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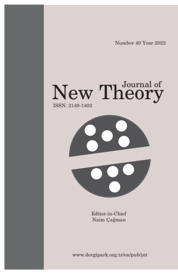

Some New Estimates for Maximal Commutator and Commutator of Maximal Function in $L_{p,\lambda}(\Gamma)$

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Abstract — The theory of boundedness of classical operators of real analyses on Morrey spaces defined on Carleson curves has made significant progress in recent years as it allows for various applications. This study obtains new estimates about the boundedness of the maximal commutator operator M_b and the commutator of the maximal function $[M, b]$ in Morrey spaces defined on Carleson curves.

Keywords — *Maximal commutator, commutator of maximal function, Morrey space, BMO*

Mathematics Subject Classification (2020) — 42B25, 42B35

1. Introduction

$L_{p,\lambda}(\mathbb{R}^n)$ Morrey spaces, proposed by Morrey in 1938, were used in the study of the local behavior of the solutions of elliptic partial differential equations. Morrey spaces are quite used for problems in the analysis of variations theory. In addition, Navier-Stokes and Schrödinger's equations have many applications in the potential theory of elliptic problems with discontinuous coefficients. Over the years, many researchers have done various studies on Morrey spaces (For more details, see [1–5]).

In recent years, there has been an increasing interest in various spaces on Carleson curves, such as Lebesgue spaces, Morrey spaces. We only mention [3, 6–9]. There are many applications for Maximal operators, which have a capital place in real and fractional analysis. These nonlinear operators, which are informative in differentiation theory, have inspired the studies of classical operators of harmonic analysis in various spaces. Moreover, they are also featured in many valuable studies (see, for example, [7, 10–12]). Samko [5] studied the boundedness of the M maximal operator in Morrey spaces defined on quasi-metric measure spaces, especially in $L_{p,\lambda}(\Gamma)$ Morrey spaces defined on Carleson curves.

The commutator operation and properties of maximal integrals have been extensively studied in various spaces, and there are many important consequences associated with them (see, for example [13–22]). The maximal commutator M_b is of great importance in studying the commutators of the BMO symbol and singular integral operators. There are important studies on the properties and boundedness of M_b (For more details, see [13, 14, 18, 19]). The commutator of maximal operator $[M, b]$ was studied by many authors (For more details, see [13–15, 18]). This operator is the product of two functions from BMO and H^1 Hardy space. It emerged when it was wanted to give meaning (Let us note that the product of these two functions may not be locally integrable).

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The fact that the operator M_b is a positive linear operator, unlike the $[M,b]$ operator, indicates that these operators are quite different from each other. However, b is bound to the operator M_b manage the operator $[M,b]$, with some additional conditions.

This study consists of three sections. Section 2 presents preliminaries of the commutator of the maximal operator and maximal commutator on Carleson curves. Section 3 investigates the boundedness of these operators on $L_{p,\lambda}(\Gamma)$.

2. Preliminaries

In this section, before giving the basic definitions and theorems, let us briefly talk about Carleson curves, which are the focus of the paper. For every $t \in \Gamma$ and $r > 0$, if a locally rectifiable Jordan curve Γ satisfies the following condition;

$$\nu\Gamma(t,r) \leq c_0 r$$

it is called the Carleson curve (regular curve), where $\Gamma(t,r) := \Gamma \cap B(t,r)$, $t \in \Gamma$, $r > 0$. Let $\Gamma: \ \gamma(t) = x(t) + iy(t)$, $a \leq t \leq b$. Length of a regular curve is given with:

$$\nu\Gamma = \ell(\gamma) = \int_\gamma |dz| = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}dt$$

Let Γ, equipped with Lebesgue length measure, be a locally rectifiable compound arc. The measure of a measurable subset $\gamma \subset \Gamma$ is displayed with $\gamma \subset \Gamma$. In particular, $|\Gamma(t,r)|$ is the sum of the lengths of the countably many arcs that make up $\Gamma(t,r)$. Let Γ be a locally rectifiable composite curve and $\Gamma_1, \ldots, \Gamma_N$ be a finite number of arcs combined such that $\Gamma = \Gamma_1 \cup \ldots \cup \Gamma_N$. Since

$$\Gamma_j(t,\varepsilon) \subset \Gamma(t,\varepsilon) \subset \Gamma_1(t,\varepsilon) \cup \ldots \cup \Gamma_N(t,\varepsilon)$$

and

$$|\Gamma_j(t,\varepsilon)| \leq |\Gamma(t,\varepsilon)| \leq |\Gamma_1(t,\varepsilon)| + \ldots + |\Gamma_N(t,\varepsilon)|$$

from the above expression

$$\Gamma \text{ is a Carleson curve} \Leftrightarrow \text{Each } \Gamma_j \text{ is a Carleson curve}$$

is obtained. In order to indicate the dependence on t, we shall denote this family by $\{\Gamma_{t,j}\}$. For almost every $\xi \notin \Gamma_{t,j}$ is $f(\xi) \leq t$. Now, we need to give below the necessary definitions for the case of spaces on Carleson curves.

Definition 2.1. [5] $L_{p,\lambda}(\Gamma)$ Morrey spaces with $0 \leq \lambda < 1$, $1 \leq p < \infty$ and $f \in L_p^{loc}(\Gamma)$ is the space of functions such that

$$\begin{aligned} \|f\|_{L_{p,\lambda}(\Gamma)} &= \sup_{r>0,\ t\in\Gamma} r^{-\frac{\lambda}{p}} \|f\|_{L_p(\Gamma(t,r))} \\ &= \sup_{r>0,\ t\in\Gamma} \left(r^{-\lambda} \int_{\Gamma(t,r)} |f(\tau)|^p \, d\nu(\tau) \right)^{1/p} < \infty \end{aligned}$$

When $\lambda < 0$ or $\lambda > 1$, $L_{p,\lambda}(\Gamma) = \theta$ where θ denotes the set of functions on Γ that is equivalent to 0. İf $\lambda = 0$, then $L_{p,0}(\Gamma) = L_p(\Gamma)$ is obtained.

Definition 2.2. [8] The space of functions with bounded mean oscillation $BMO(\Gamma)$ is defined as the set of locally integrable functions f with a finite norm

$$\|f\|_{BMO(\Gamma)} = \sup_{r>0, t\in\Gamma} (v\Gamma(t,r))^{-1} \int_{\Gamma(t,r)} \left| f(\tau) - f_{\Gamma(t,r)} \right| dv(\tau) < \infty$$

here $f_{\Gamma(t,r)}$ is displayed with;

$$f_{\Gamma(t,r)} := (v\Gamma(t,r))^{-1} \int_{\Gamma(t,r)} f(\tau) \, dv(\tau)$$

Definition 2.3. [8] Let $1 \leq p < \infty$ and Γ be a Carleson curve. Then,

$$L_{\infty}(\Gamma) = \sup_{t\in\Gamma, r>0} r^{-\frac{1}{p}} \|f\|_{L_p(\Gamma)}$$

and

$$\|f\|_{L_{\infty}(\Gamma)} \leq \sup_{t\in\Gamma, r>0} r^{-\frac{1}{p}} \|f\|_{L_p(\Gamma)} \leq c_0^{1/p} \|f\|_{L_{\infty}(\Gamma)}$$

The application of the Lebesgue Differentiation Theorem on Carleson curves is as follows:
Let $f \in L_1^{loc}(\Gamma)$, then the following statement applies

$$\lim_{r\to 0} (\nu\Gamma(t,r))^{-1} \int_{\Gamma(t,r)} f(\tau) d\nu(\tau) = f(t) \tag{1}$$

Note that in expression (1) supremum instead of limit and $|f|$ instead of function f is taken, the maximal function is defined.

The definition of the maximal operator on Carleson curves is as follows;

Definition 2.4. [5] Let Γ be a simple Carleson curve and $f \in L_1^{loc}(\Gamma)$. The maximal operator M on Γ defined by

$$Mf(t) = \sup_{t>0} (v\Gamma(t,r))^{-1} \int_{\Gamma(t,r)} |f(\tau)| dv(\tau)$$

The boundedness of the maximal function on $L_p(\Gamma)$ has been studied by Guliyev in [3].

Theorem 2.5. [3] Let Γ be a Carleson curve, $1 \leq p < \infty$ and $t_0 \in \Gamma$. Then, for $p > 1$ and any $r > 0$ in Γ, the inequality

$$\|Mf\|_{L_p(\Gamma(t_0,r))} \lesssim r^{\frac{1}{p}} \sup_{\tau>2r} \tau^{-\frac{1}{p}} \|f\|_{L_p(\Gamma(t_0,r))}$$

holds for all $f \in L_p^{loc}(\Gamma)$.

The definitions of the maximal commutator and commutator of maximal operator on Carleson curves are as follows, respectively.

Definition 2.6. [9] Given a locally integrable function b, the maximal commutator is defined by

$$M_b(f)(t) := \sup_{t>0} \frac{1}{v\Gamma(t,r)} \int_{\Gamma(t,r)} |b(t) - b(\tau)||f(\tau)| dv(\tau), \ \ for\ all\ t \in \Gamma$$

Definition 2.7. [9] Given a locally integrable function b, the commutator of the maximal operator is defined by

$$[M,b] f(t) := M(bf)(t) - b(t)Mf(t) \text{ for all } t \in \Gamma$$

Türkay and Mursaleen proved the following statement in [9].

Theorem 2.8. [9] Let $b \in BMO(\Gamma)$ and $0 < \delta < 1$. In this case, there is a positive constant $C = C_\delta$ where the following inequalities holds for all $f \in L_1^{loc}(\Gamma)$;

$$i) M_\delta(M_b(f))(\varsigma) \leq C \|b\|_{BMO(\Gamma)} M^2 f(\varsigma), \ \varsigma \in \Gamma$$

$$ii) M_b(f)(\varsigma) \leq C \|b\|_* M^2 f(\varsigma), \ \varsigma \in \Gamma$$

Lemma 2.9. [9] Let b be any non-negative locally integrable function on Γ. Then for all $f \in L_1^{loc}(\Gamma)$, the following inequalities are provided;

$$|[M,b] f(t)| \leq M_b(f)(t), t \in \Gamma \tag{2}$$

and

$$|[M,b] f(t)| \leq M_b(f)(t) + 2b^-(t)Mf(t), \ t \in \Gamma \tag{3}$$

Theorem 2.10. [9] Let $b \in BMO(\Gamma)$ such that $b^- \in L_\infty(\Gamma)$. Then, there exists a positive constant C such that

$$|[M,b] f(t)| \leq C \left(\left\|b^+\right\|_* + \left\|b^-\right\|_\infty\right) M^2 f(t) \ \text{ for all } f \in L_1^{loc}(\Gamma)$$

3. Some New Estimates

This section studies the $L_{p,\lambda}(\Gamma)$ −boundedness of the operator M_b and the operator $[M,b]$. Since it is easier to examine the boundedness of the operator M_b than the operator $[M,b]$, we will first investigate the boundedness of the operator M_b. Before the main results are given, the auxiliary theorems that will help in the proof will be reminded. Türkay and Murselen examined the boundedness of the M_b operator in $L_p(\Gamma)$ in [9].

Theorem 3.1. [9] Let $1 < p < \infty$. The operator M_b is bounded on $L_p(\Gamma)$ if and only if $b \in BMO(\Gamma)$.

Inspired by Theorem 3.1, we establish the following theorem.

Theorem 3.2. Let $1 < p < \infty$, $0 \le \lambda \le 1$. $b \in BMO(\Gamma)$ if and only if The operator M_b is bounded on $L_{p,\lambda}(\Gamma)$.

Proof. (⇒) Let $\Gamma(t_0, r_0)$ be a constant Carleson curve. Suppose that $b \in \mathrm{BMO}(\Gamma)$. By Theorem **??** and Theorem 2.5 the following inequality holds for $f \in L_p(\Gamma)$:

$$\begin{aligned} ||M_b(f)\|_{L_{p,\lambda}(\Gamma)} &\approx \sup_{\Gamma(t,r)} \left(|\Gamma(t,r)|^{\lambda-1} \int\limits_{\Gamma(t,r)} M_b(f)(\tau)\, dv(\tau) \right)^{\frac{1}{p}} \\ &= |\Gamma(t_0,r_0)|^{\frac{\lambda-1}{p}} \sup_{\Gamma(t,r)\subset\Gamma(t_0,r_0)} \left(\int\limits_{\Gamma(t,r)} M_b(f)(\tau)\, dv(\tau) \right)^{\frac{1}{p}} \\ &\le |\Gamma(t_0,r_0)|^{\frac{\lambda-1}{p}} \|b\|_* \|f\|_{L_p(\Gamma)} \end{aligned}$$

(⇐) Let the operator M_b be bounded on $L_{p,\lambda}(\Gamma)$, that is, for every $f \in L_{p,\lambda}(\Gamma)$ there is such a positive constant c that the following inequality is obtained;

$$\|M_b(f)\|_{L_{p,\lambda}(\Gamma)} \le c\|f\|_{L_{p,\lambda}(\Gamma)}$$

Obviously,

$$\|f\|_{L_{p,\lambda}(\Gamma)} \approx \sup_{\Gamma(t,r)} \left(|\Gamma(t,r)|^{\lambda-1} \int\limits_{\Gamma(t,r)} |f(\tau)|^p dv(\tau) \right)^{\frac{1}{p}}$$

Let $f = \chi_{\Gamma(t_0,r_0)}$ such that $\Gamma(t_0, r_0)$ is a constant Carleson curve. In this case, the following expression is easily written;

$$\begin{aligned} ||\chi_{\Gamma(t_0,r_0)}\|_{L_{p,\lambda}(\Gamma)} &\approx \sup_{\Gamma(t,r)} \left(|\Gamma(t,r)|^{\lambda-1} \int\limits_{\Gamma(t,r)} x_{\Gamma(t_0,r_0)}(\tau)\, dv(\tau) \right)^{\frac{1}{p}} \\ &= \sup_{\Gamma(t,r)} \left(|\Gamma(t,r) \cap \Gamma(t_0,r_0)|\, |\Gamma(t,r)|^{\lambda-1} \right)^{\frac{1}{p}} \\ &= \sup_{\Gamma(t,r)\subset\Gamma(t_0,r_0)} \left(|\Gamma(t,r)|\, |\Gamma(t,r)|^{\lambda-1} \right)^{\frac{1}{p}} \\ &= |\Gamma(t_0,r_0)|^{\frac{\lambda}{p}} \end{aligned} \tag{4}$$

In addition, since

$$M_b\left(\chi_{\Gamma(t_0,r_0)}\right)(t) \gtrsim \frac{1}{|\Gamma(t_0,r_0)|} \int\limits_{\Gamma(t_0,r_0)} \left|b(\tau) - b_{\Gamma(t_0,r_0)}\right| dv(\tau), \text{ for all } t \in \Gamma(t_0,r_0)$$

then

$$\begin{aligned} \left\|M_b\left(\chi_{\Gamma(t_0,r_0)}\right)\right\|_{L_{p,\lambda}(\Gamma)} &\approx \sup_{\Gamma(t,r)} \left(|\Gamma(t,r)|^{\lambda-1} \int\limits_{\Gamma(t,r)} \left|M_b\left(\chi_{\Gamma(t_0,r_0)}\right)(\tau)\right|^p dv(\tau) \right)^{\frac{1}{p}} \\ &\gtrsim |\Gamma(t_0,r_0)|^{\frac{\lambda}{p}} \frac{1}{|\Gamma(t_0,r_0)|} \int\limits_{\Gamma(t_0,r_0)} \left|b(\tau) - b_{\Gamma(t_0,r_0)}\right| dv(\tau) \end{aligned} \tag{5}$$

Since by assumption

$$\left\|M_b\left(\chi_{\Gamma(t_0,r_0)}\right)\right\|_{L_{p,\lambda}(\Gamma)} \lesssim \left\|\chi_{\Gamma(t_0,r_0)}\right\|_{L_{p,\lambda}(\Gamma)}$$

and by (4) and (5), we get that

$$\frac{1}{|\Gamma\left(t_0,r_0\right)|}\int_{\Gamma(t_0,r_0)} |b\left(\tau\right)-b_{\Gamma(t_0,r_0)}|dv\left(\tau\right) \leq c \tag{6}$$

Thus, the desired result is obtained. □

Milman and Schonbek obtained the $L_p\left(\mathbb{R}^n\right)$-boundedness of the operator $[M,b]$ with real interpolation techniques in [20]. Inspired by this study, Türkay and Murselen examined the boundedness of the operator $[M,b]$ in $L_p\left(\Gamma\right)$ in [9].

Theorem 3.3. [9] Let $1<p<\infty$. The operator $[M,b]$ is bounded on $L_p\left(\Gamma\right)$ if and only if $b\in BMO\left(\Gamma\right)$ and $b^-\in L_\infty\left(\Gamma\right)$.

Applying Theorem 3.3, the below theorem is obtained.

Theorem 3.4. Let $1<p<\infty$, $0\leq\lambda\leq 1$. $b\in BMO\left(\Gamma\right)$ such that $b^-\in L_\infty\left(\Gamma\right)$ if and only if the operator $[M,b]$ is bounded on $L_{p,\lambda}\left(\Gamma\right)$.

PROOF. ($\Rightarrow$) Suppose that $b\in BMO\left(\Gamma\right)$ such that $b^-\in L_\infty\left(\Gamma\right)$. Thus, the following inequality is obtained from Theorem 2.10;

$$|[M,b]\,f(t)| \leq C\left(\left\|b^+\right\|_* + \left\|b^-\right\|_\infty\right) M^2 f\left(t\right) \text{ for all } f\in L_1^{loc}\left(\Gamma\right)$$

Moreover, from Theorem 3.3, the operator $[M,b]$ is bounded on $L_p\left(\Gamma\right)$, with $1<p<\infty$, so the following inequality is satisfied for all $f\in L_p\left(\Gamma\right)$;

$$\begin{aligned}
&\|[M,b]\|_{L_{p,\lambda}(\Gamma)}\\
&\sup_{\Gamma(t,r)}\left(|\Gamma\left(t,r\right)|^{\lambda-1}\int\limits_{\Gamma(t,r)}|[M,b]\,f(\tau)|^p\,dv\left(\tau\right)\right)^{\frac{1}{p}}\\
&\leq|\Gamma\left(t_0,r_0\right)|^{\frac{\lambda-1}{p}}\sup_{\Gamma(t,r)\subset\Gamma(t_0,r_0)}\left(\int\limits_{\Gamma(t,r)}|[M,b]\,f(\tau)|^p\,dv\left(\tau\right)\right)^{\frac{1}{p}}\\
&\leq C\,|\Gamma\left(t_0,r_0\right)|^{\frac{\lambda-1}{p}}\left(\|b^+\|_{BMO(\Gamma)}+\|b^-\|_{L_\infty(\Gamma)}\right)M^2 f\left(t\right)\\
&\leq C_\Gamma
\end{aligned}$$

Hence,

$$\|\,[M,b]\,f\|_{L_{p,\lambda}(\Gamma)} \leq C\|f\|_{L_{p,\lambda}(\Gamma)} \tag{7}$$

The inequality (7) gives the desired result easily.

($\Leftarrow$) Assume that $[M,b]$ is bounded on $L_{p,\lambda}\left(\Gamma\right)$. Let $\Gamma\left(t_0,r_0\right)$ be a fixed Carleson curve.

Denote by M_b the local maximal function of f:

$$M_{\Gamma(t_0,r_0)}f(x) := \sup_{t\in\Gamma(t,r):\ \Gamma(t,r)\subset\Gamma(t_0,r_0)}\frac{1}{|\Gamma\left(t,r\right)|}\int\limits_{\Gamma(t,r)}|f\left(y\right)|\,dv\left(\tau\right),\ \ (t\in\Gamma)$$

Since

$$M(b\chi_{\Gamma(t_0,r_0)})\chi_{\Gamma(t_0,r_0)} = M_{\Gamma(t_0,r_0)}(b)$$

and

$$M\left(\chi_{\Gamma(t_0,r_0)}\right)\chi_{\Gamma(t_0,r_0)} = \chi_{\Gamma(t_0,r_0)}$$

then, we get the following inequality

$$\begin{aligned} \left|M_{\Gamma(t_0,r_0)}(b) - b\chi_{\Gamma(t_0,r_0)}\right| &= \left|M(b\chi_{\Gamma(t_0,r_0)})\chi_{\Gamma(t_0,r_0)} - bM\left(\chi_{\Gamma(t_0,r_0)}\right)\chi_{\Gamma(t_0,r_0)}\right| \\ &\leq \left|M(b\chi_{\Gamma(t_0,r_0)}) - bM\left(\chi_{\Gamma(t_0,r_0)}\right)\right| \\ &= \left|[M,b]\,\chi_{\Gamma(t_0,r_0)}\right| \end{aligned}$$

Hence,

$$\left\|M_{\Gamma(t_0,r_0)}(b) - b\chi_{\Gamma(t_0,r_0)}\right\|_{L_{p,\lambda}(\Gamma)} \leq \left\|[M,b]\,\chi_{\Gamma(t_0,r_0)}\right\|_{L_{p,\lambda}(\Gamma)} \tag{8}$$

Thus, from Equation (8), we get

$$\begin{aligned} \tfrac{1}{|\Gamma(t_0,r_0)|}\int\limits_{\Gamma(t_0,r_0)} \left|\left(b - M_{\Gamma(t_0,r_0)}(b)\right)(\tau)\right| dv(\tau) &\leq \left(\tfrac{1}{\Gamma(t_0,r_0)}\int\limits_{\Gamma(t_0,r_0)} \left|\left(b - M_{\Gamma(t_0,r_0)}(b)\right)(\tau)\right|^p dv(\tau)\right)^{\frac{1}{p}} \\ &\leq |\Gamma(t_0,r_0)|^{-\frac{1}{p}} |\Gamma(t,r)|^{\frac{1-\lambda}{p}} \left\|b\chi_{\Gamma(t_0,r_0)} - M_{\Gamma(t_0,r_0)}(b)\right\|_{L_{p,\lambda}(\Gamma)} \\ &\leq |\Gamma(t_0,r_0)|^{-\frac{1}{p}} |\Gamma(t,r)|^{\frac{1-\lambda}{p}} \left\|[M,b]\,\chi_{\Gamma(t_0,r_0)}\right\|_{L_{p,\lambda}(\Gamma)} \\ &\leq c\,|\Gamma(t_0,r_0)|^{-\frac{1}{p}} |\Gamma(t,r)|^{\frac{1-\lambda}{p}} \left\|\chi_{\Gamma(t_0,r_0)}\right\|_{L_{p,\lambda}(\Gamma)} \\ &\approx c\,|\Gamma(t_0,r_0)|^{-\frac{1}{p}} |\Gamma(t,r)|^{\frac{1-\lambda}{p}} |\Gamma(t_0,r_0)|^{\frac{1}{p}} |\Gamma(t,r)|^{\frac{\lambda-1}{p}} \\ &= c \end{aligned}$$

Since

$$\int\limits_{\Gamma(t_0,r_0)} |\left(b - b_{\Gamma(t_0,r_0)}\right)(\tau)|dv(\tau) = \begin{cases} -\int\limits_{\Gamma(t_0,r_0)} b(\tau) - b_{\Gamma(t_0,r_0)}dv(\tau), & b(t) \leq b_{\Gamma(t_0,r_0)} \text{ and } t \in \Gamma(t_0,r_0) \\ \int\limits_{\Gamma(t_0,r_0)} b(\tau) - b_{\Gamma(t_0,r_0)}dv(\tau), & b(t) > b_{\Gamma(t_0,r_0)} \text{ and } t \in \Gamma(t_0,r_0) \end{cases} \tag{9}$$

for the following sets

$$I_1 := \left\{t \in \Gamma(t_0,r_0) : b(t) \leq b_{\Gamma(t_0,r_0)}\right\}$$

and

$$I_2 := \left\{t \in \Gamma(t_0,r_0) : b(t) > b_{\Gamma(t_0,r_0)}\right\}$$

are valid. Thus, the expression

$$-\int_{I_1} \left[b(\tau) - b_{\Gamma(t_0,r_0)}\right] dv(\tau) = \int_{I_2} \left[b(\tau) - b_{\Gamma(t_0,r_0)}\right] |dv(\tau)$$

can be written. In view of the inequality

$$b(x) \leq b_{\Gamma(t_0,r_0)} \leq M_{\Gamma(t_0,r_0)}(b), x \in E$$

we get that

$$\begin{aligned} &\tfrac{1}{|\Gamma(t_0,r_0)|}\int\limits_{\Gamma(t_0,r_0)} |\left(b - b_{\Gamma(t_0,r_0)}\right)(\tau)|dv(\tau) \\ &= \tfrac{2}{|\Gamma(t_0,r_0)|}\int_{I_2} \left[\left(b - b_{\Gamma(t_0,r_0)}\right)(\tau)\right] dv(\tau) \\ &\leq \tfrac{2}{|\Gamma(t_0,r_0)|}\int_{I_2} \left[\left(b - M_{\Gamma(t_0,r_0)}(b)\right)(\tau)\right] dv(\tau) \\ &\lesssim c \end{aligned}$$

Consequently, $b \in BMO(\Gamma)$.

Besides, the following expression is valid

$$0 \leq b^- = |b| - b^+ \leq M_{\Gamma(t_0,r_0)}(b) - b^+ + b^- = M_{\Gamma(t_0,r_0)}(b) - b \tag{10}$$

where $M_{\Gamma(t_0,r_0)}(b) \geq |b|$. From our assumption, $[M,b]$ is bounded on $L_{p,\lambda}(\Gamma)$, thus the following inequality is valid;

$$\begin{aligned} \|[M,b]\,f\|_{L_{p,\lambda}(\Gamma)} &\leq C\left(\|b^+\|_{BMO(\Gamma)} + \|b^-\|_{L^\infty(\Gamma)}\right)\|f\|_{L_{p,\lambda}(\Gamma)} \\ &< C_\Gamma \end{aligned}$$

Moreover, we obtained $b \in BMO(\Gamma)$ before, with these data we obtain $b^- \in L_\infty(\Gamma)$. □

4. Conclusion

In recent years, studies on Morrey spaces defined in $\mathbb{R}^n$ metric space and their reinterpretation with Carleson curves have presented a different field of study to scientists working on Morrey spaces. In this article, we have given new estimates about the boundedness of the maximal commutator operator M_b and the commutator of the maximal function $[M, b]$ in Morrey spaces defined on Carleson curves. By making some generalizations on Morrey spaces defined on Carleson curves of this study, it is thought that it will inspire the obtaining of new inequalities boundedness of the maximal commutator operator M_b and the commutator of the maximal function $[M, b]$.

Author Contributions

The author read and approved the last version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

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