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# DIRAC'S LINEARIZATION APPLIED TO THE FUNCTIONAL, WITH MATRIX ASPECT, FOR THE TIME OF FLIGHT OF LIGHT

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Abstract. In this paper, we adopt a matrix treatment to solve the variational problem that consists of determining the physical path traveled by light between two points in a medium whose refractive index depends on a spatial coordinate. The considered treatment begins with the trivial repetition of the expression of the value of the considered functional, repetition expressed in the form of a matrix. Next, we adopt the trick (of Dirac) originally used as part of the construction of the dynamic equation of relativistic quantum mechanics, which allows us to rewrite the (now) matrix integrand in the expression of the value of the functional in terms of the sum of two (non-diagonal) matrices brought externally to the problem, which are determined based on some requirements. As a result of this development, we obtain two equivalent versions of Snell's law.

## 1. Introduction

In the context of the variational formulation of optics [1],[2], the functional $T$ is defined, whose value $T[y]$, for an arbitrary curve $y$, corresponds to the time of flight of light along the referred curve, between two fixed points $P_1$ and $P_2$, located within a medium whose refractive index may depend, in the most general situation, on the spatial coordinates $x; y; z$. Suppose, for simplicity, that the medium traversed by light has an index of refraction that depends only on the spatial coordinate $y$; that is, $n(y)$. Therefore, the value of the functional $T[y]$ is written as [3],

$$T[y] \;=\; \frac{1}{c}\int_a^b n(y)\Big(1+\Big(y'(x)\Big)^2\Big)^{1/2}dx \;=\; \frac{1}{c}\int_a^b F\Big(x,y(x),y'(x)\Big)\,dx. \tag{1.1}$$

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where $c$ is the magnitude of speed of light in vacuum. The functional $T$ above will have as an extremal a curve that can be traversed by light in a stationary[1] time, compared to the travel times corresponding to the other curves in the $T$ domain.

The functional $T$, whose value is defined in (1.1), corresponds to the following Euler equation [4],

$$F_y \;-\; \frac{d}{dx}F_{y'} \;=\; 0. \tag{1.2}$$

In the next section we will give $T$ a "matrix clothing", without changing its essence, which will be convenient.

On the other hand, the expression "Dirac linearization" is used here to indicate a simplified version [5], adapted to our problem, of the linearization of a second-order differential operator raised to the 1/2 power [6]. This idea[2] was used by Dirac to obtain a relativistic dynamic equation [8],[9] for specific quantum particles from a (non-linear) expression for their energy.

## 2. "Matrix Clothing" for $T[y]$ and Dirac's Trick

It is trivial to recognize that the factor that multiplies $n(y)$ in the integrand in (1.1) is the square root of the sum of two quadratic terms. This factor may receive a matrix clothing by multiplying each member in (1.1) by any matrix; in particular, by the identity matrix or by the square root of the identity; that is, we would have,

$$\left(\,1+\Big(y'(x)\Big)^2\,\right)^{1/2}\mathbf{I} \qquad \text{or} \qquad \left(\,1+\Big(y'(x)\Big)^2\,\right)^{1/2}\mathbf{I}^{1/2}, \tag{2.1}$$

with the difference that the left term in (2.1) corresponds to a diagonal matrix, but the right expression corresponds to a matrix that will not necessarily be diagonal[3]. As we will see, it would be more convenient to consider a matrix that could contain as many degrees of freedom as possible.

Then the expression (1.1) will receive such a "matrix clothing". However, it will only be about the matrix appearance of (1.1), since the functional $T$ continues to match functions $y$ with numbers $T[y]$, and not functions with matrices. In fact, such a matrix clothing will correspond to the trivial repetition of the same expression (1.1). In concrete terms, what we do is multiply (1.1) by the matrix $\mathbf{I}^{1/2}$, of order $N$, which in principle can have an arbitrary value. Soon,

$$\mathbf{T}[y] \;\equiv\; T[y]\,\mathbf{I}^{1/2} \;=\; \frac{1}{c}\int_a^b n(y)\,\Big(1+(y'(x))^2\Big)^{1/2}\mathbf{I}^{1/2}\,dx. \tag{2.2}$$

In the case of considering matrices of order $N=2$ we can write,

$$\mathbf{I}^{1/2} \;=\; \frac{1}{\sqrt{1+ab}}\begin{pmatrix} 1 & b \\ a & -1 \end{pmatrix}, \tag{2.3}$$

(with arbitrary $a$ and $b$, but such that $ab \neq -1$) whose square corresponds precisely to the identity matrix $\mathbf{I}$; then we would explicitly have the expression,

$$\sqrt{1+ab}\;\mathbf{T}[y] \;=\; \begin{pmatrix} T[y] & b\,T[y] \\ a\,T[y] & -T[y] \end{pmatrix} =$$

[1] This can be a minimum, as in situations typically found in books.

[2] Already acknowledged by O. Heaviside who, according to [7], would have stated that: "... *the square root of a differential operator is intrinsic to Physics*".

[3] For there are non-diagonal matrices whose square is diagonal.

$$= \begin{pmatrix} \frac{1}{c}\int_a^b n(y)\left(1+(y'(x))^2\right)^{1/2} dx & \frac{b}{c}\int_a^b n(y)\left(1+(y'(x))^2\right)^{1/2} dx \\ \frac{a}{c}\int_a^b n(y)\left(1+(y'(x))^2\right)^{1/2} dx & -\frac{1}{c}\int_a^b n(y)\left(1+(y'(x))^2\right)^{1/2} dx \end{pmatrix} \quad (2.4)$$

which, as already said, corresponds to the trivial repetition of (1.1).

Expression (2.2)(2.2) can be linearized. It is easy to verify that for matrices of order 2, non-diagonal and, in general, dependent on $y$, which we represent here by $\mathbf{A}(y)$ and $\mathbf{B}(y)$, which must be properly defined, one can write[4],

$$\left(1+\left(y'(x)\right)^2\right)^{1/2} \mathbf{I}^{1/2} \;=\; \mathbf{A}(y) \;+\; y'(x)\,\mathbf{B}(y), \quad (2.5)$$

Provided that the following requirements are met,

$$\mathbf{A}^2(y) \;=\; \mathbf{I}, \quad \mathbf{B}^2(y) \;=\; \mathbf{I}, \quad \mathbf{A}(y)\mathbf{B}(y) \;+\; \mathbf{B}(y)\mathbf{A}(y) \;=\; \mathbf{0}, \quad (2.6)$$

Furthermore, it must be taken into account, for a mathematical consistency argument, that the sum of the matrix on the right side of (2.5) must be non-diagonal, since the matrix $\mathbf{I}^{1/2}$ has this characteristic, as indicated in (2.3).

Now we can rewrite the value of the functional $T$ with "matrix clothing" as follows,

$$\mathbf{T}[y] \;=\; \frac{1}{c}\int_a^b n(y)\left(\mathbf{A}(y)+(y'(x))\,\mathbf{B}(y)\right)dx, \quad (2.7)$$

## 3. The "matrix clothing" for the Euler equation

The integrand in $\mathbf{T}[y]$, in the considered context, is written as,

$$\mathbf{F}\Big(x, y(x), y'(x)\Big) \;=\; n(y)\left(\mathbf{A}(y)+y'(x)\,\mathbf{B}(y)\right). \quad (3.1)$$

On the other hand, the matrix clothing for the corresponding Euler equation is obtained directly: multiplying the expression (1.2) by the matrix, $\mathbf{I}^{1/2}$, that is,

$$\mathbf{I}^{1/2}\left(F_y - \frac{d}{dx}F_{y'}\right) \;=\; \left(\mathbf{F}_y - \frac{d}{dx}\mathbf{F}_{y'}\right) \;=\; \mathbf{I}^{1/2}\,0 \;\equiv\; \mathbf{0}. \quad (3.2)$$

This clothing is purely formal and obviously trivial, without changing the nature of the initial problem: we have a functional $T$ that assigns the number $T[y]$ to a function $y$, which belongs to the domain of $T$.

So, from (3.1) we get,

$$\mathbf{F}_y \;=\; n(y)\left(\mathbf{A}'(y)+y'(x)\,\mathbf{B}'(y)\right)+n'(y)\left(\mathbf{A}(y)+y'(x)\,\mathbf{B}(y)\right), \quad (3.3)$$

and,

$$\mathbf{F}_{y'} \;=\; n(y)\,\mathbf{B}(y), \quad (3.4)$$

From expression (3.4) we have that,

$$\frac{d}{dx}\,\mathbf{F}_{y'} \;=\; n(y)\,\mathbf{B}'(y)\,y'(x)+n'(y)\,y'(x)\,\mathbf{B}(y). \quad (3.5)$$

[4]This is essentially the trick used by Dirac [5], which we mentioned earlier.

Substituting (3.3) and (3.5) into Euler's equation (3.2) and after simplifications we obtain the following relation,

$$n(y)\mathbf{A}'(y) + n'(y)\mathbf{A}(y) \;=\; \mathbf{0}, \tag{3.6}$$

which corresponds to the development of the derivative of a product,

$$\Big(n(y)\;\mathbf{A}(y)\Big)' = \mathbf{0}, \qquad \Rightarrow \qquad n(y)\;\mathbf{A}(y) = \mathbf{C}, \tag{3.7}$$

where $\mathbf{C}$ is a constant matrix.
Note: An aspect not discussed here refers to the existence of possible symmetries associated with the functional $\mathbf{T}$, which would be revealed through its invariance under specific transformations. In a broader context, we would see that expression (3.7) can be obtained directly from Noether's Theorem [10].

## 4. Determination of matrices A and B

Note that expression (3.7) is independent of matrix $\mathbf{B}$; which we can take advantage of considering that $\mathbf{B}$ is constant, which will simplify its determination.

Now we explicitly write the non-diagonal matrices $\mathbf{A}$ and $\mathbf{B}$,

$$\mathbf{A}(y) = \begin{pmatrix} a_{11}(y) & a_{12}(y) \\ a_{21}(y) & a_{22}(y) \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \tag{4.1}$$

The algebraic equations resulting from the requirements in (2.6) correspond, if placed in terms of the elements of $\mathbf{A}$ and $\mathbf{B}$, in (4.1), to the following:

$$\begin{aligned} &a_{11}^2(y) \;+\; a_{12}(y)\,a_{21}(y) = 1,\\ &\Big(a_{11}(y) + a_{22}(y)\Big)\,a_{12}(y) = 0,\\ &\Big(a_{11}(y) + a_{22}(y)\Big)\,a_{21}(y) = 0,\\ &a_{22}^2(y) \;+\; a_{12}(y)\,a_{21}(y) = 1, \end{aligned} \tag{4.2}$$

$$\begin{aligned} &b_{11}^2 \;+\; b_{12}\,b_{21} = 1,\\ &\Big(b_{11} + b_{22}\Big)\,b_{12} = 0,\\ &\Big(b_{11} + b_{22}\Big)\,b_{21} = 0,\\ &b_{22}^2 \;+\; b_{12}\,b_{21} = 1, \end{aligned} \tag{4.3}$$

$$\begin{aligned} &2a_{11}b_{11} + a_{12}b_{22} + a_{21}b_{12} = 0,\\ &a_{11}b_{12} + a_{12}b_{22} + a_{12}b_{11} + a_{22}b_{12} = 0,\\ &a_{21}b_{11} + a_{22}b_{21} + a_{11}b_{21} + a_{21}b_{22} = 0,\\ &2a_{22}b_{22} + a_{21}b_{12} + a_{12}b_{21} = 0, \end{aligned} \tag{4.4}$$

The groups of equations in (4.2)−(4.4) are solved with the elements of the following matrices,

$$\mathbf{A}(y) = \begin{pmatrix} \sin(\theta(y)) & \cos(\theta(y)) \\ \cos(\theta(y)) & -\sin(\theta(y)) \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \tag{4.5}$$

where $\theta(y)$ can be freely defined, with $y$ taking an arbitrary value. For example: $\theta(y) = \operatorname{arccot}(dy/dx)$, which is to say it corresponds to the angle between the coordinate direction $y$ is the path of light.

From (3.7) and (4.5) we extract the two independent relations,

$$n(y)\,\sin(\theta(y)) = c_1 \qquad e \qquad n(y)\,\cos(\theta(y)) = c_2 \tag{4.6}$$

with $y$ being free.

## 5. Conclusion

It is easy to recognize what might be called Dirac's "matrix trick" in the solution presented. In the problem analyzed here, it was possible to consider that one of the two matrices freely brought to the problem is constant. Note that the expression on the left in (4.6), with the definition of $\theta(y)$ given above, corresponds to Snell's law; in this case, the expression on the right, in the same expression, is spurious. But if we define the angle between the light path and a direction orthogonal to $y$, like $x$, as follows,

$$\theta(y) = \arctan\left(\frac{dy}{dx}\right),$$

then the expression of Snell's law is given by the one on the right, in (4.6), and the one on the left, in the same expression (4.6), would be spurious. Finally, the solution presented shows us that Dirac's trick can find application in other problems, which should be expected considering that the physical results do not depend on the mathematical tools used. As an illustration, the "Dirac linearization", as used here, can also be applied in the construction of a variant of the Feynman temporal propagator that, instead of using the action functional (and the Planck constant), the length functional of path is used (and the Compton wavelength of the considered particle).

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