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Research Article

Some Important Classes of the Continuous and Complex Interval-Valued Functions

Halise Levent [1]*, **Yılmaz Yılmaz** [1]
[1] İnönü University, Faculty of Arts and Sciences, Department of Mathematics, Malatya, Türkiye
yyilmaz44@gmail.com

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Abstract: This paper presents some important classes of the continuous functions defined from the set of real numbers to the space of complex intervals. These function spaces have an algebraic structure named as a quasilinear space which is suggested by Aseev in 1986. In this work, we analysis the quasilinear structure on the classes of the continuous and complex interval-valued functions. Further, we show that these spaces are the normed Ω-spaces. Finally, we examine the dimension of these function spaces.

Keywords: Normed quasilinear space, complex interval, continuous function, dimension of a quasilinear space.

1. Introduction

As is known the Fourier transform is the main building block of many application areas, especially in the electrical engineering. This transform that is used for analyzing the signals in the frekans domain has a wide range of applications in the digital signal processing.

Many real world problems may contain uncertainties due to environmental factors, especially in signal processing [7–9, 14]. Such problems are modelled with intervals. For this reason there has been increasing interest in interval-valued functions [1, 2, 4]. We need the space of the continuous functions defined from $\mathbb{R}$ to the set of complex intervals to analyzes the signals with inexact data.

An interval x is the compact-convex subset of real numbers and x is denoted by $x = [\underline{x}, \overline{x}]$ where $\underline{x}$ and $\overline{x}$ are the left and right endpoints of x, respectively [13]. Further, if $\underline{x} = \overline{x}$, then we say that x is a degenerate interval and it can be shown by $\{x\}$ or $[x, x]$. The set of all real intervals is denoted by $\mathbb{I}_{\mathbb{R}}$.

To get a comprehensive and healthy interval-valued signal processing we need the notion of the complex interval. Therefore, we defined the space $\mathbb{I}_{\mathbb{C}}$ which is the set of all complex intervals

*Correspondence: halisebanazili44@gmail.com
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in [11]. A complex interval is defined by

$$u=\left[\underline{u_r},\overline{u_r}\right]+i\left[\underline{u_s},\overline{u_s}\right],$$

where $\left[\underline{u_r},\overline{u_r}\right]$ and $\left[\underline{u_s},\overline{u_s}\right]$ are real intervals and $i=\sqrt{-1}$ is the complex unit. $\left[\underline{u_r},\overline{u_r}\right]$ and $\left[\underline{u_s},\overline{u_s}\right]$ are called real and imaginary part of u, respectively. Unfortunately, both $\mathbb{I}_{\mathbb{R}}$ and $\mathbb{I}_{\mathbb{C}}$ have an algebraic structure which is not linear space which is called as a "quasilinear space"by Aseev in 1986 (for details, see [3]). The most popular examples are $\Omega(E)$ and $\Omega_C(E)$ which are defined as the sets of all nonempty closed bounded and nonempty convex closed bounded subsets of any normed linear space E, respectively. Both are a quasilinear space with the inclusion relation "$\subseteq$", the algebraic sum operation

$$A+B=\overline{\{a+b: a\in A,\ b\in B\}},$$

where the closure is taken on the norm topology of E. The real-scalar multiplication

$$\lambda A=\{\lambda a: a\in A\}.$$

Especially, $\mathbb{I}_{\mathbb{R}}$ is a quasilinear space with the Minkowski sum and scalar multiplication operations are defined by

$$x+y=[\underline{x},\overline{x}]+[\underline{y},\overline{y}]=[\underline{x}+\underline{y},\overline{x}+\overline{y}]$$

and

$$\lambda x=\begin{cases}[\lambda\underline{x},\lambda\overline{x}] & ,\quad \lambda\geq 0\\ [\lambda\overline{x},\lambda\underline{x}] & ,\quad \lambda<0,\end{cases}$$

$x,y\in\mathbb{I}_{\mathbb{R}}$ and $\lambda\in\mathbb{R}$, respectively.

The Minkowski sum and scalar multiplication on $\mathbb{I}_{\mathbb{C}}$ are defined by

$$\begin{aligned}u+v&=\left[\underline{u_r},\overline{u_r}\right]+i\left[\underline{u_s},\overline{u_s}\right]+\left[\underline{v_r},\overline{v_r}\right]+i\left[\underline{v_s},\overline{v_s}\right]\\&=\left[\underline{u_r}+\underline{v_r},\overline{u_r}+\overline{v_r}\right]+i\left[\underline{u_s}+\underline{v_s},\overline{u_s}+\overline{v_s}\right]\\&=\left\{a+ib: a\in\left[\underline{u_r}+\underline{v_r},\overline{u_r}+\overline{v_r}\right],\ b\in\left[\underline{u_s}+\underline{v_s},\overline{u_s}+\overline{v_s}\right]\right\}\end{aligned}$$

and

$$\begin{aligned}\lambda u&=\lambda\left[\underline{u_r},\overline{u_r}\right]+i\left(\lambda\left[\underline{u_s},\overline{u_s}\right]\right)\\&=\left\{\lambda a+i\lambda b: a\in\left[\underline{u_r},\overline{u_r}\right],\ b\in\left[\underline{u_s},\overline{u_s}\right]\right\}\end{aligned}$$

on $\mathbb{I}_{\mathbb{C}}$, where $i=\sqrt{-1}$ and $\lambda\in\mathbb{C}$. Further, the relation

$$u\preceq v \text{ iff } \left[\underline{u_r},\overline{u_r}\right]\subseteq\left[\underline{v_r},\overline{v_r}\right] \text{ and } \left[\underline{u_s},\overline{u_s}\right]\subseteq\left[\underline{v_s},\overline{v_s}\right]$$

is a partial order relation on $\mathbb{I}_{\mathbb{C}}$. Thus, $\mathbb{I}_{\mathbb{C}}$ is a quasilinear space.

This article is organized as follows: In Section 2, we present some definitions and theorems with respect to the normed quasilinear spaces. In Section 3, we introduce some the classes of the continuous complex interval-valued functions defined from $\mathbb{R}$ to $\mathbb{I}_{\mathbb{C}}$. Further, we prove that these function spaces are the consolidate spaces and we investigate the dimensions of these spaces.

2. Preliminaries

We will start by giving some main definitions and notions.

Suppose that X is a quasilinear space and $Y \subseteq X$. Then Y is called a *subspace of* X whenever Y is a quasilinear space with the same partial order and the restriction to Y of the operations on X. Y is subspace of a quasilinear space X if and only if for every $x, y \in Y$ and $\alpha, \beta \in \mathbb{K}$, $\alpha x + \beta y \in Y$. Proof of this theorem is quite similar to its classical linear space analogue. Let Y be a subspace of a quasilinear space X and suppose each element x in Y has an inverse in Y. Then the partial order on Y is determined by the equality. In this case, Y is a linear subspace of X [16].

An element x in a quasilinear space X is said to be *symmetric* if $-x = x$ and X_{sym} denotes the set of all symmetric elements. Also, X_r stands for the set of all regular elements of X while X_s stands for the sets of all singular elements and zero in X. Further, it can be easily shown that X_r, X_{sym} and X_s are subspaces of X. They are called *regular, symmetric* and *singular subspaces* of X, respectively. Furthermore, it isn't hard to prove that summation of a regular element with a singular element is a singular element and the regular subspace of X is a linear space while the singular one is nonlinear at all. Further, $\mathbb{I}_{\mathbb{C}}$ is a closed subspace of $\Omega(\mathbb{C})$ [6].

A real-valued function $\|.\|$ on the quasilinear space X is called a *norm* if the following conditions hold;

$$\|x\| > 0 \text{ if } x \neq 0, \tag{1}$$

$$\|x + y\| \leq \|x\| + \|y\|, \tag{2}$$

$$\|\alpha x\| = |\alpha| \|x\|, \tag{3}$$

$$\text{if } x \preceq y, \text{ then } \|x\| \leq \|y\|, \tag{4}$$

$$\text{if for any } \varepsilon > 0 \text{ there exists an element } x_\varepsilon \in X \text{ such that} \tag{5}$$

$$x \preceq y + x_\varepsilon \text{ and } \|x_\varepsilon\| \leq \varepsilon, \text{ then } x \preceq y,$$

here x, y, x_ε are arbitrary element in X and α is any scalar. A quasilinear space X with a norm defined on it, is called *normed quasilinear space* [3].

For a normed linear space E, a norm on $\Omega(E)$ is defined by

$$\|A\|_{\Omega} = \sup_{a\in A} \|a\|_E \ .$$

Hence $\Omega_C(E)$ and $\Omega(E)$ are normed quasilinear spaces. A norm on $\mathbb{I}_{\mathbb{R}}$ is defined by

$$\|x\| = \|[\underline{x}, \overline{x}]\| = \sup_{t\in[\underline{x},\overline{x}]} |t| \, .$$

Moreover, $\mathbb{I}_{\mathbb{C}}$ is a normed quasilinear space with the norm

$$\begin{aligned} \|X\|_{\mathbb{I}_{\mathbb{C}}} &= \sup\{|z| : z \in X\} \\ &= \sup\{|a+ib| : a \in [\underline{x_r}, \overline{x_r}] \, , b \in [\underline{x_s}, \overline{x_s}]\} \end{aligned}$$

for $X = [\underline{x_r}, \overline{x_r}] + i\,[\underline{x_s}, \overline{x_s}]$ [15].

Now we will give the notion of consolidate quasilinear space defined in [15]. Thanks to this definition, we were able to give a representation to every element in a quasilinear space and we were able to define an inner-product quasilinear space.

Definition 2.1 *[15] Let X be a quasilinear space and $y \in X$. The floor of y is the set of all regular elements y of X such that $x \preceq y$. It is denoted by F_y^X and $F_y^X \subset X$. Hence $F_y^X = \{x \in X_r : x \preceq y\}$.*

For example, $[3,7]$ is an element of $(\mathbb{I}_{\mathbb{R}})_s$ and hence of $\mathbb{I}_{\mathbb{R}}$ since $(\mathbb{I}_{\mathbb{R}})_s \subset \mathbb{I}_{\mathbb{R}}$. The floor of $[3,7]$ in $(\mathbb{I}_{\mathbb{R}})_s$ is empty set, that is,

$$F_{[3,7]}^{(\mathbb{I}_{\mathbb{R}})_s} = \{x \in ((\mathbb{I}_{\mathbb{R}})_s)_r : x \subseteq [3,7]\} = \{x \in \{0\} : x \subseteq [3,7]\} = \varnothing$$

since $((\mathbb{I}_{\mathbb{R}})_s)_r = \{0\}$. But, the floor of $[3,7]$ in $\mathbb{I}_{\mathbb{R}}$ is

$$F_{[3,7]}^{\mathbb{I}_{\mathbb{R}}} = \{x \in (\mathbb{I}_{\mathbb{R}})_r : x \subseteq [3,7]\} \equiv [3,7]$$

since $(\mathbb{I}_{\mathbb{R}})_r \equiv \mathbb{R}$.

Definition 2.2 *[15] A quasilinear space X is called consolidate or Solid-Floored whenever*

$$\sup_{\preceq}\{x \in X_r : x \preceq y\} = \sup_{\preceq} F_y^X$$

exists and

$$y = \sup_{\preceq}\{x \in X_r : x \preceq y\}$$

for each $y \in X$. Otherwise, X is called a nonconsolidate QLS, or briefly, a nc-QLS.

From above example immediately we can see that $\mathbb{I}_{\mathbb{R}}$ is consolidate while $(\mathbb{I}_{\mathbb{R}})_s$ is not. Analogous results are also true for the spaces $\mathbb{I}_{\mathbb{C}}$ and $(\mathbb{I}_{\mathbb{C}})_s$.

3. Main Results

In this section, we present some important class of the continuous functions defined from $\mathbb{R}$ to $\mathbb{I}_{\mathbb{C}}$ and we show that these sets are the normed quasilinear spaces.

Definition 3.1 *The support of the set-valued function $F : \mathbb{R} \to \mathbb{I}_{\mathbb{C}}$ is the smallest closed set outside which the function is equal to zero:*

$$suppF = \overline{\{x \in \mathbb{R} : F(x) \neq \{0\}\}}.$$

If $suppF$ is a bounded set, then we say that F has compact support.

Definition 3.2 *(Classes of Continuous Set-Valued Functions) Consider a set-valued function $F : \mathbb{R} \to \mathbb{I}_{\mathbb{C}}$.*

(i) *The set $C_c(\mathbb{R}, \mathbb{I}_{\mathbb{C}})$ consists of all continuous set-valued functions having compact support:*

$$C_c(\mathbb{R}, \mathbb{I}_{\mathbb{C}}) = \{F : \mathbb{R} \to \mathbb{I}_{\mathbb{C}} \mid F \text{ is continuous and has compact support } \}.$$

(ii) *The set $C_0(\mathbb{R}, \mathbb{I}_{\mathbb{C}})$ consists of all continuous set-valued functions that $F(x) \to \{0\}$ with respect to Hausdorff metric on $\mathbb{I}_{\mathbb{C}}$ as $x \to \pm\infty$:*

$$C_0(\mathbb{R}, \mathbb{I}_{\mathbb{C}}) = \{F : \mathbb{R} \to \mathbb{I}_{\mathbb{C}} \mid F \text{ is continuous and } F(x) \to \{0\} \text{ as } x \to \pm\infty\}.$$

Example 3.3 *Consider the complex interval-valued functions $F, G : \mathbb{R} \to \mathbb{I}_{\mathbb{C}}$ given by*

$$F(t) = \begin{cases} \{i\} & , \quad \text{for } t \in [0,1]; \\ \{0\} & , \quad \text{otherwise} \end{cases}$$

and

$$G(t) = \begin{cases} [0,1] & , \quad \text{for } t \in [-1,1); \\ \{0\} & , \quad \text{otherwise,} \end{cases}$$

respectively. Since F and G are continuous and $suppF = [0,1]$, $suppG = [-1,1]$ we say that $F, G \in C_c(\mathbb{R}, \mathbb{I}_{\mathbb{C}})$. In fact, F is a regular element of $C_c(\mathbb{R}, \mathbb{I}_{\mathbb{C}})$ while G is a singular element of $C_c(\mathbb{R}, \mathbb{I}_{\mathbb{C}})$.

Theorem 3.4 *$C_0(\mathbb{R}, \mathbb{I}_{\mathbb{C}})$ is a quasilinear space with the operations of algebraic sum, multiplication by complex numbers and partial order relation are defined as follows;*

$$(F_1 + F_2)(x) = F_1(x) + F_2(x),$$

$$(\alpha F) = \alpha F(x)$$

and

$$F_1 \preccurlyeq F_2 \Leftrightarrow F_1(x) \subseteq F_2(x) \text{ for any } x \in \mathbb{R}.$$

Proof Verification of first five axioms to be a quasilinear space is to straighforward. Further, the function $F = \{0\}$ is the identity element of the addition. Obviously, $1.F = F$ and $0.F = 0$, for $1, 0 \in \mathbb{C}$ and $F \in C_0(\mathbb{R}, \mathbb{I}_\mathbb{C})$, easily see that $\alpha(\beta F) = (\alpha\beta)F$ and $\alpha(F + G) = \alpha F + \alpha G$ for $\alpha, \beta \in \mathbb{C}$ and $F, G \in C_0(\mathbb{R}, \mathbb{I}_\mathbb{C})$. For any $x \in \mathbb{R}$,

$$((\alpha + \beta)F)(x) = (\alpha + \beta)F(x) \subseteq \alpha F(x) + \beta F(x) = (\alpha F)(x) + (\beta F)(x)$$

and so $(\alpha + \beta)F \preccurlyeq \alpha F + \beta F$. If $F_1 \preccurlyeq F_2$ and $F_3 \preccurlyeq F_4$, then $F_1(x) \preceq F_2(x)$ and $F_3(x) \preceq F_4(x)$ for any $x \in \mathbb{R}$. Since $F_1(x), F_2(x), F_3(x), F_4(x) \in \mathbb{I}_\mathbb{C}$, we write $F_1(x) + F_3(x) \preceq F_2(x) + F_4(x)$. This means $F_1 + F_3 \preccurlyeq F_2 + F_4$. Suppose that $F_1 \preccurlyeq F_2$. Then $\alpha F_1(x) \preceq \alpha F_2(x)$ for any $x \in \mathbb{R}$, $\alpha \in \mathbb{C}$ since $\mathbb{I}_\mathbb{C}$ is a quasilinear space. Thus, we have $\alpha F_1 \preccurlyeq \alpha F_2$. □

Lemma 3.5 $C_c(\mathbb{R}, \mathbb{I}_\mathbb{C})$ *is a subspace of the quasilinear space* $C_0(\mathbb{R}, \mathbb{I}_\mathbb{C})$.

Proof It is not hard to see that $C_c(\mathbb{R}, \mathbb{I}_\mathbb{C}) \subset C_0(\mathbb{R}, \mathbb{I}_\mathbb{C})$. Suppose that $\lambda_1, \lambda_2 \in \mathbb{C}$ and $F, G \in C_c(\mathbb{R}, \mathbb{I}_\mathbb{C})$. Let us take an arbitrary $y \in A = \{x \in \mathbb{R} : \lambda_1 F(x) + \lambda_2 G(x) \neq 0\}$. Then we say that $\lambda_1 F(y) + \lambda_2 G(y) \neq 0$. In this case it is either $\lambda_1 F(y) \neq 0$ or $\lambda_2 G(y) \neq 0$. If $\lambda_1 F(y) \neq 0$, then $y \in B = \{x \in \mathbb{R} : F(x) \neq 0\}$. This means $A \subseteq B$. Thus,

$$\bar{A} = \text{supp}(\lambda_1 F + \lambda_2 G) \subseteq \bar{B} = \text{supp} F.$$

Further, there exists at least an interval $[a, b]$ such that supp$F \subseteq [a, b]$ since $F \in C_c(\mathbb{R}, \mathbb{I}_\mathbb{C})$. Consequently, we say that supp$(\lambda_1 F + \lambda_2 G) \subseteq [a, b]$ and so $\lambda_1 F + \lambda_2 G \in C_c(\mathbb{R}, \mathbb{I}_\mathbb{C})$. If $\lambda_2 G(y) \neq 0$, then the proof is similar. Now suppose that both $\lambda_1 F(y) \neq 0$ and $\lambda_2 G(y) \neq 0$ are satisfied. Then we have that

$$\{x \in \mathbb{R} : \lambda_1 F(x) + \lambda_2 G(x) \neq 0\} \subseteq \{x \in \mathbb{R} : F(x) \neq 0\} \cap \{x \in \mathbb{R} : G(x) \neq 0\}$$

since $y \in \{x \in \mathbb{R} : \lambda_1 F(x) + \lambda_2 G(x) \neq 0\}$. This implies $A \subseteq B$. Because of the fact that $\bar{A} \subset \bar{B}$ we write $\bar{A} =$supp$(\lambda_1 F + \lambda_2 G) \subseteq \bar{B} =$supp$F$. Thus, supp$(\lambda_1 F + \lambda_2 G)$ is bounded and $\lambda_1 F + \lambda_2 G \in C_c(\mathbb{R}, \mathbb{I}_\mathbb{C})$. □

Theorem 3.6 *The expression*

$$\|F\|_\infty = \max_{x \in \mathbb{R}} \|F(x)\|_{\mathbb{I}_\mathbb{C}}$$

defines a norm on $C_0(\mathbb{R}, \mathbb{I}_\mathbb{C})$ *and this space is a normed quasilinear space.*

Proof It is obvious that the above equality is well-defined. It can be shown similarly to the classical analysis that the first three conditions of norm are satisfied. Let us only verify the last two conditions. Let F_1 and F_2 be arbitrary elements of $C_0(\mathbb{R},\mathbb{I}_{\mathbb{C}})$. If $F_1 \preccurlyeq F_2$, then $F_1(x) \preceq F_2(x)$ for every $x \in \mathbb{R}$. This implies $\|F_1(x)\|_{\mathbb{I}_{\mathbb{C}}} \leq \|F_2(x)\|_{\mathbb{I}_{\mathbb{C}}}$ and so $\|F_1\|_\infty = \max_{x\in\mathbb{R}} \|F_1(x)\|_{\mathbb{I}_{\mathbb{C}}} \leq \max_{x\in\mathbb{R}} \|F_2(x)\|_{\mathbb{I}_{\mathbb{C}}} = \|F_2\|_\infty$. For the last condition of the norm, let $\varepsilon > 0$ be arbitrary and there exists an element $F_\varepsilon \in C_0(\mathbb{R},\mathbb{I}_{\mathbb{C}})$ such that $F \preccurlyeq G + F_\varepsilon$ and $\|F_\varepsilon\|_\infty = \max_{x\in\mathbb{R}} \|F_\varepsilon(x)\|_{\mathbb{I}_{\mathbb{C}}} \leq \varepsilon$. By the assumption, we write that $F(x) \preceq G(x) + F_\varepsilon(x)$ and $\|F_\varepsilon(x)\|_{\mathbb{I}_{\mathbb{C}}} \leq \varepsilon$. By the last condition of norm on $\mathbb{I}_{\mathbb{C}}$ we say that $F(x) \preceq G(x)$ for every $x \in \mathbb{R}$. Thus, we obtain that $F \preccurlyeq G$. □

Now we will show that $C_c(\mathbb{R},\mathbb{I}_{\mathbb{C}})$ and $C_0(\mathbb{R},\mathbb{I}_{\mathbb{C}})$ are consolidate spaces. Thus, we can give a representation to every element in these spaces.

Lemma 3.7 *$C_c(\mathbb{R},\mathbb{I}_{\mathbb{C}})$ and $C_0(\mathbb{R},\mathbb{I}_{\mathbb{C}})$ are the consolidate quasilinear spaces.*

Proof We will give only the proof for the space $C_c(\mathbb{R},\mathbb{I}_{\mathbb{C}})$ since a similar proof can be made for $C_0(\mathbb{R},\mathbb{I}_{\mathbb{C}})$. Let us take an arbitrary $g \in C_c(\mathbb{R},\mathbb{I}_{\mathbb{C}})$. Because of the fact that $\mathbb{I}_{\mathbb{C}}$ is consolidate, we write for $t \in \mathbb{R}$

$$\sup_{\preceq}\{x \in (\mathbb{I}_{\mathbb{C}})_r : x \preceq G(t)\} = \sup_{\preceq} F^{\mathbb{I}_{\mathbb{C}}}_{G(t)} = G(t) = [\underline{G_r(t)}, \overline{G_r(t)}] + i[\underline{G_s(t)}, \overline{G_s(t)}].$$

Now let us choose an element $\{x_{G_r(t)}\} + i\{x_{G_s(t)}\} \in \mathbb{I}_{\mathbb{C}}$ for each $t \in \mathbb{R}$ such that let be

$$\{x_{G_r(t)}\} + i\{x_{G_s(t)}\} \preceq [\underline{G_r(t)}, \overline{G_r(t)}] + i[\underline{G_s(t)}, \overline{G_s(t)}]. \tag{6}$$

Consider the function $h : \mathbb{R} \to (\mathbb{I}_{\mathbb{C}})_r$ given by

$$h(t) = \{x_{G_r(t)}\} + i\{x_{G_s(t)}\}, \tag{7}$$

where $\{x_{G_r(t)}\} + i\{x_{G_s(t)}\}$ is the regular element of $\mathbb{I}_{\mathbb{C}}$ that satisfies the condition (6). Now we will prove that $\sup F^{C_c(\mathbb{R},\mathbb{I}_{\mathbb{C}})}_G = G$, i.e.,

$$\sup_{\preceq}\{h \in C_c(\mathbb{R},\mathbb{I}_{\mathbb{C}})_r : h(t) \preceq G(t), \forall t \in \mathbb{R}\}.$$

First we have $h \preccurlyeq G$ since $h(t) = \{x_{G_r(t)}\} + i\{x_{G_s(t)}\} \preceq G(t)$. This means $F^{C_c(\mathbb{R},\mathbb{I}_{\mathbb{C}})}_G \neq \varnothing$. Further, the set $F^{C_c(\mathbb{R},\mathbb{I}_{\mathbb{C}})}_G$ is the upper bounded since $h \preccurlyeq G$ for $h \in F^{C_c(\mathbb{R},\mathbb{I}_{\mathbb{C}})}_G$. Suppose that the function F is another upper bound of the set $F^{C_c(\mathbb{R},\mathbb{I}_{\mathbb{C}})}_G$. Now let us assume that $G \not\preccurlyeq F$. Then there exists an element $t_0 \in \mathbb{R}$ such that $G(t_0) \not\preceq F(t_0)$. This implies that it is either

$[\underline{G_r(t_0)}, \overline{G_r(t_0)}] \nsubseteq [\underline{F_r(t_0)}, \overline{F_r(t_0)}]$ or $[\underline{G_s(t_0)}, \overline{G_s(t_0)}] \nsubseteq [\underline{F_s(t_0)}, \overline{F_s(t_0)}]$. If $[\underline{G_r(t_0)}, \overline{G_r(t_0)}] \nsubseteq [\underline{F_r(t_0)}, \overline{F_r(t_0)}]$, then there exists the singleton $\{x_{G_r(t_0)}\}$ such that $\{x_{G_r(t_0)}\} \subseteq [\underline{G_r(t_0)}, \overline{G_r(t_0)}]$ while $\{x_{G_r(t_0)}\} \nsubseteq [\underline{F_r(t_0)}, \overline{F_r(t_0)}]$. Further, we have that $\{x_{G_r(t_0)}\} + i\{x_{G_s(t_0)}\} \preceq G(t_0)$ and $\{x_{G_r(t_0)}\} + i\{x_{G_s(t_0)}\} \npreceq F(t_0)$. Thus, we write that $h(t_0) \preceq G(t_0)$ while $h(t_0) \npreceq F(t_0)$ for the function h defined in (7). Therefore, $h \npreceq F$. This is a contradiction. If $[\underline{G_s(t_0)}, \overline{G_s(t_0)}] \nsubseteq [\underline{F_s(t_0)}, \overline{F_s(t_0)}]$, then the proof is given in a similar way. Consequently, the proof is complete. □

Now we will examine the dimension of the quasilinear spaces $C_c(\mathbb{R}, \mathbb{I}_{\mathbb{C}})$ and $C_0(\mathbb{R}, \mathbb{I}_{\mathbb{C}})$. For this purpose, firstly let us give some algebraic definitions in a quasilinear space (for details, see [5]). Let X be a quasilinear space and $\{x_k\}_{k=1}^n$ be a subset of X, where n is a positive integer. A (linear) combination of the set $\{x_k\}_{k=1}^n$ is an element z of X in the form

$$\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n = z,$$

where the coefficients $\alpha_1, \alpha_2, ..., \alpha_n$ are real scalars. On the other hand, a quasilinear combination of the set $\{x_k\}_{k=1}^n$ is an element $z \in X$ such that

$$\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n \preceq z$$

for some real scalars $\alpha_1, \alpha_2, ..., \alpha_n$. Hence,the quasilinear combination, briefly ql-combination, is defined by the partial order relation on X. Further, for any nonempty subset A of a quasilinear space X, span of A is given by following known definition

$$SpA = \{\sum_{k=1}^{n} \alpha_k x_k : x_1, x_2, ..., x_n \in A,\ \alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{R},\ n \in \mathbb{N}\}.$$

However, $QspA$, the *quasispan (q-span,* for short) *of* A, is defined by the set of all possible quasilinear combinations of A, that is,

$$QspA = \{x \in X : \sum_{k=1}^{n} \alpha_k x_k \preceq x,\ x_1, x_2, ..., x_n \in A,\ \alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{R},\ n \in \mathbb{N}\}.$$

A given set $A = \{x_1, x_2, ..., x_n\}$ in a quasilinear space X is called *quasilinear independent (ql-independent,* briefly) whenever the inequality

$$\theta \preceq \lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_n x_n \tag{8}$$

holds if and only if $\lambda_1 = \lambda_2 = ... = \lambda_n = 0$. Otherwise, A is called *quasilinear dependent (ql-dependent,* briefly*).* A ql-independent subset A of a quasilinear space X which q-spans X is called *a basis (or Hamel basis)* for X.

Let S be a ql-independent subset of a quasilinear space X. S is called *maximal ql-independent subset* of X whenever S is ql-independent, but any superset of S is ql-dependent.

Definition 3.8 *[5] Regular (Singular) dimension of any quasilinear space X is the cardinality of any maximal ql-independent subsets of $X_r(X_s)$. If this number is finite then X is said to be finite regular (singular)-dimensional, otherwise; is said to be infinite regular (singular)-dimensional. Regular dimension is denoted by $r\text{-}\dim X$ and singular dimension is denoted by $s\text{-}\dim X$. If $r\text{-}\dim X = a$ and $s\text{-}\dim X = b$, then we say that X is an (a_r, b_s)-dimensional quasilinear space.*

Using these information we can give the following theorem.

Theorem 3.9 *The quasilinear spaces $C_c(\mathbb{R}, \mathbb{I}_{\mathbb{C}})$ and $C_0(\mathbb{R}, \mathbb{I}_{\mathbb{C}})$ are the (∞_r, ∞_s)-dimensional spaces.*

Proof Consider the functions $x_n : \mathbb{R} \to \mathbb{I}_{\mathbb{C}}$ given by

$$x_n(t) = \begin{cases} \{t^n\} & , \quad \text{for } t \in [-1,1]; \\ \{0\} & , \quad \text{otherwise} \end{cases}$$

for $n = 0, 1, ...$ and the set $M = \{x_0, x_1, ...\}$. It is obvious that M is a subset of the regular subspace of $C_c(\mathbb{R}, \mathbb{I}_{\mathbb{C}})$. Now we will prove that M is ql-independent. Let us take an arbitrary and finite subset $\{x_{k_1}, x_{k_2}, ..., x_{k_n}\}$ of M. Suppose that

$$c_{k_1} x_{k_1} + c_{k_2} x_{k_2} + ... + c_{k_n} x_{k_n} = 0$$

for $c_{k_1}, c_{k_2}, ..., c_{k_n} \in \mathbb{C}$. Then we write

$$c_{k_1}\{t^{k_1}\} + c_{k_2}\{t^{k_2}\} + ... + c_{k_n}\{t^{k_n}\} = \{0\}$$

and so $c_{k_1} t^{k_1} + c_{k_2} t^{k_2} + ... + c_{k_n} t^{k_n} = 0$. This implies that $c_{k_1} = c_{k_2} = ... = c_{k_n} = 0$. Thus, we say that $r\text{-}\dim C_c(\mathbb{R}, \mathbb{I}_{\mathbb{C}}) = \infty$. Further, $s\text{-}\dim C_c(\mathbb{R}, \mathbb{I}_{\mathbb{C}}) = \infty$ since $C_c(\mathbb{R}, \mathbb{I}_{\mathbb{C}})$ is a consolidate quasilinear space. Furthermore, we can say that $r\text{-}\dim C_0(\mathbb{R}, \mathbb{I}_{\mathbb{C}}) = s - \dim C_0(\mathbb{R}, \mathbb{I}_{\mathbb{C}}) = 0$ since $C_0(\mathbb{R}, \mathbb{I}_{\mathbb{C}})$ is a subspace of $C_c(\mathbb{R}, \mathbb{I}_{\mathbb{C}})$. □

Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Authors Contributions

Author [Halise Levent]: Thought and designed the research/problem, contributed to research method or evaluation of data, wrote the manuscript (%70).

Author [Yılmaz Yılmaz]: Collected the data, contributed to research method or evaluation of data (%30).

Conflicts of Interest

The authors declare no conflict of interest.

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