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VULNERABILITY OF BANANA TREES VIA CLOSENESS AND RESIDUAL CLOSENESS PARAMETERS

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Abstract. One of the most important research topics about complex networks is examination of their vulnerability. Therefore, there are many studies in the literature about analyzing the robustness and reliability of networks using graph theoretical parameters. Among these parameters, the centrality parameters play an important role. The closeness parameters and its derivatives are widely discussed. In this study, the closeness parameter and the more sensitive parameter residual closeness which is based on closeness parameter have been considered. Furthermore, the closeness and residual closeness of banana tree structure have been calculated.

1. Introduction

With the developments in network science, computer science and its applications are developing rapidly. Many problems that can appear in real life can be modeled as a network and the system can be analyzed utilizing relationship between vertices and edges. Therefore, graph theory is an important scientific tool for determining vulnerability of a network. Analyzing a network in terms of vulnerability is one of the main purpose of the graph theory problems. Thus, utilizing graph theory parameters and techniques, a network can be investigated in terms of robustness and reliability from many researchers. One of the most important goal of network analysis is to research concept of centrality. There are several centrality parameters in the literature yet closeness centrality is one of the quite significant index that measures how capital position a node is in the network.

Closeness parameter have also changed to provide more sensitive approaches to network analyzing. First important closeness definition provided in [13]. Nevertheless, it can not suitable for disconnected graphs. The other closeness definition is given by Latora and Marchiori in [14]. The definition is formulated as $C(i) = \sum\limits_{j \neq i} \frac{1}{d(i,j)}$. Here $d(i,j)$ denotes distance between vertices i and j. This new

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definition can be applied to disconnected graphs. Afterwards, Danglachev introduced new closeness definition which is varied Latora and Marchiori's definition to provide calculation and formulation convention [9]. Dangalachev's new closeness definition is $\sum_{j \neq i} \frac{1}{2^{d(i,j)}}$. Derived from Dangalchev's closeness definition, many vulnerability parameters have been appeared to measure resilience of a network. Among of these new indexes, *residual vertex and edge closeness* parameters calculate closeness value of a graph after vertex or edge extracted from a graph [9]. The concept of residual closeness, a more sensitive parameter based on this definition of closeness, also emerged meanwhile from Dangalchev again [9]. The all-important point here is to find how a vertex removed from the graph influences the vulnerability of the graph. In order to evaluate closeness value after vertex k is removing such as $C_k =. \sum_i \sum_{j \neq i} \frac{1}{2^{d_k(i,j)}}$ where $d_k(i,j)$ is distance between vertices i and j after removing vertex k. Then the vertex residual closeness ,denoted by R, defined as $R = \min_k \{C_k\}$.Another measure is *additional closeness* which determine maximal potential of graph's closeness via adding an edge. It can be referred the readers to get detailed information about these new sensitive parameters [1–6, 10–12, 15–17].

In this paper, the graph G is taken as a simple, finite and undirected graph with vertex set $V(G)$ and edge set $E(G)$. The *open neighborhood* of any vertex in $V(G)$, denoted by $N(v) = \{u \in V(G) : (uv) \in E(G)\}$. *The degree of a vertex* v denoted by $deg(v)$, is cardinality of its neighborhood. *The distance* between two vertices u and v is shortest path between them, denoted by $d(u,v)$. A vertex of degree one is called *pendant vertex* and its incident edge is called support edge [7].

In this work, we investigate results about closeness and residual closeness of Banana Trees. Banana tree is a structure introduced by Chen et al. [8] as obtained by linking one leaf of each of n copies of an k vertices star graph structure with a single root vertex that is distinct from all the stars and the tree is denoted by $B_{n,k}$.

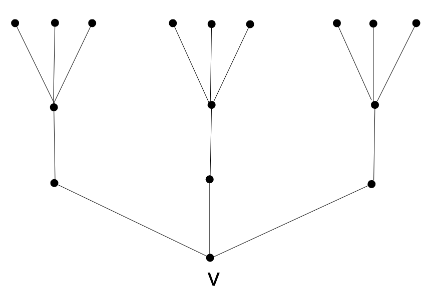

FIGURE 1. Banana Tree graph illustration with three copy of five vertices star graph, $B_{3,5}$

Calculating closeness value for huge graph structures is detailed process. In order to facilitate this process, it will be easier to use the method of splitting the graph into subgraphs in some structures, as in [11]. Utilizing this idea, we will denote root vertex as v, vertex 1_i be neighbour of vertex v in i^{th} copy of star graph, vertex 2_i is hub vertex of i^{th} copy of star graph and $3_i, ..., (3+j)_i$ are leaf vertices in banana graph where $1 \leq i \leq n$ and $1 \leq j \leq k-2$ in order to formulate closeness and residual closeness value for banana trees.

2. Closeness of Banana Trees

In this section we will get closeness value of Banana Tree structure. In order to ease of formulation, graph can be split into subforms and relationship between them. Next theorem gives us closeness value of Banana tree graph in terms of number of copy, denoted by n and number of vertices of star graph, denoted by k.

Theorem 2.1. *Let $B_{n,k}$ be banana tree with $nk+1$ vertices.The closeness value of $B_{n,k}$ is*

$$C(B_{n,k}) = \frac{16n(k^2+2k+2)+(n^2-n)(k+4)^2}{64}.$$

Proof. Due to form of Banana tree, graph can be splitted into three subforms such as $C(v)$ where v is root vertex, $C(S_k)$ and $C(S_k \sim S_k)$ where $C(v)$ is closeness value of vertex v, $C(S_k) = \frac{(k-1)(k+2)}{4}$ is closeness value of star graph with k vertices [9], $C(S_k \sim S_t)$ is closeness value of vertices in a copy of star graphs to other copies. Let 1_i be neighbour of root vertex v in i^{th} copy of star graph, 2_i is hub vertex of i^{th} copy of star graph and $3_i, ..., (3+j)_i$ are leaf vertices in banana graph where $1 \leq i \leq n$ and $1 \leq j \leq k-2$. Distace between v and 1_i is one, distance between v and 2_i is two and distance between v and all leaves notated by $3_i, ..., (3+j)_i$ is three for all $1 \leq i \leq n$ and $1 \leq j \leq k-2$. Therefore, closeness value of root vertex v is

$$C(v) = \sum_{\substack{i \in V(B_{n,k}) \\ i \neq v}} 2^{-d(i,v)} = (\frac{n}{2} + \frac{n}{2^2} + \frac{n(k-2)}{2^3})$$

and in order to calculate $C(S_k \sim S_k)$ value, without loss of generality, we can consider distance between vertices of 1^{st}*copy* of the S_k and 2^{st}*copy* of the S_k initially. Then, distance of vertex 1_1 to all vertices of 2^{st}*copy* of star graph is $A = \frac{1}{2^2} + \frac{1}{2^3} + \frac{(k-2)}{2^4}$ and distance of vertex 2_1 to all vertices of 2^{st}*copy* of star graph is $\frac{1}{2}A$ and distance of any leaf vertex to all vertices of 2^{st}*copy* of star graph is $\frac{1}{2^2}A$. Thus, we can formulate closeness value of vertices between first and second copy of star graphs as follows

$$\begin{aligned} C(S_k^{(1)} \sim S_k^{(2)}) &= A + \frac{1}{2}A + \frac{1}{2^2}A \\ &= A(1 + \frac{1}{2} + \frac{1}{2^2}) \\ &= \frac{1}{2^2}(1 + \frac{1}{2} + \frac{1}{2^2})^2. \end{aligned}$$

We will consider closeness value to every other vertices in both direction and there are n copies of star graph and also there are $n(n-1)$ relationship between star graph structures. Therefore, we obtain closeness value of banana tree

$$\begin{aligned} C(B_{n,k}) &= 2C(v) + nC(S_k) + n(n-1)(C(S_k \sim S_k)) \\ &= \frac{16n(k^2+2k+2) + (n^2-n)(k+4)^2}{64}. \end{aligned}$$

□

3. Residual Closeness of Banana Trees

In order to evaluate residual closeness value, a vertex will be removed from the graph and the minimum closeness value will be calculated after removing. Therefore, the most sensitive vertex will be determined in the graph. In the banana tree structure, we will obtain four distinct value after removing. These modification can be get from removing vertex v, leaf vertex of a banana graph, a center of an star graph and a leaf of star graph that is connected with vertex v. After determining the effect of these modifications on the graph in the next theorem, we will get residual closeness value of banana trees.

Theorem 3.1. *Let $B_{n,k}$ be banana tree with $nk+1$ vertices.The residual closeness value of $B_{n,k}$ is*

$$R = \frac{n(k-1)(k+2)}{4}.$$

Proof. First determine notation of vertices that will removed from the graph in order to evaluate residual closeness value. Let v be root vertex, 1_i be neighbour of vertex v in i^{th} copy of star graph, 2_i is hub vertex of i^{th} copy of star graph and $3_i, ..., (3+j)_i$ are leaf vertices in banana graph where $1 \leq i \leq n$ and $1 \leq j \leq k-2$. Therefore, we will get four different value after vertex removing.

- If root vertex v will be removed then
$$\begin{aligned} R_1 &= nC(S_k) \\ &= \frac{n(k-1)(k+2)}{4} \end{aligned} \tag{3.1}$$
- If an 1_i removed from the graph for any i where $1 \leq i \leq n$ then
$$R_2 = C(B_{n-1,k}) + C(S_{k-1}) \tag{3.2}$$
- If an 2_i removed from the graph for any i where $1 \leq i \leq n$ then
$$R_3 = C(B_{n-1,k}) + 2C(1_i \sim B_{n-1,k}). \tag{3.3}$$
Here the notation $1_i \sim B_{n-1,k}$ denote the closeness value of a vertex 1_i after modification
- Let any leaf be removed from the graph. Due to removing any leaf has same effect on residual value, we can choose one of them. Without loss of generality, let choose any 3_i as removed vertex, where $1 \leq i \leq n$.
$$R_4 = C(B_{n,k}) - 2C(3_i). \tag{3.4}$$
If we compare equations 3.1, 3.2, 3.3 and 3.4 , then it can be seen that the value comes from equation 3.1 is the minimum value. Since the value $C(B_{n-1,k})$ include at least n closeness of star graph value. Hence,
$$R = \frac{n(k-1)(k+2)}{4}.$$

□

4. Conclusion

In this article, we have calculated closeness value and residual closeness value of banana tree graphs. In order to evaluate closeness of a graph easily, the graph can be splitted into subgraphs if we know the closeness of the underlying graph. Utilizing this idea, we can split the banana tree into subgraphs and using closeness value of components and relation between them. In addition, we have considered residual closeness value of banana tree graph. In order to do this calculation, we acted from the thought of how much a change in the closeness value of a vertex removed from the graph would cause. We obtained four different values, and the minimum value we obtained among them was obtained by removing the root vertex from the graph.

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