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Pseudo Simplicial Algebras, Crossed Modules and 2-Crossed Modules

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Abstract

In this paper we define pseudo 2-crossed module of commutative algebras and we give relations between the pseudo 2-crossed modules of commutative algebras and pseudo simplicial algebras with Moore complex of length 2.

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1. Introduction

Simplicial commutative algebras play an important role in homological algebras, homotopy theory, algebraic K-theory. In each theory, the internal structure has been studied relatively little.

Crossed modules were introduced by Whitehead in[16] with a view to capturing the relationship between π_1 and π_2 of a space. Homotopy systems (Which would now be called free crossed complexes [7] or totally free crossed chain complexes [5],[6]) were introduced, again by Whitehead, to incorporate the action of π_1 on the higher relative homotopy groups of a CW - complex. They consist of a crossed module at the base and a chain complex of modules over π_1 further up.

In [10] Inasaridze, H.N constructed homotopy groups of pseudosimplicial groups and in [11] using the homotopy groups of pseudosimplicial groups, he construct nonabelian derived functors with values in the category of groups.

In [1] it is examined the low dimensional parts of the Moore complex of a pseudosimplicial group. It is proved that the category of crossed modules is equivalent to the category of pseudosimplicial groups with Moore complex of length 1. Also it is defined pseudo 2 - crossed module and proved that the category of pseudo 2 - crossed modules is equivalent to the category of pseudosimplicial groups with Moore complex of length 2.

In this paper we define the pseudosimplicial algebras and pseudo 2-crossed modules of algebras, also we show that given a pseudosimplicial algebra with Moore complex of length 2 it can be construct a pseudo 2 - crossed module.

The above theorems in some sense, are well known. We give details of the proofs as analogous proofs can be found in the literature [8], [12], [2].

2. Pseudo Simplicial Algebras

Let k be a commutative ring with identity. We will use the term *commutative algebra* to mean a commutative algebra over k. The category of commutative algebras will be denoted by CA. All algebras will be commutative and will be over the same fixed but unspecified ground ring. A *pseudo simplicial (commutative) algebra* $\mathbf{E}$ consists of $\{E_n\}$ together with boundary homomorphisms $\partial_i^n : E_n \to E_{n-1}$, $0 \leq i \leq n$, $(n \neq 0)$ and pseudo degeneracies $s_i^n : E_n \to E_{n+1}$, $0 \leq i \leq n$, satisfying the following pseudosimplicial identities:

$$\begin{aligned}
\partial_i^{n-1}\partial_j^n &= \partial_{j-1}^{n-1}\partial_i^n && \text{for } i<j \\
\partial_i^{n+1}s_j^n &= s_{j-1}^{n-1}\partial_i^n && \text{for } i<j \\
\partial_j^{n+1}s_j^n &= 1=\partial_{j+1}^{n+1}s_j^n && \\
\partial_i^{n+1}s_j^n &= s_j^{n-1}\partial_{i-1}^n && \text{for } i>j+1
\end{aligned}$$

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To obtain the definition of *simplicial algebra*, we must add the condition that $s_i^{n+1}s_j^n = s_{j+1}^{n+1}s_i^n$ for $i \leq j$ (see [13]).
A topological interpretation is, for example, the F-construction of Milnor [14], which gives the simplicial group of loops of the suspension of a complex. For an arbitrary simplicial set **K** with pole ψ, the group of n-simplicies FK_n is the free group on a family of generators σ in one-to-one correspondence with the n-simplicies $\sigma \in K_n$, with the single relation $(s_{n-1}s_{n-2}\dots s_0(\psi)) = e_n$, while the boundary and degeneracy homomorphisms are induced by the corresponding mappings of the set **K**.
For any pseudosimplicial algebra E, put $NE_n = E_n \cap Ker\ \partial_0^n \cap \dots \cap Ker\ \partial_{n-1}^n$, $n \geq 0$, and let d_n be the restriction of ∂_n^n to NE_n, $n > 0$. Then $im\ d_n$ is an ideal of E_{n-1}, and $im\ d_{n+1} \subset Ker\ d_n$ for $n > 0$. This determines the Moore complex $\mathbf{NE} = \{NE_n, d_n\}$. Clearly **NE** is independent of the pseudodegeneracies, depending only on the boundary homomorphisms.
The n-dimensional homology of the Moore complex **NE** is called the *n-dimensional homotopy module* $\pi_n(\mathbf{E})$ of the pseudosimplicial algebra $\mathbf{E}$, $n \geq 0$.
A mapping $\mathbf{f} : \mathbf{E} \longrightarrow \mathbf{E}'$ induces, in a natural fashion, homomorphisms $\pi_n(\mathbf{f}) : \pi_n(\mathbf{E}) \longrightarrow \pi_n(\mathbf{E}')$, $n \geq 0$.
Let **f** and **g** be two mappings from **E** to $\mathbf{E}'$. The following definition is due to Inassaridze [11]. **f** is *pseudohomotopic* to **g** if there exist homomorphisms $h_i^n : G_n \longrightarrow G'_{n+1}$, $0 \leq i \leq n$, such that

$$
\begin{array}{rcllc}
\partial_0^{n+1}h_0^n & = & f_n & & \partial_{n+1}^{n+1}h_n^n = g_n, \\
\partial_i^{n+1}h_j^n & = & h_{j-1}^{n-1}\partial_i^n & \text{for} & i < j, \\
\partial_{j+1}^{n+1}h_{j+1}^n & = & \partial_{j+1}^{n+1}h_j^n, & & \\
\partial_i^{n+1}h_j^n & = & h_j^{n-1}\partial_{i-1}^n & \text{for} & i > j+1.
\end{array}
$$

to obtain the definition of homotopy of **f** to **g**, we must add the following conditions:

$s_i^{n+1}h_j^n = h_{j+1}^{n+1}s_i^n$ for $i \leq j$, and $s_i^{n+1}h_j^n = h_j^{n+1}s_{i-1}^n$ for $i > j$.

A mapping $\mathbf{f} : \mathbf{E} \longrightarrow \mathbf{E}'$ of pseudosimplicial algebras is called *simplicial* if it satisfies the condition $f_{n+1}s_i^n = s_i^n f_n$ for $n \geq 0$, $0 \leq i \leq n$. A simplicial map $\mathbf{f} : \mathbf{E} \longrightarrow \mathbf{E}'$ is called a *weak equivalence* if it induces isomorphisms $\pi_n(\mathbf{E}) \cong \pi_n(\mathbf{E}')$ for $n \geq 0$. A simplicial map $\mathbf{f} : \mathbf{E} \longrightarrow \mathbf{E}'$ called a *fibration* if $f_n : E_n \longrightarrow E'_n$ is surjective for $n \geq 0$.
By a k-truncated pseudosimplicial algebra we mean a collection of algebras $\{E_0, \dots, E_k\}$ and boundary homomorphisms $\partial_i^n : E_n \longrightarrow E_{n-1}$ for $0 \leq i \leq n$, $0 \leq n \leq k$ and pseudodegeneracies $s_i^n : E_n \longrightarrow E_{n+1}$ for $0 \leq i \leq n$, $0 \leq n \leq k$ which satisfy the pseudosimplicial identities. Clearly by forgetting higher dimensions, any pseudosimplicial algebra **E** yields a k-truncated pseudosimplicial algebra $tr^k\mathbf{E}$. The functor tr^k admits a right adjoint $\cos k^k$, called the *k-coskeleton functor*, and a left adjoint functor sk^k called the *k-skeleton functor.* We recall from [9] a brief description of these functors.
Suppose $tr^k(\mathbf{E}) = \{E_0, \dots, E_k\}$ is a pseudosimplicial algebra. A family of homomorphisms

$$
(\delta_0, \dots, \delta_{k+1}) : X_{k+1} \quad \begin{array}{c} \xrightarrow{\delta_{k+1}} \\ \vdots \\ \xrightarrow[\delta_0]{} \end{array} \quad E_k
$$

is the *simplicial kernel* of the family of boundary homomorphisms $(\partial_0, \dots, \partial_k)$ if it has the following universal property: given any family $(\partial_0, \dots, \partial_{k+1})$ of $k+2$ homomorphisms $\partial_i : Y \longrightarrow E_k$ satisfying the identities $\partial_i\partial_j = \partial_{j-1}\partial_i$ $(0 \leq i < j \leq k+1)$ with the last part of the truncated pseudosimplicial algebra, there exists a unique homomorphism $f : Y \longrightarrow X_{k+1}$ such that $\delta_i f = \partial_i$. Given the simplicial kernel

X_{k+1} the family of homomorphisms $(\alpha_{n+1,j}, \dots, \alpha_{1j}\alpha_{0j})$ defined by

$$
\alpha_{ij} = \begin{cases} s_{j-1} & i < j \\ id & i = j, i = j+1 \\ s_j d_{i-1} & i > j+1 \end{cases}
$$

satisfies the pseudosimplicial identities with the last part of the truncated pseudosimplicial algebra; hence there exists a unique $s_j : E_k \longrightarrow X_{k+1}$ such that $\delta_i s_j = \alpha_{ij}$. We thus have a $(k+1)$-truncated pseudosimplicial algebra $\{E_0, \dots, E_k, X_{k+1}\}$. By iterating this construction we get a pseudosimplicial algebra $\cos k^k(tr^k(\mathbf{E})) = \{E_0, \dots, E_k, X_{k+1}\}$ called the *coskeleton* of the truncated pseudosimplicial algebra. If $\mathbf{E}$, $\mathbf{E}'$ are any pseudosimplicial algebras, than any truncated simplicial map $\mathbf{f} : tr^k\mathbf{E} \longrightarrow tr^k\mathbf{E}'$ extends uniquely to a simplicial map $\mathbf{f} : \mathbf{E} \longrightarrow \cos k^k(tr^k(\mathbf{E}'))$.
The k-skeleton functor can be constructed by a dual process involving pseudosimplicial cokernels

$$
(s_0, \dots, s_k) : E_k \quad \begin{array}{c} \xrightarrow{s_k} \\ \vdots \\ \xrightarrow[s_0]{} \end{array} \quad X_{k+1}.
$$

(That is, universal systems of $k+1$ arrows which satisfy pseudosimplicial identities.)

3. Crossed Modules

J.H.C. Whitehead (1949) [16] described crossed modules in various contexts especially in his investigation into the group structure of relative homotopy groups. We recalled the definitions of crossed modules of commutative algebras given by T. Porter [15].

Definition 3.1. *Let R be a k-algebra with identity. A pre-crossed module of commutative algebras is an R-algebra C together with a commutative action of R on C and a morphism*

$\partial : C \longrightarrow R$

such that for all $c \in C$, $r \in R$

CM1) $\partial(r \cdot c) = r\partial c$.

This is a crossed R-module if in addition for all $c, c' \in C$

CM2) $\partial c \cdot c' = cc'$.

The last condition is called the Peiffer identity. We denote such a crossed module by (C, R, ∂).

3.1. Examples of Crossed Modules

1. Any ideal I in R gives an inclusion map, $inc : I \longrightarrow R$ which is a crossed module. Conversely given an arbitrary R-module $\partial : C \longrightarrow R$ one easily sees that the Peiffer identity implies that ∂C is an ideal in R.
2. Any R-module M can be considered as an R-algebra with zero multiplication and hence the zero morphism $0 : M \to R$ sending everything in M to the zero element of R is a crossed module. Conversely: If (C, R, ∂) is a crossed module, $\partial(C)$ acts trivially on $\ker\partial$, hence $\ker\partial$ has a natural $R/\partial(C)$-module structure.
As these two examples suggest, general crossed modules lie between the two extremes of ideal and modules. Both aspects are important.
3. Let be $M(R)$ multiplication algebra as given example. Then $(R, M(R), \mu)$ is multiplication crossed module. $\mu : R \to M(R)$ is defined by $\mu(r) = \delta_r$ with $\delta_r(r') = rr'$ for all $r, r' \in R$, where δ is multiplier $\delta : R \to R$ such that for all $r, r' \in R$, $\delta(rr') = \delta(r) r'$. Also $M(R)$ acts on R by $\delta \cdot r = \delta(r)$.(See [3] for details).

Definition 3.2. *A morphism of crossed modules from (C, R, ∂) to (C', R', ∂') is a pair of k-algebra morphisms $\phi : C \longrightarrow C'$, $\psi : R \longrightarrow R'$ such that*

$$\partial' \circ \phi = \psi\partial \qquad \text{and} \qquad \phi(r \cdot c) = \psi(r) \cdot \phi(c).$$

Thus we get a category $\mathbf{XMod}_k$ *of crossed modules (for fixed k).*

The following lemma is a straightforward modification of Theorem 1.3 in [8].

Lemma 3.3. *Let $\mathbf{E}$ be a pseudosimplicial algebra. The Moore complex of its k -coskeleton $\cos k^k(tr^k\mathbf{E})$ is of length $k+1$, and is identical to the Moore complex of $\mathbf{E}$ in dimensions $\le k$. Moreover, in dimensions $k-1$ to $k+2$ the Moore complex of $\cos k^k(tr^k\mathbf{E})$ is an exact sequence*

$$0 \longrightarrow N(\cos k^k(tr^k\mathbf{E}))_{k+1} \xrightarrow{\partial_{k+1}} NE_k \xrightarrow{\partial_k} NE_{k-1}.$$

where N_k is the kth term of the Moore complex of $\mathbf{E}$.

Proof. The $(k+1)-$dimensional part of $\cos k^k\left(tr^k\mathbf{E}\right)$ can be identified with the subalgebra of the $(k+2)-$ fold direct sum E_k^{k+2} consisting of those elements $(x_0, ..., x_{k+1})$ such that $d_j x_k = d_{k-1} x_j$ for $j < k$; the face maps are given by $d_j(x_0, ..., x_{k+1}) = x_j$ Thus $N(\cos k^k(tr^k\mathbf{E}))_{k+1}$consists of elements $(0, ..., 0, x_{k+1})$ such that $d_j x_{k+1} = 0$ for all j. In other words $N(\cos k^k(tr^k\mathbf{E}))_{k+1}$ is the kernel of $\partial_k : NE_k \longrightarrow NE_{k-1}$, and hence we have the exact sequence of the lemma.
The injectivity of ∂_{k+1} and the isomorphism

$$\cos k^{n-1}\left(tr^{n-1}\left(\cos k^k\left(tr^k E\right)\right)\right) \simeq \cos k^k\left(tr^k E\right)$$

for $n \ge k+2$ shows that the Moore complex of $\cos k^k\left(tr^k E\right)$ is of length $k+1$. □

The following theorem is well known. In [4] and [12] this theorem was proved.

Theorem 3.4. *([4], [12]) The category of crossed modules is equivalent to the category of simplicial algebras with Moore complex of length 1.*

Now, we shall give the pseudo version of this theorem.

Theorem 3.5. *The category of crossed modules of algebras is equivalent to the category of pseudosimplicial algebras with Moore complex of length 1.*

Proof. Let $\mathbf{E}$ be a pseudosimplicial algebra with Moore complex of length 1. Put $P = NE_0 = E_0$, $M = NE_1 = \ker(d_0 : E_1 \longrightarrow E_0)$ and $\partial = d_1$ (restricted to M). Then $p \in P$ acts on $m \in M$ by $p \cdot m = s_0(p) m$, and $\partial(p \cdot m) = d_1(s_0(p) m)$. Since the Moore complex $\cdots \longrightarrow 0 \longrightarrow M \xrightarrow{\partial} P \longrightarrow 0$ is of length 1 we have $\partial_2 NE_2 = 0$. It then follows that for all $m, m' \in M$ and $p \in P$,

$$\begin{aligned}
(i) \quad \partial_1(p \cdot m) &= d_1(p \cdot m) \\
&= d_1(s_0(p) m) \\
&= d_1 s_0(p) d_1(m) \\
&= p\partial_1(m)
\end{aligned}$$

$$
\begin{array}{llll}
(ii) & (\partial_1 m)\cdot m' & = & s_0(\partial_1(m))m' \\
 & & = & s_0(d_1(m))m' \\
 & & = & s_0(d_1(m))m' - mm' + mm' \\
 & & = & (s_0(d_1(m)) - m)m' + mm' \\
 & & = & (d_2 s_0(m) - d_2 s_1(m))d_2 s_1(m') + mm' \\
 & & = & d_2[(s_0(m) - s_1(m))s_1(m')] + mm' \\
 & & = & mm'
\end{array}
$$

for $m,m' \in M$ because $(s_0(m)-s_1(m))s_1(m')$ lies in NE_2. Thus $\partial : M \longrightarrow P$ is a crossed module of algebras.
Conversely, let $\partial : M \longrightarrow P$ be a crossed module of algebras. By using the action of P on M we can form the semi-direct product $M \rtimes P = \{(m,p) : m \in M\ ,\ p \in P\}$, in which multiplication

$$(m,p)\left(m',p'\right) = \left(p\cdot m' + p'\cdot m + m'm, pp'\right)$$

for $m,m' \in M$, $p,p' \in P$. There are homomorphisms

$$
\begin{array}{ll}
d_o : M \rtimes P \longrightarrow P, & (m,p) \longmapsto p, \\
d_1 : M \rtimes P \longrightarrow P, & (m,p) \longmapsto (\partial m) + p, \\
s_0 : P \longrightarrow M \rtimes P, & p \longmapsto (0,p).
\end{array}
$$

Let $E_0 = P$, $E_1 = M \rtimes P$. We have a 1 - truncated pseudosimplicial algebra $\{E_0, E_1\}$ whose 1- coskeleton we denote by E^1. The algebra $M \rtimes P$ acts on M via the action of P on M and the homomorphism d_1. We can thus form the semi-direct product $M \rtimes (M \rtimes P)$ and construct homomorphisms

$$
\begin{array}{ll}
d_0 : M \rtimes (M \rtimes P) \longrightarrow M \rtimes P, & (m,m',p) \longmapsto (m',p), \\
d_1 : M \rtimes (M \rtimes P) \longrightarrow M \rtimes P, & (m,m',p) \longmapsto (mm',p), \\
d_2 : M \rtimes (M \rtimes P) \longrightarrow M \rtimes P, & (m,m',p) \longmapsto (m,(\partial m')\,p), \\
s_0 : M \rtimes P \longrightarrow M \rtimes (M \rtimes P), & (m,p) \longmapsto (0,m,p), \\
s_1 : M \rtimes P \longrightarrow M \rtimes (M \rtimes P), & (m,p) \longmapsto (m,0,p).
\end{array}
$$

Conditions (i) and (ii) of a crossed module ensure that these are homomorphisms (Condition (ii) is needed for d_2). Let $E_2 = M \rtimes (M \rtimes P)$. We then have a 2 - truncated pseudosimplicial algebra $\{E_0, E_1, E_2\}$ whose 2 - coskeleton we denote by E^2. There is a unique simplicial map $E^2 \longrightarrow E^1$ which in dimensions 0 and 1 is the identity. We let $\overline{E}^2$ denote the image of E^2 in E^1. It is readily checked that the Moore comple of G^2 is trivial in dimension 2; it follows from Lemma 2.3 that $\overline{E}^2$ is a pseudosimplicial algebra whose Moore complex is of length 1. □

3.2. Pseudo 2-crossed Modules of algebras

Conduchě [8] in 1984 described the notion of 2 -crossed module as a model for (homotopy connected) 3 -types.

Definition 3.6. *A pseudo 2- crossed module $(L,M,P,\partial_1,\partial_2,\{,\})$ of (commutative) algebras is given by a chain complex of algebras*

$$L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} P$$

together with left actions $\cdot$ of P on M and L (and also on P by multiplication), preserved by ∂_1 and ∂_2, and an P -linear function(called the Peiffer lifting)

$$\{,\} : M \rtimes M \longrightarrow L$$

denoted by $\{m,m\prime\}$ satisfying the following axioms:

$$
\begin{array}{lllll}
\mathbf{P-2CM1)} & & \partial_2\{m,m'\} & = & mm' - \partial_1(m')\cdot m \\
\mathbf{P-2CM2)} & & \{\partial_2 l, \partial_2 l'\} & = & ll' \\
\mathbf{P-2CM3)} & & \{m, m'm''\} & = & \{mm', m''\} + \partial_1(m'')\cdot\{m,m'\} \\
\mathbf{P-2CM4)} & (a) & \{\partial_2 l, m\} & = & m\cdot l - \partial_1(m)\cdot l, \\
 & (b) & \{m, \partial_2 l\} & = & m\cdot l
\end{array}
$$

for all $l,l' \in L$, $m,m',m'' \in M$. To obtain the definition of 2 - crosssed modules, we must add the condition that **:**

$$\mathbf{2CM5)} \qquad p\cdot\{m,\acute{m}\} = \{p\cdot m,\acute{m}\} = \{m, p\cdot\acute{m}\} \ .$$

A morphism of pseudo 2 - crossed modules of algebras may be pictured by diagram

$$
\begin{array}{ccccc}
L & \xrightarrow{\partial_2} & M & \xrightarrow{\partial_1} & P \\
{\scriptstyle f_2}\downarrow & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_0}\downarrow \\
L' & \xrightarrow{\partial_2'} & M' & \xrightarrow{\partial_1'} & P'
\end{array}
$$

of algebras and homomorphisms such that $f_0\partial_1 = \acute{\partial_1} f_1$, $f_1\partial_2 = \acute{\partial_2} f_2$ and such that

$$f_1(p\cdot m) = f_0(p)\cdot f_1(m),\ f_2(p\cdot l) = f_0(p)\cdot f_2(l)$$

and

$$\{,\} = f_1 \times f_1 = f_2\{,\},$$

for all $l \in L$, $m \in M$, $p \in P$. We thus define the category of pseudo 2 - crossed modules, denoting it by $\mathfrak{pX}_2\mathfrak{Mod}$.

The category of simplicial (commutative) algebras with Moore complex of length 2 is equivalant to that of 2 -crossed modules. This equivalence was proved by Conduchě in [8]. Now, we shall give the following theorem.

Theorem 3.7. *Let* **G** *be a pseudosimplicial algebra with Moore complex of length 2. Then we can construct a pseudo 2 - crossed module.*

Proof. Let **G** be a pseudosimplicial group with Moore complex of length 2. We construct a pseudo 2 - crossed module as follows: $P = G_0$, $M = \ker(d_0 : G_1 \longrightarrow G_0)$, and $L = \ker(d_0 : G_2 \longrightarrow G_1) \cap (d_1 : G_2 \longrightarrow G_1)$. Then $p \in P$ acts on $m \in M$ by $p \cdot m = s_0(p)m$, and on $l \in L$ by $p \cdot l = s_1 s_0(p)l$ and $m \in M$ acts on $l \in L$ by $m \cdot l = s_1(m)l$. For $m, \acute{m} \in M$ set $\{m, \acute{m}\} = (s_1(m') - s_0(m'))s_1(m)$. Let $\partial_1 = d_1$ (restricted to M) and $\partial_2 = d_2$ (restricted to L).

$$
\begin{aligned}
\mathbf{P-2CM1)} \quad \partial_2\{m,m'\} &= \partial_2((s_1(m') - s_0(m'))s_1(m)) \\
&= (d_2 s_1(m') - d_2 s_0(m'))d_2 s_1(m) \\
&= d_2 s_1(m') d_2 s_1(m) - d_2 s_0(m') d_2 s_1(m) \\
&= m'm - s_0 d_1(m')m \\
&= mm' - s_0(\partial_1(m'))m \\
&= mm' - \partial_1(m') \cdot m
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{P-2CM2)} \quad \{\partial_2 l, \partial_2 l'\} &= \{d_2 l, d_2 l'\} \\
&= (s_1 d_2(l') - s_0 d_2(l'))s_1 d_2(l) \\
&= (d_3 s_1(l') - d_3 s_0(l'))d_3 s_1(l) \\
&= d_3 s_1(l') d_3 s_1(l) - d_3 s_0(l') d_3 s_1(l) - ll' + ll' \\
&= d_3 s_1(l') d_3 s_1(l) - d_3 s_0(l') d_3 s_1(l) - d_3 s_2(l) d_3 s_2(l') + ll' \\
&= d_3(s_1(l') s_1(l) - s_0(l') s_1(l) - s_2(l) s_2(l')) + ll' \\
&= ll'
\end{aligned}
$$

where $s_1(l')s_1(l) - s_0(l')s_1(l) - s_2(l)s_2(l')$ lies in $\partial_3 NG_3$.

$$
\begin{aligned}
\mathbf{P-2CM3)} \quad (i) \quad \{m, m'm''\} &= (s_1(m'm'') - s_0(m'm''))s_1(m) \\
&= s_1(m')s_1(m'')s_1(m) - s_0(m')s_0(m'')s_1(m) \\
&\quad + d_3[s_1 s_0(m'') s_2 s_1(m') s_2 s_1(m) - s_1 s_0(m'') s_2 s_0(m') s_2 s_1(m) \\
&\quad - s_2 s_1(m) s_2 s_1(m') s_2 s_0(m'') + s_2 s_0(m') s_2 s_0(m'') s_2 s_1(m)] \\
&= s_1(m')s_1(m'')s_1(m) - s_0(m')s_0(m'')s_1(m) \\
&\quad [d_3 s_1 s_0(m'') s_1(m') s_1(m) - d_3 s_1 s_0(m'') s_0(m') s_1(m) \\
&\quad - s_1(m) s_1(m') s_0(m'') + s_0(m') s_0(m'') s_1(m)] \\
&= s_1(m)s_1(m')s_1(m'') - s_1(m)s_1(m')s_0(m'') \\
&\quad + d_3(s_1 s_0(mm''))(s_1(m') - s_0(m'))s_1(m) \\
&= \{mm', m''\} + s_1 s_0 d_1(m'')(s_1(m') - s_0(m'))s_1(m) \\
&= \{mm', m''\} + \partial_1(m'') \cdot \{m, m'\}
\end{aligned}
$$

where $s_1 s_0(m'') s_2 s_1(m') s_2 s_1(m) - s_1 s_0(m'') s_2 s_0(m') s_2 s_1(m) - s_2 s_1(m) s_2 s_1(m') s_2 s_0(m'') + s_2 s_0(m') s_2 s_0(m'') s_2 s_1(m)$ lies in $\partial_3 NG_3$,

$$
\begin{aligned}
\mathbf{P-2CM4)} \quad (a) \quad \{\partial_2 l, m\} &= (s_1(m) - s_0(m))s_1 d_2(l) \\
&= (s_1(m) - s_0(m))d_3 s_1(l) \\
&= (d_3 s_2 s_1(m) d_3 s_1(l) - d_3 s_2 s_0(m) d_3 s_1(l) - d_3 s_2 s_1(m) d_3 s_2(l) \\
&\quad + d_3 s_1 s_0(m) d_3 s_2(l)) + s_1(m)l - s_1 s_0 d_1(m)l \\
&= d_3(s_2 s_1(m) s_1(l) - s_2 s_0(m) s_1(l) - s_2 s_1(m) s_2(l) \\
&\quad + s_1 s_0(m) s_2(l)) + s_1(m)l - s_1 s_0 d_1(m)l \\
&= s_1(m)l - s_1 s_0 d_1(m)l \\
&= m \cdot l - \partial_1(m) \cdot l,
\end{aligned}
$$

where $s_2 s_1(m) s_1(l) - s_2 s_0(m) s_1(l) - s_2 s_1(m) s_2(l) + s_1 s_0(m) s_2(l)$ lies in $\partial_3 NG_3$.

$$
\begin{aligned}
(b) \quad \{m, \partial_2 l\} &= (s_1 \partial_2(l) - s_0 \partial_2(l))s_1(m) \\
&= s_1 d_2(l) s_1(m) - s_0 d_2(l) s_1(m) - s_1(m)l + s_1(m)l \\
&= d_3 s_1(l) d_3 s_2 s_1(m) - d_3 s_0(l) d_3 s_2 s_1(m) - d_3 s_2 s_1(m) d_3 s_2(l) + s_1(m)l \\
&= d_3(s_1(l) s_2 s_1(m) - s_0(l) s_2 s_1(m) - s_2 s_1(m) s_2(l)) + s_1(m)l \\
&= s_1(m)l \\
&= m \cdot l
\end{aligned}
$$

where $s_1(l)s_2s_1(m) - s_0(l)s_2s_1(m) - s_2s_1(m)s_2(l)$ lies in $\partial_3 NG_3$.
Therefore $(L, M, P, \partial_1, \partial_2, \{,\})$ is a pseudo 2- crossed module of (commutative) algebras. □

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