu.

Anahtar kelimeler: Kesirli Bagley-Torvik denklemi, yeni iterativ metod, Caputo, Riemann-Liouville kesirli integral operatörü, Maple 18 yazılım paketi.

1. Introduction

Initial value problems (IVPs) and boundary value problems (BVPs) of fractional order occur in the description of many physical processes of stochastic transportation, investigation of liquid filtration in a strongly porous medium, diffusion wave, cellular systems, signal processing, control theory, and oil industries. The analytical and numerical solutions to these fractional differential equations have attracted a lot of attention from researchers over the last decade. The Bagley-Torvik equation was proposed by Bagley and Torvik to described it is one of the many leading mathematical models of viscoelasticity damped structures with fractional derivatives. Therefore, it is meaningful to investigate the nature of the Bagley-Torvik equation with time delay. In particular, the $\frac{1}{2}$ -order derivative or $\frac{3}{2}$ -order derivative describes the frequency-dependent damping materials quite satisfactorily and the Bagley-Torvik equation with $\frac{1}{2}$ -order derivative or $\frac{3}{2}$ -order derivative describes the motion of real physical systems, an immersed plate in a Newtonian fluid and a gas in a fluid respectively [1-7].

In this paper, we investigate the relationship between the surface area and stiffness of the spring constants appearing in the Bagley-Torvik equation which arises in the application of fractional calculus of the theory of viscoelasticity and it can also describe the motion of real physical systems, the modeling of the motion of a rigid plate immersed in a viscous fluid, and a gas in a fluid respectively [8, 9,10].

$$MD^2y(x) + SD^{\frac{3}{2}}y(x) + Ky(x) = g(x) \qquad 0 < x \leq 1 \qquad (1)$$

with initial conditions

$$\begin{cases} y(0) = \Omega \\ y_x(0) = \Phi \end{cases} \quad (2)$$

Where $y(x)$ represents the displacement of the plate of mass M, S is the surface area, K is the stiffness of the spring to which the plate is attached, Ω and Φ are constant parameters, $D^{\frac{3}{2}}$ is the Caputo fractional derivative of order $\frac{3}{2}$,and $g(x)$ represents the loading force.

Several numerical methods have been proposed for analytically and numerically solutions of this typical equation in area formulation of mathematical model of viscoelastic damped structures with fractional derivative such as [11] who presented a theoretical frame work on basis for the application of fractional calculus to viscoelasticity, [12] studied the fractional calculus in the transient analysis of viscoelastically damped structures, [14] obtained numerical solution of the Bagley-Torvik equation, [14] proposed and applied generalized Taylor collocation method for numerical solutions of the Bagley-Torvik equation, [15] employed discrete spline methods for solving two point fractional Bagley-Torvik equation, [16] applied fractional iteration method to obtain approximate analytical solutions to the Bagley-Torvik equation, [17] proposed and applied Chebyshev wavelet operational matrix of fractional derivative for the numerical solution of Bagley-Torvik equation, [18] employed Adomian decomposition method for analytical solution of the Bagley-Torvik equation, [19] formulated numerical scheme for solving two point fractional Bagley-Torvik equation using Chebyshev collocation method, [20] presented and applied Bessel collocation method for the numerical solution of the Bagley-Torvik equation, [21] applied shifted Chebyshev operational matrix for the numerical solution of the Bagley–Torvik equation and [22] investigated the existence and uniqueness as well as approximations of the solutions for the Bagley-Torvik equation.

We aim to investigate and obtain numerical solutions for the relationship between surface area S and stiffness of the spring constants K of equation (1) and coupled with given initial conditions. Although some of the methods stated in our references have done great work in solving the Bagley-Torvik equation, however, there is still a need for a robust, fast, and suitable algorithm using MAPLE 18 software codes command to overcome the complexity of Riemann–Liouville fractional integral and Liouville–Caputo fractional derivative for better simplicity and efficiency.

2. The fractional integration and differentiation

The definition of fractional derivatives are defined in many ways such as Riemann-Liouville, Grunwald-Letnikove and Caputo. In the present paper, we consider the Caputo fractional derivative and Riemann–Liouville fractional integral as follow [21]:

Definition 1

The Riemann–Liouville fractional integral operator J^{α}, $\alpha > 0$ is defined as follows:

$$J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(s) ds, \qquad \alpha > 0 \quad (3)$$

Definition 2

The Liouville–Caputo fractional derivative operator D^{α} is defined as follows:

$$\begin{cases} D^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-s)^{m-\alpha-1} f^m(s)\, ds \\ for \;\; m-1 < \alpha \le m, \qquad m \in \mathbb{N}, \qquad x > 0, f(x) \in C_{-1}^m \end{cases} \quad (4)$$

And the Liouville–Caputo derivative (4), we have

$$D^{\alpha} x^{\beta} = \begin{cases} 0 & for\ \beta \in \mathbb{N}_0\ and\ \beta < \lceil \alpha \rceil; \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha} & for\ \beta \in \mathbb{N}_0\ and\ \beta \ge \lfloor \alpha \rfloor; \\ & or\ \beta \notin \mathbb{N}_0\ and\ \beta > \lfloor \alpha \rfloor \end{cases} \quad (5)$$

Here, $\lceil\alpha\rceil$ and $\lfloor\alpha\rfloor$ are the ceilings and floor functions respectively, and $\mathbb{N}_0 = \{0,1,2 \ldots\}$.

3. Description of the new iterative method (NIM)

The new iterative method was proposed by Daftardar-Gejji and Jafari [23]. It is an efficient technique for solving linear and nonlinear functional differential equations which arise in a wide area of applications in nonlinear problems without linearization or small perturbation [24, 25]. We consider the basic idea of the new iterative method for the general functional equation of the form:

$$y(x) = f(x) \; + N(y(x)) \tag{6}$$

where N Na nonlinear operator from a Banach is space $B \rightarrow B$ and $f(x)$ is a known function. Looking for a solution of (6) having the series form:

$$y(x) = \sum_{i=0}^{\infty} y_i(x) \tag{7}$$

The nonlinear operator which is on the right-hand side of (6) can be decomposed as follow:

$$N\left(\sum_{i=0}^{\infty} y_i(x)\right) = N(y_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} y_j\right) - N\left(\sum_{j=0}^{i-1} y_j\right)\right\} \tag{8}$$

Substituting equations (7) and (8) into equation (6); becomes;

$$\sum_{i=0}^{\infty} y_i(x) = f(x) + N(y_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} y_j\right) - N\left(\sum_{j=0}^{i-1} y_j\right)\right\} \tag{9}$$

The recurrence relation is given by:

$$\begin{cases} y_0 = f \\ y_1 = N(y_0) \\ y_{m+1} = N(y_0 + y_1 + \cdots + y_m) - N(y_0 + y_1 + \cdots + y_{m-1}) \quad m = 1,2,3,\ldots \end{cases} \tag{10}$$

Then,

$$N(y_0 + y_1 + \cdots + y_m) = N(y_0 + y_1 + \cdots + y_{m-1}) \qquad m = 1,2,3 \ldots \tag{11}$$

and

$$\sum_{i=0}^{\infty} y_i = f + N\left(\sum_{i=0}^{\infty} y_i\right), \tag{12}$$

The $k-$ term approximate solution of (6) is given by;

$$y = y_0 + y_1 + \cdots + y_{k-1}. \tag{13}$$

3.1 Suitable Algorithm

In this section, we formulate three steps algorithm using MAPLE 18 software codes for solving fraction Bagley-Torvik equation (1) coupled with initial conditions given in equation (2). We utilize definition 2 and the new iterative method discussed in the previous section as follows:

restart:
Step 1:
$withplots$:
$Digits := \mathbb{R}^+$;
$N := \mathbb{R}^+$;
$S := \mathbb{R}^+$;
$K := \mathbb{R}^+$;
$g := g(x)$;
$y[0] := \Omega$;
$dy[0] := \Phi$;

$Y[0] := \left(y[0] + dy[0] * x + int(g, x\$2)\right);$
$m := \alpha + \frac{1}{2};$
Step 2:
for n from 0 to N do
$f := diff(Y[n], x\$2);$
$G := simplify\left(\frac{1}{\Gamma(m-\alpha)} * int((x-s)^{m-\alpha-1} * subx(s = x, f), s = 0 \ldots x), assume = nonnegative\right);$
$H := simplify(-int(S * G + K * Y[n], x\$2), assume = nonnegative);$
$Y[n+1] := expand(H);$
end do;
Step 3:
$NIA := sum(Y[k], k = 0..N+1);$
for x from 0 by 0.1to 1 do
$NIA[x] := evalf\left(eval(NIA)\right);$
end do;
$y[x][2Dplot] := plot([NIA], x = 0 \ldots 1, color\ [red], axes = boxed, title = fractional$ Bagley − Torvik equation $IVPs);$
$y[x][2Dplot] := logplot([NIA], x = 0 \ldots 1, color\ [red], axes = boxed, title = fractional$ Bagley − Torvik equation $IVPs);$

4. Numerical experiments

To illustrate the effectiveness of the proposed algorithm, three test cases are considered to examine the relationship between surface area S and the stiffness of the spring constants K of equation (1) as follows:

Case 1. The stiffness of the spring constant *K* is greater than surface area *S* [20]

$$D^2y(x) + \frac{2}{5}D^{\frac{3}{2}}y(x) - \frac{1}{2}y(x) = \frac{x^2}{4} - \frac{x}{4} - \frac{8}{5}\frac{\sqrt{x}}{\sqrt{\pi}} - 2, \tag{14}$$

with initial conditions:

$$\begin{cases} y(0) = 0, \\ y_x(0) = 1, \end{cases} \tag{15}$$

The exact solution is

$$y(x) = x(1-x), \tag{16}$$

restart:
Step 1:
$withplots$:
$N := 5;$
$S := \frac{2}{5};$
$K := \frac{1}{2};$
$g := \frac{x^2}{4} - \frac{x}{4} - \frac{8}{5}\frac{\sqrt{x}}{\sqrt{\pi}} - 2;$
$y[0] := 0;$
$dy[0] := 1;$
$Y[0] := \left(0 + 1 * x + int\left(\frac{x^2}{4} - \frac{x}{4} - \frac{8}{5}\frac{\sqrt{x}}{\sqrt{\pi}} - 2, x\$2\right)\right);$

```
m := 3/2 + 1/2;
Step 2:
for n from 0 to N do
f := diff(Y[n], x$2);
G := simplify(1/Γ(m − α) * int((x − s)^(m−α−1) * subx(s = x, f), s = 0 ... x), assume = nonnegative);
H := simplify(−int(2/5 * G − 1/2 * Y[n], x$2), assume = nonnegative);
Y[n + 1] := expand(H);
end do;
Step 3:
NIA := sum(Y[k], k = 0..N + 1);
for x from 0 by 0.1to 1 do
NIA[x] := evalf(eval(NIA));
end do;
y[x][2Dplot] := plot([NIA], x = 0 ... 1, color [red], axes = boxed, title
                = fractional Bagley − Torvik equation IVPs Case 1);
y[x][2Dplot] := logplot([NIA], x = 0 ... 1, color [red], axes = boxed, title
                = fractional Bagley − Torvik equation IVPs Case 1);
Output: See Table 1 and Figures 1, 2
```

Case 2. The surface area *S* is greater than the stiffness of the spring constant *K*

$$D^2y(x) + \frac{1}{2}D^{\frac{3}{2}}y(x) - \frac{2}{5}y(x) = \frac{x^2}{4} - \frac{x}{4} - \frac{8}{5}\frac{\sqrt{x}}{\sqrt{\pi}} - 2, \qquad (17)$$

with initial conditions

$$\begin{cases} y(0) = 0, \\ y_x(0) = 1, \end{cases} \qquad (18)$$

```
restart:
Step 1:
withplots:
N := 5;
S := 1/2;
K := 2/5;
g := x^2/4 − x/4 − (8/5)(√x/√π) − 2;
y[0] := 0;
dy[0] := 1;
Y[0] := (0 + 1 * x + int(x^2/4 − x/4 − (8/5)(√x/√π) − 2, x$2));
m := 3/2 + 1/2;
Step 2:
for n from 0 to N do
f := diff(Y[n], x$2);
G := simplify(1/Γ(m − α) * int((x − s)^(m−α−1) * subx(s = x, f), s = 0 ... x), assume
                = nonnegative);
```

```
H := simplify(-int(1/2 * G - 2/5 * Y[n], x$2), assume = nonnegative);
Y[n + 1] := expand(H);
end do;
Step 3:
NIA := sum(Y[k], k = 0..N + 1);
for x from 0 by 0.1to 1 do
NIA[x] := evalf(eval(NIA));
end do;
y[x][2Dplot] := plot([NIA], x = 0 ... 1, color [red], axes = boxed, title
        = fractional Bagley − Torvik equation IVPs Case 2);
y[x][2Dplot] := logplot([NIA], x = 0 ... 1, color [red], axes = boxed, title
        = fractional Bagley − Torvik equation IVPs Case 2);
Output: See Table 1 and Figures 1, 2.
```

Case 3. The surface area *S* is equal to the stiffness of the spring constant *K*

$$D^2 y(x) + \frac{1}{2} D^{\frac{3}{2}} y(x) - \frac{1}{2} y(x) = \frac{x^2}{4} - \frac{x}{4} - \frac{8}{5}\frac{\sqrt{x}}{\sqrt{\pi}} - 2, \tag{19}$$

with initial conditions

$$\begin{cases} y(0) = 0, \\ y_x(0) = 1, \end{cases} \tag{20}$$

```
restart:
Step 1:
withplots:
N := 5;
S := 1/2;
K := 1/2;
g := x^2/4 - x/4 - 8/5 * sqrt(x)/sqrt(π) - 2;
y[0] := 0;
dy[0] := 1;
Y[0] := (0 + 1 * x + int(x^2/4 - x/4 - 8/5 * sqrt(x)/sqrt(π) - 2, x$2));
m := 3/2 + 1/2;
Step 2:
for n from 0 to N do
f := diff(Y[n], x$2);
G := simplify(1/Γ(m − α) * int((x − s)^(m−α−1) * subx(s = x, f), s = 0 ... x), assume
        = nonnegative);
H := simplify(-int(1/2 * G - 1/2 * Y[n], x$2), assume = nonnegative);
Y[n + 1] := expand(H);
end do;
Step 3:
NIA := sum(Y[k], k = 0..N + 1);
for x from 0 by 0.1to 1 do
```

$NIA[x] := evalf(eval(NIA))$;
end do;
$y[x][2Dplot] := plot([NIA], x = 0 \ldots 1, color[red], axes = boxed, title = fractional$ Bagley − Torvik equation $IVPs$ ***Case* 3**);
$y[x][2Dplot] := logplot([NIA], x = 0 \ldots 1, color[red], axes = boxed, title = fractional$ Bagley − Torvik equation $IVPs$ ***Case* 3**);
Output: See Table 1 and Figures 1, 2.

Table 1. Numerical solutions for fractional Bagley-Torvik equation

x	**Case 1** $K > S$	**Case 2** $S > K$	**Case 3** $K = S$
0	0.00000000000000000000	0.00000000000000000000	0.0000000000000000000
0.1	0.09003714575609670000	0.09019689275910890000	0.0902115355713342000
0.2	0.16027451051370710000	0.16111925446865500000	0.1612269944499560000
0.3	0.21085875278105220000	0.21308057719348000000	0.2134168770485690000
0.4	0.24188680735470110000	0.24629374672292200000	0.2470320576228550000
0.5	0.25341043900550710000	0.26091502542611400000	0.2622500177590770000
0.6	0.24543906121089720000	0.25705962169323800000	0.2591917700789600000
0.7	0.21794167866797620000	0.23480976534771500000	0.2379308325787490000
0.8	0.17084825859778810000	0.19421956791560900000	0.1984986925198820000
0.9	0.10405067979332910000	0.13531816759601700000	0.1408883409487810000
1.0	0.01740334450319540000	0.05811182557896970000	0.0650565938996210000

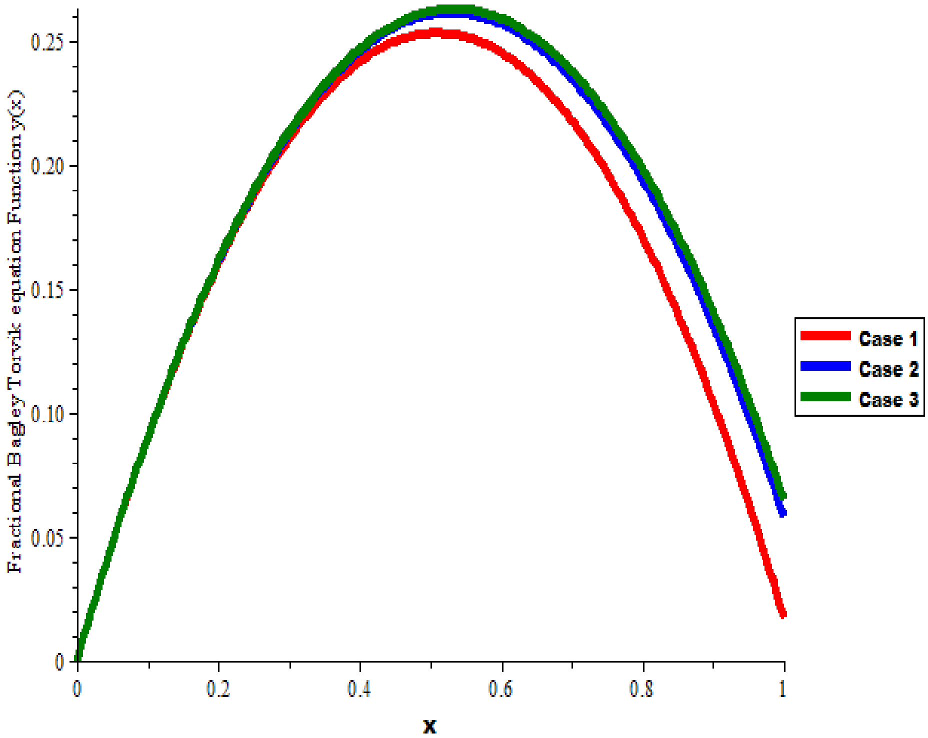

Figure 1. Depict the numerical solution $y(x)$ for the relationship between three cases of the stiffness of the spring constant K and surface area S on interval $0 \leq x \leq 1$ for $K > S, S > K, and\ K = S$

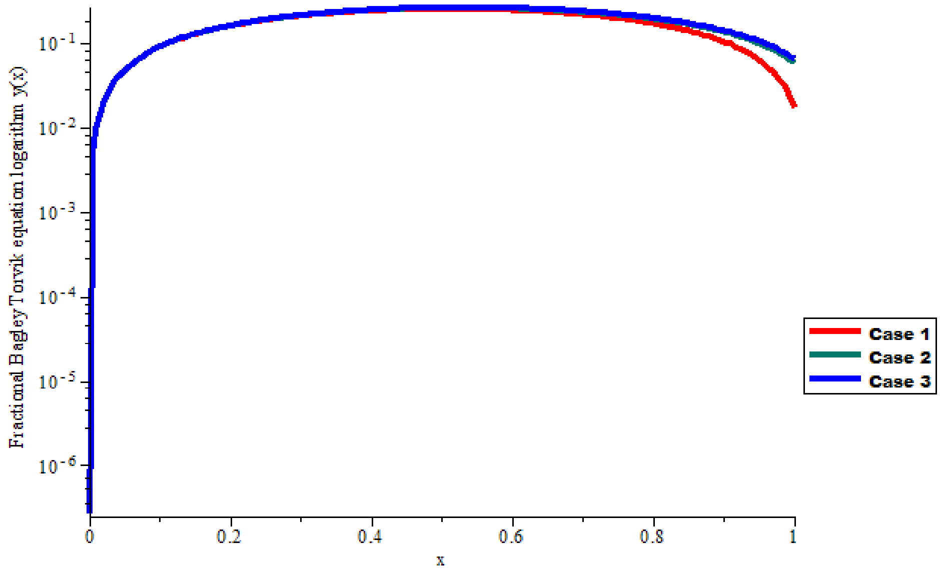

Figure 2. Depict the numerical solution $y(x)$ (log) for the relationship between three cases of stiffness of the spring constant K and surface area S on interval $0 \leq x \leq 1$ for $K > S, S > K, and\ K = S.$

5. Results and Discussion

The surface area S and stiffness of the spring constants K of fractional Bagley-Torvik equation are examined and from the numerical solutions obtained, we observed the following:

i. The proposed algorithm demonstrated a good alternative to the conventional approach to solving the fractional differential equations.

ii. The peak (maximum) results obtained for all cases within interval $0.5 \leq x \leq 0.55$.

iii. The highest numerical results are obtained when the surface area S is equal to the stiffness of the spring constants K (see case 3: Table 1. and Figure 1. Blue).

iv. The last results have obtained the stiffness of the spring constant K is greater than surface area S (see case 1: Table 1. and Figure 1. Red).

v. Intermediary solutions are obtained when surface area S is greater than the stiffness of the spring constant K (see case 2: Table 1. and Figure 1. Green).

6. Conclusion

An efficient new iterative algorithm was formulated and applied for the numerical solutions of fractional Bagley-Torvik equation with constant parameters of the surface area S and stiffness of the spring K. Numerical results show that the computational cost of the technique has decreased significantly and through the numerical experiments, we observed that the solutions obtained are considered as the sum of finite series $y(x) = \sum_{i=0}^{5} y_i$ converges to exact solutions which demonstrated that new iterative algorithm is a suitable and powerful tool for solving fractional differential equations arising in the calculus of the theory of viscoelasticity.

References

[1] Agrawal OP, Kumar P. Comparison of five schemes for fractional differential equations, J. Sabatier et al. (eds.), Advances in Fractional Calculus: Theoretical Developments and App in Phy and Eng 2007; 43-60.

[2] Chen W, Ye L, Sun H. Fractional diffusion equation by the Kansa method, Comp Math App 2010;59:1614-1620.

[3] Jiang CX, Carletta JE, Hartley TT. Implementation of fractional order operators on field programmable gate arrays, J. Sabatier et al. (eds.), Advances in Fractional Calculus:Theoretical Developments and App in Phy and Eng 2007; 333-346.

[4] Kilbas AA, Srivastava HM, Trujillo JJ. Theory of application of fractional differential equations, first ed., Belarus, 2006.

[5] Lakshmikantham V, Vatsala AS. Basic theory of fractional differential equations, Noon Anal. 2008;(69): 2677-2682.

[6] Miller KS, Ross B. An Introduction to the fractional calculus and differential equations, John Wiley, New York, 1993.

[7] Momani S, Noor MN. Numerical methods for fourth-order fractional integrodifferential equations, App Math and Comp 2006 (182);754-760.

[8] Torvik PJ, Bagley RL. On the appearance of the fractional derivative in the behavior of real materials. J Appl Mech. 1984; (51): 294-298.

[9] Azhar AZ, Grzegorz K, Jan A. An investigation of fractional Bagley–Torvik equation, MDPI Entropy 2020; (22): 1-13.

[10] Witkowski K, Kudra G, Wasilewski G, Awrejcewicz J. Modelling and experimental validation of 1-degree-of-freedom impacting oscillator. Journal of System Control Engineering 2019; (233): 418–430.

[11] Bagley RL, Torvik PJ. A theoretical basis for the application of fractional calculus to viscoelasticity, J Rheol 1983; (27): 201-210.

[12] Torvik PJ, Bagley RL. Fractional calculus in the transient analysis of viscoelastically damped structures. AIAA J 1985(23): 918-925.

[13] Diethelm K, Ford NJ, Numerical solution of the Bagley-Torvik equation, BIT Numerical Mathematics, 2002; (42): 490-507.

[14] Çenesiz Y, Keskin Y, Kurnaz A. The solution of the Bagley-Torvik equation with the generalized Taylor collocation method, J Frankl Inst 2010:(347), 452-466.

[15] Zahra WK. Van Daele M. Discrete spline methods for solving two-point fractional Bagley-Torvik equation, Appl Math Comput 2017; (296):42-56.

[16] Mekkaoui T, Hammouch Z. Approximate analytical solutions to the Bagley-Torvik equation by the fractional iteration method, Ann Univ Craiova Math Comput Sci Ser 2012; 39 (2): 251-256.

[17] Mohammadi F. Numerical solution of Bagley-Torvik equation using Chebyshev wavelet operational matrix of fractional derivative, Int J Adv in Appl Math and Mech 2014; 2(1): 83-91.

[18] Ray S. S., Bera R. K. Analytical solution of the Bagley-Torvik equation by Adomian decomposition method, Appl Math Comput 2005; 168 (1): 398-410.

[19] Vijay S, Sushil K. Numerical scheme for solving two-point fractional Bagley-Torvik equation using Chebyshev collocation method, WSEAS Transactions on Systems 2018; (17):166-177.

[20] Uzbas S. Y. Numerical solution of the Bagley-Torvik equation by the Bessel collocation method, Mathematical Methods in the Applied Sciences 2013; 36:300-312.

[21] Tianfu J, Jianhua H and Changqing Y. Numerical solution of the Bagley–Torvik equation using shifted Chebyshev operational matrix, Advances in Difference Equations 2020; 3-14.

[22] Hossein F, Juan J. N. An investigation of fractional Bagley-Torvik equation, DE GRUYTER Open Math. 2019, 17: 499–512.

[23] Daftardar-Gejji. V and Jafari H. An iterative method for solving nonlinear functional equations, Journal of Mathematical Analysis and Applications 2006; 316(2): 753–763.

[24] Falade KI, Tiamiyu AT. Numerical solution of partial differential equations with fractional variable coefficients using new iterative method (NIM), IJ Mathematical Sciences and Computing, 2020; 3: 12-21.

[25] Hemeda AA. New iterative method: an application for solving fractional physical differential equations, Hindawi Publishing Corporation Abstract and Applied Analysis 2013; 1-10.