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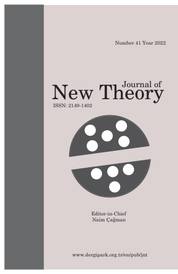

Vertices of Suborbital Graph $F_{u,N}$ under Lorentz Matrix Multiplication

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Research Article

Abstract — In this study, suborbital graphs, $G_{u,N}$ and $F_{u,N}$ are examined. Modular group Γ and its act on $\widehat{\mathbb{Q}}$ are studied. Lorentz matrix that gives the vertices obtained under the classical matrix multiplication in the suborbital graph $F_{u,N}$ is analysed with the Lorentz matrix multiplication. Lorentz matrix written as Möbius transform is normalized and the type of the transform is researched. Moreover, a different element of Modular group Γ is scrutinized. The vertices on the path starting with ∞ are obtained under this element and the Lorentz matrix multiplication. For this path, it is shown that the vertices obtained in $F_{u,N}$ under the Lorentz matrix multiplication with the Lorentz matrix satisfied the farthest vertex condition for the previous vertex.

Keywords — *Lorentz matrix multiplication, modular group, Mobius transform*

Mathematics Subject Classification (2020) — 22E43, 11F06

1. Introduction

Graph theory and its elements are at the core of our work. With the discovery of non-Euclidean geometry in the 18^{th} and 19^{th} centuries, Graph theory began to be studied in this field as well. Elliptic and hyperbolic geometries are both of non-Euclidean geometries.Moreover, with the discovery of invariant theory and non-Euclidean geometries, linear fractional transformation groups achieve special importance. Since linear fractional transformation groups are suitable for the topological group structure, they have been extensively studied in recent years with different methods. In [1], some ideas were put forward about graph action firstly. These ideas found an important place in the work of [2] and [3] in applications for finite groups. Modular group Γ and its subgroups,which play an important role in the last theorem by proved Fermat, have been researched extensively in recent years. In [4], suborbital graphs, $G_{u,N}$ and $F_{u,N}$ obtained by element of Modular group Γ are examined and they presented some conclusions. In [5], it is provided that suborbital graph is a forest if and only if it does not have triangles. Elliptic elements and elliptic circuits are investigated in [6]. In [7], it is shown that each vertex in the suborbital graph $F_{u,N}$ has a continued fraction structure for $(u, N) = 1$ and $u \leq N$ and investigated the vertices on path with minimal lengths. Suborbital graphs are studied for invariant groups in [8]. The vertices obtained with help of continued fractions and recurrence relations are generalized and associated with Fibonacci numbers in [9]. In [10], Gündoğan and Keçilioğlu defined Lorentz matrix multiplication.

This study investigates corollaries of suborbital graph act by different matrix multiplication.Then,we use Lorentz matrix multiplication and investigate Lorentz matrix (Equation 6) that gives the vertices

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obtained under classical matrix multiplication with Lorentz matrix multiplication. Here, it is demonstrated that Lorentz matrix (Equation 6) is not an element of Modular group Γ.

However, we examined an element (Equation 7) of Modular group Γ. We obtained a path starting with ∞ using Equation 7 in this article.Then, we demonstrated that the vertices on this path satisfy the minimal length condition. In addition,we assumed k as 1 in Equation 7. Therefore, we associated the vertices of path with Fibonacci numbers and n^{th} vertex with golden section.

2. Preliminary

2.1. Suborbital Graphs

Assume that (G, Ω) is a transitive permutation group, $g \in G$ and $\alpha, \beta \in \Omega$. Then G provides

$$g : (\alpha, \beta) \to (g(\alpha), g(\beta))$$

on $\Omega \times \Omega$.The orbitals of this transformation are called suborbitals of G. $O(\alpha, \beta)$ represents the suborbital covering (α, β).

$$O(\alpha, \beta) = \{g(\alpha, \beta) | g \in G\}$$

$$(x,y) \in O(\alpha, \beta) \Leftrightarrow g \in G : (x, y) = g(\alpha, \beta) = (g(\alpha), g(\beta))$$

The suborbital graph $G(\alpha, \beta)$ can be obtained from the suborbital $O(\alpha, \beta)$. Assume that γ and δ vertices in $\widehat{\mathbb{Q}}$, if $(\gamma, \delta) \in O(\alpha, \beta)$ exists, the orbit represents a directional edge from γ to δ and is denoted by $\gamma \longrightarrow \delta$. This edge can be drawn at $\mathbb{H} = \{z \in \mathbb{C} | Im(z) > 0\}$ as a hyperbolic geodesic.

$O(\beta, \alpha)$ is also an orbit and can be equal to or different from the $O(\alpha, \beta)$. If the orbits are different from each other, the suborbital graph $G(\beta, \alpha)$ is the opposite direction of the edges of the suborbital graph $G(\alpha, \beta)$ and in this case the suborbital graphs are called paired suborbital graphs.If the orbits are equal, $G(\beta, \alpha) = G(\alpha, \beta)$ and the graph includes the opposite pair of edges; in this case, by replacing each pair with a simple directed edge, an undirected edge paired with it is obtained.In other words, if $O(\beta, \alpha) = O(\alpha, \beta)$ and $(\gamma, \delta) \in O(\alpha, \beta)$, the edge between γ and δ vertices is denoted by $\gamma - \delta$ instead of $\gamma \leftrightarrow \delta$.

Assume that equivalence relation is "$\approx$ " and for all $\alpha, \beta \in \Omega$, for all $g \in G$, if is provided $g(\alpha) \approx g(\beta)$ when $\alpha \approx \beta$, "$\approx$ " is called "G-invariant equivalence relation "on Ω, and equivalent classes formed in this way are called "blocks". Examples of these relations are identity and universal relation:

i. identity relation, "$\alpha = \beta \Leftrightarrow \alpha \approx \beta$" for all $\alpha, \beta \in \Omega$

ii. universal relation, "$\alpha \approx \beta$" for all $\alpha, \beta \in \Omega$.

Unlike these relations, if there is a G -invariant equivalence relation on Ω, (G, Ω) is called imprimitive, otherwise primitive.The transitive act of the primitive group (G, Ω) is necessary, otherwise the orbits do not constitute a system block and its reverse is not true.

Theorem 2.1. [4] Assume that (G, Ω) is an transitive permutation group. In this case (G, Ω) is primitive $\Leftrightarrow G_\alpha$ the stabilizer of a point $\alpha \in \Omega$ is a maximal subgroup of G for all $\alpha \in \Omega$.

Theorem 2.2. [4] **G** is a suborbital graph for transitive permutation group (G, Ω). In this case,

i. G acts as a group of automorphism of **G**.

ii. G acts as transitive on vertices of **G**.

iii. If **G** is self-paired, then G acts transitively on ordered pairs of consecutive vertices of **G**.

iv. If **G** is not self-paired, then G acts transitively on the edges of **G**.

2.2. Modular Group and its Act on $\widehat{\mathbb{Q}}$ Under Classical Matrix Multiplication

The Modular group is the quotient group of the $SL(2,\mathbb{Z})$ with $\{\mp I\}$. Specially if the Modular group is denoted by Γ, it is written as

$$\Gamma = PSL(2,\mathbb{Z}) \cong SL(2,\mathbb{Z})/\{\mp I\}$$

$$\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{Z}, ad - bc = 1 \right\}$$

Γ is consist of $\mp \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ matrices pairs. Here, the $+$ and $-$ symbols are ignored and matrices considered equivalent.

Some equivalent subgroups of Γ are given below:

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid c \equiv 0 \,(\mathrm{mod}\ N) \right\}$$

$$\Gamma^0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid b \equiv 0 \,(\mathrm{mod}\ N) \right\}$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid a \equiv d \equiv 1 \,(\mathrm{mod}\ N), c \equiv 0 \,(\mathrm{mod}\ N) \right\}$$

The hyperbolic plane is defined by $\mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\}$. Möbius transformations are known transformations with the elements of the Modular group in the upper half plane $\mathbb{H}$. Transformation is defined by for all $z \in \mathbb{C}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \to \frac{az+b}{cz+d}$$

f and g are elements of Modular group Γ for $f(z)$ and $g(z)$ linear Möbius transformations.

$$f(z) = \frac{-1}{z} = \frac{0z-1}{1z+0} \Rightarrow f = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \Gamma$$

$$g(z) = z + \lambda = \frac{1z+\lambda}{0z+1} \Rightarrow g = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \in \Gamma$$

In addition, Möbius transformations are used to describe the elements of $\widehat{\mathbb{Q}}$. Especially for $\frac{a}{c} \in \widehat{\mathbb{Q}}$, if $c = 0$, it is accepted as $\frac{a}{c} = \infty$. For $x, y \in \mathbb{Z}$ and $(x, y) = 1$, each element of $\widehat{\mathbb{Q}}$ can be expressed as reduced fraction $\frac{x}{y}$.Since $\frac{x}{y} = \frac{-x}{-y}$, the notation is not uniform. ∞ would be considered as $\frac{1}{0} = \frac{-1}{0}$.For $z = \frac{x}{y}$, Möbius transformation is written as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \frac{x}{y} \to \frac{ax+by}{cx+dy} \tag{1}$$

The reduced result in Equation 1 is shown as follows:

$$c(ax+by) - a(cx+dy) = cax + cby - acx - ady = (cb - ad)y = -y$$

$$d(ax+by) - b(cx+dy) = dax + dby - bcx - bdy = (ad - bc)x = x$$

$$(ax+by, cx+dy) = 1$$

Lemma 2.3. [4]

i. Act of Γ is transitive on $\widehat{\mathbb{Q}}$.

ii. Element of Γ that fixed a vertex on $\widehat{\mathbb{Q}}$ is infinitely period.□

Assume that examine what has been given so far about suborbital graphs if G is the Modular group Γ and Ω is $\widehat{\mathbb{Q}}$. Γ_∞ which fixed ∞ is a subgroup of Γ produced by $Z = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then we can generate Γ-invariant equivalence relations on $\widehat{\mathbb{Q}}$ by obtaining the subgroups H of Γ containing Γ_∞ or equivalently Z. Since $Z = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, the congruence groups $\Gamma_0(N)$ can be selected as H, with $N \in \mathbb{N}$. Clearly, $\Gamma_\infty < \Gamma_0(N) \leqslant \Gamma$ for all $N \in \mathbb{N}$ and $\Gamma_\infty < \Gamma_0(N) < \Gamma$ for $N > 1$. Hence, act of Γ on $\widehat{\mathbb{Q}}$ is imprimitive.

Assume that denote the reduced Γ- invariant equivalence relation on $\widehat{\mathbb{Q}}$ of $\Gamma_0(N)$ with "$\approx_N$". Transformations $v = g(\infty)$ and $w = g'(\infty)$ are provided for $v = \frac{r}{s}$,$w = \frac{x}{y} \in \widehat{\mathbb{Q}}$ and $g = \begin{pmatrix} r & * \\ s & * \end{pmatrix}$, $g' = \begin{pmatrix} x & * \\ y & * \end{pmatrix} \in \Gamma$.

Since

$$v \approx_N w \Leftrightarrow g(v) \approx_N g'(w) \Leftrightarrow g^{-1}g' \in H = \Gamma_0(N)$$

and

$$g^{-1} = \begin{pmatrix} * & * \\ -s & r \end{pmatrix}$$

$$g^{-1}g' = \begin{pmatrix} * & * \\ -s & r \end{pmatrix}\begin{pmatrix} x & * \\ y & * \end{pmatrix} = \begin{pmatrix} * & * \\ ry - sx & * \end{pmatrix} \in H = \Gamma_0(N)$$

$$v \approx w \Leftrightarrow ry - sx \equiv 0 \pmod N$$

results are obtained. In other words, $v = \frac{r}{s}$ and $w = \frac{x}{y}$ are equivalent $\Leftrightarrow \exists u \in H : x \equiv ur \pmod N$, $y \equiv us \pmod N$

Similarly, $\Gamma^0(N)$ which fixed 0 is a subgroup of Γ produced by $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Thus, we can generate Γ-invariant equivalence relations on $\widehat{\mathbb{Q}}$ by finding the subgroups K of Γ containing $\Gamma^0(N)$ or equivalently B.

The number of equivalence classes under "$\approx_N$" is given by

$$\Psi(N) =\mid \Gamma : \Gamma_0(N) \mid$$

equation.

2.3. Investigation of $G_{u,N}$ and $F_{u,N}$

Since the act of Γ on $\widehat{\mathbb{Q}}$ is transitive, each suborbit contains the pair (∞, v) for $v \in \widehat{\mathbb{Q}}$. If $v = \frac{u}{N}$ for $N \geqslant 0$ and $(u, N) = 1$, suborbit is denoted by $O_{u,N}$, suborbital graph $G(\infty, v)$ corresponding to $O_{u,N}$ is denoted by $G_{u,N}$. If $v = \infty$, $G_{1,0} = G_{-1,0}$ is trivial suborbital graph, so we can assume $v \in \widehat{\mathbb{Q}}.v, v' \in \widehat{\mathbb{Q}}$ and $O(\infty, v) = O(\infty, v') \Leftrightarrow v$ and v' are in orbit of Γ_∞. Since Γ_∞ produced by $Z : v \longrightarrow v + 1$, equivalent to $v' = \frac{u'}{N}$ for $u \equiv u' \pmod N$.

Theorem 2.4. [4] $\frac{r}{s} \to \frac{x}{y} \in G_{u,N}$ if and only if

i. $x \equiv ur \pmod N$, $y \equiv us \pmod N$ and $ry - sx = N$

ii. $x \equiv -ur \pmod N$, $y \equiv -us \pmod N$ and $ry - sx = -N$

Corollary 2.5. [4] Suborbital graph which is paired with $G_{u,N}$ is $G_{-\overline{u},N}$ for $\overline{u}$ providing $u\overline{u} \equiv 1 \pmod N$.

Corollary 2.6. [4] $G_{u,N}$ is self-paired $\Leftrightarrow u^2 \equiv -1 \pmod N$.

$G_{u,N}$ is the discrete union of $\Psi(N)$ subgraphs and the vertices of each subgraph form a single block according to the $\approx_N$ Γ-invariant equivalence relation defined by $ry - sx \equiv 0 \,(\mathrm{mod}\ N)$. Since Γ acts transitively on $\widehat{\mathbb{Q}}$, Γ permutates these blocks as transitive and all subgraphs are isomorphic. $F_{u,N}$ be the subgraph of $G_{u,N}$ consisting ∞ on vertices and

$$[\infty] = \left\{ \frac{x}{y} | y \equiv 0 \,(\mathrm{mod}\ N), x, y \in \mathbb{Q} \right\}$$

Thus, $G_{u,N}$ consists of $\Psi(N)$ discrete copies of $F_{u,N}$.

Theorem 2.7. [4] $\frac{r}{s} \to \frac{x}{y} \in F_{u,N}$ if and only if

i. $x \equiv ur \,(\mathrm{mod}\ N)$ and $ry - sx = N$

ii. $x \equiv -ur \,(\mathrm{mod}\ N)$ and $ry - sx = -N$

Theorem 2.8. [4] $\Gamma_0(N)$ permutates vertices and edges of $F_{u,N}$ transitively.

2.4. Continued Fractions

Continued fractions are basically divided into two groups as finite and infinite.

2.4.1. Finite Continued Fractions

A finite continued fraction is defined as follow

$$x = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_{m-2} + \cfrac{1}{a_{m-1} + \cfrac{1}{a_m}}}}}}$$

for $a_1 \geqslant 0$,$i \geqslant 2$ and a_i positive integer. It can be written as notation $x = [a_1; a_2, a_3, \ldots, a_k]$.

2.4.2. Infinite Continued Fractions

An infinite continued fraction is defined as follow

$$x = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_{m-2} + \cfrac{1}{\ddots}}}}}$$

for $a_1 \geqslant 0$,$i \geqslant 2$ and $a_i \geqslant 1$. It can be written as $x = [a_1; a_2, a_3, \ldots]$ [11].

More generally, a continued fraction is defined

$$x = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{\ddots}}} \tag{2}$$

where $\mathbb{N}$ is set of natural numbers, $\mathbb{Z}$ is set of integer numbers and $a_i \in \mathbb{Z} - \{0\}$, $b_i \in \mathbb{Z}$ for all $i \in \mathbb{N} \cup \{0\}$.

Continued fraction in Equation 2 can be written as

$$b_0 + \mathbf{K}_{i=1}^{\infty} \left(\frac{a_i}{b_i} \right) \tag{3}$$

However, n. approach for continued fraction in Equation 3 is denoted by f_n and it is written as

$$f_n = b_0 + \mathbf{K}_{i=1}^{n} \left(\frac{a_i}{b_i} \right)$$

In additon, $\{f_n\}$ sequence is obtained by $(\{a_i\}_{i\in\mathbb{N}}, \{b_i\}_{i\in\mathbb{N}\cup\{0\}})$ for $i \geqslant 1$, $a_i \neq 0$ and linear fractional transformation sequences $\{t_n(s)\}_{n\in\mathbb{N}\cup\{0\}}$ and $\{T_n(s)\}_{n\in\mathbb{N}\cup\{0\}}$ where

$$t_0(s) = s, t_n(s) = \frac{a_n}{b_n + s}, n = 1, 2, 3, \ldots$$

$$T_0(s) = t_0(s), T_n(s) = T_{n-1}(t_n(s)), n = 1, 2, 3, \ldots$$

$$f_n = T_n(0) \in \widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}, n = 1, 2, 3, \ldots$$

From here,

$$((\{a_i\}_{i\in\mathbb{N}}, \{b_i\}_{i\in\mathbb{N}\cup\{0\}}), \{f_n\})$$

can be written. This obtained sequence corresponds to the continued fraction given in Equation 2. a_i is called the partial numerator and b_i is called the partial denominator.

In accordance with the above, the linear fractional transformation $T_n(s)$ can be expressed by

$$T_n(s) = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{\ddots + \cfrac{a_n}{b_n + s}}}}$$

From the representation of continued fractions,

$$T_n(s) = (t_0 \circ t_1 \circ t_2 \circ \ldots \circ t_n)$$

can be written where $\circ$ compound function.We get

$$(t_0 \circ t_1)(s) = t_0(t_1(s))$$

and

$$t^n(s) = (t \circ t \circ t \circ \ldots \circ t)(s)$$

The number of n^{th} modified approaches is denoted by

$$T_n(S_n) \in \mathbb{R}$$

for $\{S_n\}_{n\in\mathbb{N}\cup\{0\}}$ sequence.

2.5. Paths of Minimal Length on Suborbital Graphs

In this section, some definitions and theorems are given about paths of minimal length on suborbital graph.

Definition 2.9. [7] $v_0, v_1, v_2, \ldots, v_m$ is a sequence of different vertices of suborbital graph $F_{u,N}$. If $m \geqslant 2$, $v_0 \to v_1 \to v_2 \to \ldots \to v_m \to v_0$ is called directed circuit (or closed path).If at least one (but not all) edge in this path are is the opposite direction, this path is called an undirected circuit (or reverse directed circuit). If $m = 2$, the circuit is called a triangle, directed or not.If $m = 1$, the path $v_0 \to v_1 \to v_0$ is called a self-matched edge.

Definition 2.10. [7] Since the elements of the Modular group represent Hyperbolic lines to Hyperbolic lines, the elements of the graph $F_{u,N}$ for proper visualization are shown half lines perpendicular to the real axis in the upper half plane of $\mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\}$ and half lines with the center on $\mathbb{R}$ as hyperbolic geodesics.

Definition 2.11. [7] The path $v_0 \to v_1 \to v_2 \to \ldots \to v_m$ and $v_0 \to v_1 \to v_2 \to \ldots$ is called a path and an infinite path in the graph $F_{u,N}$ respectively.

Definition 2.12. [7] $\frac{r}{s} \overrightarrow{<} \frac{x}{y} \in F_{u,N}$ ($\frac{r}{s} \overleftarrow{>} \frac{x}{y} \in F_{u,N}$), if there is no vertex greater (or smaller) than the $\frac{x}{y}$ vertex connected to the $\frac{r}{s}$ vertex in the graph $F_{u,N}$, the $\frac{x}{y}$ vertex is called the farthest (nearest) vertex.

Definition 2.13. [7] For the path $v_0 \to v_1 \to v_2 \to \ldots \to v_m$ in the graph $F_{u,N}$ to have minimal length,$v_i \nleftrightarrow v_j$ where $i < j-1$, $i \in \{0, 1, \ldots, m-2\}$,$j \in \{2, 3, \ldots, m\}$ and vertex v_{i+1} should be the farthest vertex that connects to vertex v_i.

Definition 2.14. [7] If $F_{u,N}$ contains no circuits it is called a forest. A connected non-empty graph with no circuit is a tree.

Lemma 2.15. [12] If $(u, N) = 1$, there is an integer k that satisfies the equation $u^2 + ku + 1 \equiv 0 \,(\mathrm{mod}\ N)$.

For $k \geqslant 2$ and $k \in \mathbb{Z}$, $\begin{pmatrix} -u & \frac{u^2+ku+1}{N} \\ -N & u+k \end{pmatrix} \in \Gamma_0(N)$ that the element of an equivalent subgroup of the Modular group connects the vertices in order on an infinite minimal length path in suborbital graph $F_{u,N}$ and each vertex forms a continued fractional structure.

$$\infty = \frac{1}{0} \to \frac{u}{N} \to \frac{u + \frac{1}{k}}{N} \to \frac{u + \frac{1}{k - \frac{1}{k}}}{N} \to \frac{u + \frac{1}{k - \frac{1}{k - \frac{1}{k}}}}{N} \to \ldots$$

This path is right directed. Each vertex that can be connected to the previous vertex is the farthest vertex.

It can be defined as

$$v_q = \begin{pmatrix} -u & \frac{u^2+ku+1}{N} \\ -N & u+k \end{pmatrix}^q (v_0) \tag{4}$$

for all $q \in \mathbb{Z}^+$,where $v_0 = \frac{u}{N}$.

Theorem 2.16. [12] Assume that $u^2 + ku + 1 \equiv 0 \,(\mathrm{mod}\ N)$ and $1 < k < N$ in Farey graph.

i. The farthest vertex to which $\frac{u}{N}$ can be connected becomes $\frac{u+\frac{1}{k}}{N}$ and there is no similar nearest vertex.

ii. The farthest vertex to which $\frac{u+\frac{1}{k}}{N}$ can be connected becomes $\frac{u+\frac{1}{k-\frac{1}{k}}}{N}$ and there is no similar nearest vertex.

Theorem 2.17. [7] Assume that $u^2+ku+1 \equiv 0 \,(\mathrm{mod}\ N)$ and $u^2-lu+1 \equiv 0 (\mathrm{mod}\ N)$ for $1 \leqslant k, l \leqslant N$. If the suborbital graph $F_{u,N}$ is paired with itself, it is $k = l = N$ and otherwise $l = N - k$.

Theorem 2.18. [7] Assume that $u^2 - lu + 1 \equiv 0 \,(\mathrm{mod}\ N)$ and $1 < l \leqslant N$ in Farey graph.

i. The farthest vertex to which $\frac{u}{N}$ can be connected becomes $\frac{u-\frac{1}{l}}{N}$ and there is no similar nearest vertex.

ii. The farthest vertex to which $\frac{u-\frac{1}{l}}{N}$ can be connected becomes $\frac{u-\frac{1}{l-\frac{1}{l}}}{N}$ and there is no similar nearest vertex.

Corollary 2.19. [7] If $u^2 - u + 1 \equiv 0 \,(\mathrm{mod}\ N)$ then $F_{u,N}$ has a triangle as $\frac{1}{0} \leftarrow \frac{u-1}{N} \leftarrow \frac{u}{N} \leftarrow \frac{1}{0}$.

2.6. Lorentz Matrix Multiplication

In this section, we will investigate Lorentz matrix multiplication and related concepts, which have an important place in our study.

2.6.1. Lorentz Transform

Definition 2.20. [13] Linear transform $\theta : \mathbb{R}^n \to \mathbb{R}^n$ is a Lorentz transform $\Leftrightarrow \theta(x) \circ \theta(y) = x \circ y$ for all $x, y \in \mathbb{R}^n$.

Base $\{x_1, x_2, \ldots, x_n\} \in \mathbb{R}^n$ is orthonormal if and only if $x_1 \circ x_1 = 1$, for other cases $x_i \circ x_j = \delta_{ij}$.

Theorem 2.21. [13] Linear transform $\theta : \mathbb{R}^n \to \mathbb{R}^n$ is a Lorentz transform if and only if θ is linear and $\{\theta(e_1), \theta(e_2), \ldots, \theta(e_n)\}$ is an Lorentz orthonormal base of $\mathbb{R}^n$.

Assume that θ is linear and $\{\theta(e_1), \theta(e_2), \ldots, \theta(e_n)\}$ is an Lorentz orthonormal base of $\mathbb{R}^n$. Since θ Lorentz transform,

$$\begin{aligned}
\theta(x) \circ \theta(y) &= \theta(\textstyle\sum_{i=1}^n x_i e_i) \circ \theta(\sum_{j=1}^n x_j e_j) \\
&= (\textstyle\sum_{i=1}^n x_i \theta(e_i)) \circ (\sum_{j=1}^n y_j \theta(e_j)) \\
&= \textstyle\sum_{i=1}^n \sum_{j=1}^n x_i y_j \theta(e_i) \circ \theta(e_j) \\
&= -x_1 y_1 + x_2 y_2 + \ldots + x_n y_n = x \circ y
\end{aligned}$$

Definition 2.22. [13] $x, y \in \mathbb{R}^n$ is Lorentz orthogonal $\Leftrightarrow x \circ y = 0$.

2.6.2. Some Properties of Lorentz Matrix Multiplication

Assume that R_n^m be the set of matrices of type $m \times n$ and R_p^n be the set of matrices of type $n \times p$. Between the rows of the matrix $A = (a_{ij}) \in R_n^m$ and the columns of the matrix $B = (b_{jk}) \in R_p^n$, $A._L B = (-a_{i1} b_{1k} + \sum_{j=2}^n a_{ij} b_{jk})$ is defined with "$._L$" and this product is called the "Lorentz matrix product". $A._L B$ is a matrix of type $m \times p$. Besides, if we assume A_i as i^{th} row of A and B^j as j^{th} column of B, $\langle A_i, B^j \rangle_L$ is dot product $(i,j)^{th}$ of $A._L B$. L_n^m is denoted R_n^m that Lorentz matrix multiplication applied. $A._L B$ can be given more generally as follows:

$$A._L B = \begin{pmatrix} \langle A_1, B^1 \rangle & \ldots & \langle A_1, B^j \rangle \\ \vdots & \ddots & \vdots \\ \langle A_j, B^1 \rangle & \ldots & \langle A_j, B^j \rangle \end{pmatrix}$$

Theorem 2.23. [10] The following equations are obtained under Lorentz matrix multiplication.

i. For all $A \in L_n^m$, $B \in L_p^n$, $C \in L_r^p$, $A._L(B._L C) = (A._L B)._L C$

ii. For all $A \in L_n^m$, $B, C \in L_p^n$, $A._L(B + C) = A._L B + A._L C$

iii. For all $A, B \in L_n^m$, $C \in L_p^n$, $(A + B)._L C = A._L C + B._L C$

iv. For all $k \in \mathbb{R}$,$A \in L_n^m$, $B \in L_p^n$, $k(A._L B) = (kA)._L B = A._L(kB)$

Theorem 2.24. [10] The Lorentz unit matrix can be represented as

$$I._L = \begin{pmatrix} -1 & \ldots & 0 \\ \vdots & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & 0 \\ 0 & \ldots & 1 \end{pmatrix}_{n \times n}$$

Definition 2.25. [10] A is a matrix of type $n \times n$, if there is a B matrix of type $n \times n$ such that $A._L B = B._L A = I_n$, A is called reversible and denoted by A^{-1}.

Definition 2.26. [10] Transpose of $A = [a_{ij}] \in L_n^m$ demonstrations with A^T and define with $A^T = [a_{ji}] \in L_m^n$.

Definition 2.27. [10] If $A^{-1} = A^T$ for $A \in L_n^n$ matrix, A is called $L-$ orthogonal.

2.7. Pseudo Matrix Multiplication

Throughout this section $R^{m,n}$ is denoted the set of matrices of type $m \times n$. $R^{m,n}$ is a real vector space by addition and scalar multiplication. Each element of the matrix $A \cdot_v B$ is the inner product defined by Equation 5 where "$\cdot_v$" is the Pseudo matrix product between two matrices. The set of matrices defined pseudo matrix multiplication is denoted by $R_v^{m,n}$. $(i,j)^{th}$ Element of matrice's $A \cdot_v B$ is defined by

$$\langle x, y\rangle_v = -\sum_{j=1}^{v} a_{ij}b_{jk} + \sum_{j=v+1}^{n} a_{ij}b_{jk} \tag{5}$$

i. If $v = 0$ then it is equivalent to classic matrix multiplication.

$$\begin{aligned} \langle x, y\rangle_0 &= -\textstyle\sum_{j=1}^{0} a_{ij}b_{jk} + \sum_{j=0+1}^{n} a_{ij}b_{jk} \\ &= a_{i1}b_{1k} + a_{i2}b_{2k} + \ldots + a_{in}b_{nk} \end{aligned}$$

ii. If $v = 1$ then it is equivalent to Lorentz matrix multiplication.

$$\begin{aligned} \langle x, y\rangle_1 &= -\textstyle\sum_{j=1}^{1} a_{ij}b_{jk} + \sum_{j=1+1}^{n} a_{ij}b_{jk} \\ &= -a_{i1}b_{1k} + a_{i2}b_{2k} + \ldots + a_{in}b_{nk} \end{aligned}$$

in [14].

Theorem 2.28. [14] $det(A \cdot_v B) = (-1)^v det A \cdot det B$, for all $A, B \in R_v^{n,n}$.

Since it is equivalent to Lorentz matrix multiplication for $v = 1$,$det(A \cdot_1 B) = -det A \cdot det B$ is obtained.

2.8. Coordinate Transformations in Two Dimensional Lorentz Space

In this section, obtaining the Lorentz matrix using the displacement between two points in $\mathbb{R}^2$ is examined. Assume that $m(CAx) = \alpha$,$m(BAC) = \beta$, $m(BAx) = \theta$, $B(\sinh\theta, \cosh\theta)$ and $C(\sinh\alpha, \cosh\alpha)$. If point $C(x, y)$ is rotated counter clockwise around the origin by an angle of β, it becomes point $B(x^{'}, y^{'})$.Since the coordinates of point C are taken as $x = r\sinh\alpha$ and $y = r\cosh\alpha$, the coordinates of point B are written as $x^{'} = r\sinh(\alpha+\beta)$ and $y^{'} = r\cosh(\alpha+\beta)$.

$$\begin{aligned} x^{'} &= r\sinh(\alpha+\beta) \\ &= r(\sinh\alpha\cosh\beta + \sinh\beta\cosh\alpha) \\ &= r\sinh\alpha\cosh\beta + r\sinh\beta\cosh\alpha \\ &= x\cosh\beta + y\sinh\beta \end{aligned}$$

$$\begin{aligned} y^{'} &= r\cosh(\alpha+\beta) \\ &= r(\cosh\alpha\cosh\beta + \sinh\alpha\sinh\beta) \\ &= r\cosh\alpha\cosh\beta + r\sinh\alpha\sinh\beta \\ &= y\cosh\beta + x\sinh\beta \\ &= x\sinh\beta + y\cosh\beta \end{aligned}$$

are obtained.

The trigonometric expansions of $x^{'}$ and $y^{'}$ can be rewritten in matrix form as follows:

$$\begin{pmatrix} x^{'} \\ y^{'} \end{pmatrix} = \begin{pmatrix} x\cosh\beta + y\sinh\beta \\ x\sinh\beta + y\cosh\beta \end{pmatrix} = \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

If this matrix product is written according to the Lorentz matrix multiplication

$$\begin{pmatrix} x^{'} \\ y^{'} \end{pmatrix} = \begin{pmatrix} x\cosh\beta + y\sinh\beta \\ x\sinh\beta + y\cosh\beta \end{pmatrix} = \begin{pmatrix} -\cosh\beta & \sinh\beta \\ -\sinh\beta & \cosh\beta \end{pmatrix} \cdot_L \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} -\cosh\beta & \sinh\beta \\ -\sinh\beta & \cosh\beta \end{pmatrix} \in L_2^2$$

matrix is obtained [10].

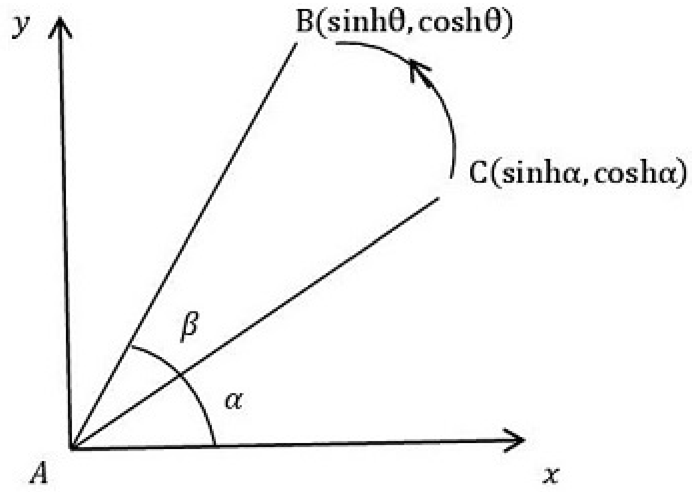

Fig. 1. Rotation diagram between two points

3. Main Results

3.1. Obtaining Vertices of a Suborbital Graph $F_{u,N}$ Under Lorentz Matrix Multiplication

In this section, we examine that the Lorentz matrix in Equation 6 that gives the vertices obtained under the classical matrix multiplication in the suborbital graph $F_{u,N}$ under the Lorentz matrix multiplication and see that the Lorentz matrix in Equation 6 is not a member of the Modular group. In Subsection 2.8, obtaining the Lorentz matrix using the displacement between two points in $\mathbb{R}^2$ was examined. We know that from Equation 4

$$v_q = \begin{pmatrix} -u & \frac{u^2+ku+1}{N} \\ -N & u+k \end{pmatrix}^q (v_0)$$

From Subsection 2.8, the Lorentz matrix giving the same vertices on the path of minimal length can be given by

$$\begin{pmatrix} u & \frac{u^2+ku+1}{N} \\ N & u+k \end{pmatrix} \in L_2^2 \tag{6}$$

From here, the vertices of the path with minimal length are provided as follows:

$$v_q = \begin{pmatrix} u & \frac{u^2+ku+1}{N} \\ N & u+k \end{pmatrix}^q \cdot_L (v_0)$$

for all $q \in \mathbb{Z}^+$,where $v_0 = \frac{u}{N}$.

Corollary 3.1. Lorentz matrix given in Equation 6 is not member of Modular group.

PROOF.

$$\begin{aligned} \begin{vmatrix} u & \frac{u^2+ku+1}{N} \\ N & u+k \end{vmatrix} &= u(u+k) - N(\tfrac{u^2+ku+1}{N}) \\ &= u^2 + uk - u^2 - uk - 1 \\ &= -1 \end{aligned}$$

□

Since the determinant is -1 in Corollary 3.1., we can normalize the relevant matrix. The relevant matrix can be written as the Möbius transform as follows:

$$m(z) = \frac{uz + \frac{u^2+ku+1}{N}}{Nz + u + k}$$

For $\alpha \in \widehat{\mathbb{C}}$,

$$m(z) = \frac{\alpha uz + \alpha\frac{u^2+ku+1}{N}}{\alpha Nz + \alpha(u + k)}$$

$$\begin{aligned} det(m(z)) &= \alpha u\alpha(u+k) - \alpha N(\alpha\frac{u^2+ku+1}{N}) \\ &= \alpha^2 u(u+k) - \alpha^2(u^2+ku+1) \\ &= \alpha^2u^2 + \alpha^2uk - \alpha^2u^2 - \alpha^2uk - \alpha^2 \\ &= 1 \end{aligned}$$

$\alpha^2 = -1, \alpha = \mp i$ If $\alpha = i$, then

$$m(z) = \frac{iuz + i\frac{u^2+ku+1}{N}}{iNz + i(u + k)}$$

Möbius transform can be writtten for $i \in \widehat{\mathbb{C}}$,

$$\begin{pmatrix} ui & \frac{u^2+ku+1}{N}i \\ Ni & (u+k)i \end{pmatrix}$$

as an element of Modular group.Similar operations can be done for $\alpha = -i$.

From here, the type of Möbius transformation can be determined. Trace of Möbius transformation $m(z) = \frac{az+b}{cz+d}$ can be written as $\tau(m) = (a + d)^2$. From the above matrix,

$$\begin{aligned} \tau(m) &= (a+d)^2 \\ &= (ui + i(u+k))^2 \\ &= (ui)^2 + 2uii(u+k) + (i(u+k))^2 \\ &= -u^2 - 2u^2 - 2uk + (-u^2 - 2uk - k^2) \\ &= -4u^2 - 4uk - k^2 = -(2u+k)^2 \end{aligned}$$

trace is obtained. m is elliptic since $\tau(m) = 0$ real for $u = -\frac{k}{2}$ when $k \geqslant 2$,$k \in \mathbb{Z}$ and u are arbitrary and m is loxodromic for $u \neq -\frac{k}{2}$ and $\tau(m)$ is real.

Corollary 3.2. Assume that $u^2 + ku + 1 \equiv 0 \pmod N$ under Lorentz multiplication in Farey graph provided for $(u, N) = 1$ and $k \geqslant 2$,$k \in \mathbb{Z}$. Under Lorentz matrix multiplication, i and ii are provided for the vertices obtained by the matrix given in [7].

i. The farthest vertex to which $\frac{u}{N}$ can be connected becomes $\frac{u+\frac{1}{k}}{N}$ and there is no similar nearest vertex.

ii. The farthest vertex to which $\frac{u+\frac{1}{k}}{N}$ can be connected becomes $\frac{u+\frac{1}{k-\frac{1}{k}}}{N}$ and there is no similar nearest vertex.

Since the vertices obtained under Lorentz matrix multiplication in the suborbital graph $F_{u,N}$ with vertices obtained in Theorem 2.16. are the same, then the proof of this Corollary is the same with proof of Theorem 2.16.

Corollary 3.3. For $(u, N) = 1$ and $k \in \mathbb{Z}$, where $u^2 + ku - 1 \equiv 0 \,(\text{mod } N)$,

$$\begin{pmatrix} u & \frac{u^2+ku-1}{N} \\ N & u+k \end{pmatrix} \in \Gamma_0(N) \tag{7}$$

provides under Lorentz matrix multiplication,

$$\infty = \frac{1}{0} \rightarrow \frac{u}{N} \rightarrow \frac{u-\frac{1}{k}}{N} \rightarrow \frac{u-\frac{1}{k+\frac{1}{k}}}{N} \rightarrow \frac{u-\frac{1}{k+\frac{1}{k+\frac{1}{k}}}}{N} \rightarrow \dots$$

path in suborbital graph $F_{u,N}$.

Proof. Here the first four vertices of the path are found, the other vertices are obtained in a similar way.

$$\begin{pmatrix} u & \frac{u^2+ku-1}{N} \\ N & u+k \end{pmatrix} \cdot_L \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} u \\ N \end{pmatrix}$$

$$\begin{pmatrix} u & \frac{u^2+ku-1}{N} \\ N & u+k \end{pmatrix} \cdot_L \begin{pmatrix} u \\ N \end{pmatrix} = \begin{pmatrix} ku-1 \\ Nk \end{pmatrix} = \begin{pmatrix} u-\frac{1}{k} \\ N \end{pmatrix}$$

$$\begin{pmatrix} u & \frac{u^2+ku-1}{N} \\ N & u+k \end{pmatrix} \cdot_L \begin{pmatrix} ku-1 \\ Nk \end{pmatrix} = \begin{pmatrix} k^2u+u-k \\ Nk^2+N \end{pmatrix} = \begin{pmatrix} u-\frac{1}{k+\frac{1}{k}} \\ N \end{pmatrix}$$

$$\begin{pmatrix} u & \frac{u^2+ku-1}{N} \\ N & u+k \end{pmatrix} \cdot_L \begin{pmatrix} k^2u+u-k \\ Nk^2+N \end{pmatrix} = \begin{pmatrix} k^3u-k^2+2uk-1 \\ Nk^3+2Nk \end{pmatrix} = \begin{pmatrix} u-\frac{1}{k+\frac{1}{k+\frac{1}{k}}} \\ N \end{pmatrix}$$

□

Example 3.4. If $u = 1$, $N = 5$ and $k = 3$, from Corollary 3.3.

$$\infty = \frac{1}{0} \rightarrow \frac{1}{5} \rightarrow \frac{2}{15} \rightarrow \frac{7}{50} \rightarrow \frac{1-\frac{1}{3+\frac{1}{3+\frac{1}{3}}}}{5} \rightarrow \dots$$

is obtained.

Corollary 3.5. Assume that $u^2 + ku - 1 \equiv 0 \,(\text{mod } N)$ under Lorentz multiplication in Farey graph provided for $(u, N) = 1$ and $k \in \mathbb{Z}$. Under Lorentz matrix multiplication, i and ii are provided for the vertices obtained by the matrix given in Equation 7.

i. The farthest vertex to which $\frac{u}{N}$ can be connected becomes $\frac{u-\frac{1}{k}}{N}$ and there is no similar nearest vertex.

ii. The farthest vertex to which $\frac{u-\frac{1}{k}}{N}$ can be connected becomes $\frac{u-\frac{1}{k+\frac{1}{k}}}{N}$ and there is no similar nearest vertex.

Proof. *i.* Assume that $\frac{u-\frac{r}{m}}{N}$ be the farthest vertex that can be connected with $\frac{u}{N}$, where $\frac{u}{N}$ is a vertex in $F_{u,N}$ under Lorentz multiplication. Hence

$$\frac{u}{N} \rightarrow \frac{u-\frac{r}{m}}{N} = \frac{um-r}{mN}$$

is obtained. $um - r \equiv u^2 \,(\text{mod } N)$,$umN - N(um - r) = N$ must be provided for edge condition. Thus,

$$umN - N(um - r) = N, umN - Num + Nr = N \Rightarrow r = 1$$

If the value of $r = 1$ is substituted in the congruence equation,

$$um - 1 \equiv u^2 \, (\text{mod } N)$$

is obtained. Then $-u^2 + um - 1 \equiv 0 \, (\text{mod } N)$. If equations $-u^2 + um - 1 \equiv 0 \, (\text{mod } N)$ and $u^2 + ku - 1 \equiv 0 \, (\text{mod } N)$ are added, $(um-1) + (uk-1) \equiv 0 \, (\text{mod } N)$. From here, $m \equiv k \, (\text{mod } N)$ is reached. As a result,

$$m = k + Nx, x \in \mathbb{N} \cup \{0\}$$

Thus, $\frac{r}{m} = \frac{1}{k+Nx}$. Here, a function can be defined as follows:

$$f(x) = \frac{u - \frac{r}{m}}{N} = \frac{u - \frac{1}{k+Nx}}{N}, f : \mathbb{N} \cup \{0\} \to \mathbb{R}$$

If the derivative of the function is taken, it is seen that it is strictly increasing.

$$f^{'}(x) = \frac{1}{(k + Nx)^2} > 0$$

Since it is a strictly increasing function, it takes the minimum value for $x = 0$.If the value of $x = 0$ is written in the relevant function, it becomes $f(0) = \frac{u - \frac{1}{k}}{N}$. It is obvious that $(uk - 1, kN) = 1$. Consequently, $\frac{u-\frac{1}{k}}{N}$ is a vertex at $F_{u,N}$ and is the farthest vertex to which $\frac{u}{N}$ can connect.

ii. Assume that $\frac{u-\frac{r}{m}}{N}$ be the farthest vertex that can be connected with $\frac{u-\frac{1}{k}}{N}$, where $\frac{u-\frac{1}{k}}{N}$ is a vertex in $F_{u,N}$ under Lorentz multiplication. Hence

$$\frac{u - \frac{1}{k}}{N} = \frac{uk - 1}{kN} \to \frac{u - \frac{r}{m}}{N} = \frac{um - r}{mN}$$

is obtained. $um - r \equiv -u(uk - 1) \, (\text{mod } N)$,$(uk - 1)mN - kN(um - r) = -N$ must be provided for edge condition. So, $(uk - 1)mN - kN(um - r) = -N, ukmN - mN - knuM + kNr = -N, -mN + kNr = -N, -m + kr = -1 \Rightarrow m = kr + 1$. If the value of $m = kr + 1$ is substituted in the congruence equation, $u(kr+1) - r \equiv -u(uk-1) \, (\text{mod } N)$ is obtained. Then $u(kr+1) - r \equiv -u(uk - 1) \, (\text{mod } N)$, $ukr + u - r + u^2k - u \equiv 0 \, (\text{mod } N)$ is provided for $ukr + u - r \equiv -u^2k + u \, (\text{mod } N)$. From here, $(uk - 1)r + u^2k \equiv 0 \, (\text{mod } N)$ and $uk - 1 \equiv -u^2 \, (\text{mod } N)$ are reached.

$$(uk - 1)r + u^2k \equiv 0 \, (\text{mod } N), -u^2r + u^2k \equiv 0 \, (\text{mod } N)$$

$$-r + k \equiv 0 \, (\text{mod } N), r \equiv k \, (\text{mod } N), r = k + Nx, x \in \mathbb{N} \cup \{0\}$$

Hence, $\frac{r}{m} = \frac{k+Nx}{k(k+Nx)+1}$. Here, a function can be defined as follows:

$$\begin{aligned} f(x) &= \frac{u - \frac{k+Nx}{k(k+Nx)+1}}{N} \\ &= \frac{u(k(k+Nx)+1) - (k+Nx)}{N(k(k+Nx)+1)} \\ &= \frac{uk^2 + ukNx + u - k - Nx}{Nk^2 + kN^2x + N}, f : \mathbb{N} \cup \{0\} \to \mathbb{R} \end{aligned}$$

If the derivative of the function is taken, it is seen that it is strictly decreasing.

$$f^{'}(x) = \frac{-1}{(k^2 + kNx + 1)^2} < 0$$

Since it is a strictly decreasing function, it takes the maximum value for $x = 0$. If the value of $x = 0$ is written in the relevant function, it becomes $f(0) = \frac{u - \frac{1}{k + \frac{1}{k}}}{N}$. We have to show that $(uk^2 + u - k, k^2 + 1) = 1$. Assume that $(u(k^2 + 1) - k, k^2 + 1) = g$. From here, $g \setminus k^2 + 1$ and

$g \setminus (u(k^2+1))$. Then $g \setminus (u(k^2+1)) - k$, $g \setminus -k$. Hence $g \setminus k^2+1$, $g=1$. We have to demonstrate that $(uk^2+u-k, N) = 1$. For $(u(k^2+1)-k, N) = z$, $u(k^2+1)-k = k(uk-1)+u \equiv 0(\text{mod } z)$ and $N \equiv 0(\text{mod } z)$ are obtained. So, $uk-1 \equiv -u^2\,(\text{mod } N)$. Then, $k(uk-1)+u = k(-u^2)+u \equiv 0(\text{mod } z)$, $u(-ku+1) \equiv 0(\text{mod } z)$, $u \equiv 0(\text{mod } z)$ or $-ku+1 \equiv 0(\text{mod } z)$. This is a contradiction. Hence $z=1$.

Consequently, $\frac{u-\frac{1}{k+\frac{1}{k}}}{N}$ is a vertex at $F_{u,N}$ and is the farthest vertex to which $\frac{u-\frac{1}{k}}{N}$ can connect. Since $f(0) = \frac{u-\frac{1}{k+\frac{1}{k}}}{N}$ is, there is no nearest vertex to which $\frac{u-\frac{1}{k}}{N}$ can connect.

□

Corollary 3.6. For $(u, N) = 1$,where $u^2+u-1 \equiv 0\,(\text{mod } N)$ for $k=1$, $\begin{pmatrix} u & \frac{u^2+u-1}{N} \\ N & u+1 \end{pmatrix} \in \Gamma_0(N)$ provides under Lorentz matrix multiplication,

$$\infty = \frac{1}{0} \to \frac{u}{N} \to \frac{u-1}{N} \to \frac{u-\frac{1}{2}}{N} \to \frac{u-\frac{2}{3}}{N} \to \ldots \to \frac{u-\frac{F_n}{F_{n+1}}}{N} \to \ldots$$

path in suborbital graph $F_{u,N}$.

Proof. $(n+1)^{th}$ vertex is obtained as follows where n^{th} vertex is $\frac{u-\frac{F_n}{F_{n+1}}}{N}$:

$$\begin{aligned}
\begin{pmatrix} u & \frac{u^2+u-1}{N} \\ N & u+1 \end{pmatrix} \cdot_L \begin{pmatrix} u-\frac{F_n}{F_{n+1}} \\ N \end{pmatrix} &= \begin{pmatrix} -u(u-\frac{F_n}{F_{n+1}}) + N(\frac{u^2+u-1}{N}) \\ -N(u-\frac{F_n}{F_{n+1}}) + N(u+1) \end{pmatrix} \\
&= \begin{pmatrix} -u^2 + u\frac{F_n}{F_{n+1}} + u^2 + u - 1 \\ -Nu + N\frac{F_n}{F_{n+1}} + Nu + N \end{pmatrix} \\
&= \begin{pmatrix} u\frac{F_n}{F_{n+1}} + u - 1 \\ N\frac{F_n}{F_{n+1}} + N \end{pmatrix} \\
&= \begin{pmatrix} uF_{n+2} - F_{n+1} \\ NF_{n+2} \end{pmatrix} \\
&= \begin{pmatrix} u - \frac{F_{n+1}}{F_{n+2}} \\ N \end{pmatrix}
\end{aligned}$$

□

Example 3.7. If $u=2$, $N=5$ and $k=1$, from Corollary 3.6.,

$$\infty = \frac{1}{0} \to \frac{2}{5} \to \frac{1}{5} \to \frac{3}{10} \to \cdots \to \frac{2-\frac{F_{n+1}}{F_{n+2}}}{5} \to \ldots$$

is obtained.

Corollary 3.8. For $(u, N) = 1$, where $u^2+u-1 \equiv 0\,(\text{mod } N)$ for $k=1$ and α is golden ratio, $\begin{pmatrix} u & \frac{u^2+u-1}{N} \\ N & u+1 \end{pmatrix} \in \Gamma_0(N)$ provides value of vertex as $\frac{u-\frac{1}{\alpha}}{N}$ for $n \to \infty$ under Lorentz matrix multiplication in suborbital graph $F_{u,N}$.

Proof. From Corollary 3.6.,

$$\begin{aligned}
\lim_{n\to\infty} \frac{u-\frac{F_n}{F_{n+1}}}{N} &= \frac{u-\lim_{n\to\infty}\frac{F_n}{F_{n+1}}}{N} \\
&= \frac{u-\frac{1}{\lim_{n\to\infty}\frac{F_{n+1}}{F_n}}}{N} \\
&= \frac{u-\frac{1}{\alpha}}{N}
\end{aligned}$$

□

Example 3.9. If $u = 2$, $N = 5$, $k = 1$, and $\alpha = 1.618$ from Corollary 3.8.,

$$\frac{2-\frac{1}{1.618}}{5} = 0,276$$

is obtained.

4. Conclusion

In this study, we especially examined suborbital graphs obtained by Lorentz matrix multiplication. It is seen that Lorentz matrix which gave vertices that are obtained by classical matrix multiplication is not an element of Modular group Γ. Moreover, we defined a matrix that is an element of Modular group Γ. Furthermore, we investigated vertices, edges and path obtained under Lorentz matrix multiplication by this matrix. It is indicated that vertices on the path provide the minimal length condition. The vertices are associated with Fibonacci numbers for $k = 1$ and value of vertex is found for $n \to \infty$.

Authors Contributions

All the authors declare that they contributed equally and adequately to this paper. They all read and approved the last version of the paper. This study was derived from the first author's PhD dissertation supervised by the second author.

Conflict of Interest

All the authors declare that they have no conflict of interest.

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