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ON SOLUTIONS OF THREE-DIMENSIONAL SYSTEM OF DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS*

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Abstract. In this study, we show that the system of difference equations

$$x_n = \frac{x_{n-2}y_{n-3}}{y_{n-1}\left(a + bx_{n-2}y_{n-3}\right)},$$
$$y_n = \frac{y_{n-2}z_{n-3}}{z_{n-1}\left(c + dy_{n-2}z_{n-3}\right)}, n \in \mathbb{N}_0,$$
$$z_n = \frac{z_{n-2}x_{n-3}}{x_{n-1}\left(e + fz_{n-2}x_{n-3}\right)},$$

where the initial values x_{-i}, y_{-i}, z_{-i}, $i = \overline{1,3}$ and the parameters a, b, c, d, e, f are non-zero real numbers, can be solved in closed form. Moreover, we obtain the solutions of above system in explicit form according to the parameters a, c and e are equal 1 or not equal 1. In addition, we get periodic solutions of aforementioned system. Finally, we define the forbidden set of the initial conditions by using the acquired formulas.

1. Introduction

In recent years, many authors have been interested in non-linear difference equations and non-linear systems of difference equations [1–3, 5, 6, 8–10, 12–14, 20–23, 25–41]. One of the important topics in this field is the solvability of non-linear difference equations or non-linear difference equations systems. There are different methods for obtaining solutions of non-linear difference equations and non-linear systems of difference equations (two-dimensional or three-dimensional). One of the

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methods for solving non-linear difference equations and non-linear difference equations systems is to use the change of variables. Then, aforementioned difference equations or their systems can be reduced to a linear difference equation with constant or variable coefficients. The other method is to use induction method. For instance, El-Metwally et al. solved the following non-linear difference equations

$$x_{n+1} = \frac{x_{n-1}x_{n-2}}{x_n\left(\pm 1 \pm x_{n-1}x_{n-2}\right)},\ n \in \mathbb{N}_0, \tag{1}$$

by using induction method in [7]. In addition, they investigated the behavior of the solutions of difference equations in (1).

In addition, Ibrahim et al. in [15] obtained the solutions of the following difference equation

$$x_{n+1} = \frac{x_{n-1}x_{n-2}}{x_n\left(a_n + b_n x_{n-1}x_{n-2}\right)},\ n \in \mathbb{N}_0, \tag{2}$$

where initial conditions x_{-2}, x_{-1}, x_0 are non-zero real numbers and $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$ are real two-periodic sequences. They used induction method to acquire the solutions of equation (2).

Ahmed et al. in [4], investigated the periodic character and the form of the solutions of the following two-dimensional difference equations systems

$$x_{n+1} = \frac{x_{n-1}y_{n-2}}{y_n\left(-1 \pm x_{n-1}y_{n-2}\right)},\ y_{n+1} = \frac{y_{n-1}x_{n-2}}{x_n\left(\pm 1 \pm y_{n-1}x_{n-2}\right)},\ n \in \mathbb{N}_0, \tag{3}$$

by induction with $x_{-j},\ y_{-j}\ , j = \overline{0,2}$ are nonzero real numbers.

A few years ago, in [16], Kara and Yazlik showed that the following two-dimensional difference equations system

$$x_n = \frac{x_{n-2}y_{n-3}}{y_{n-1}\left(a_n + b_n x_{n-2}y_{n-3}\right)},\ y_n = \frac{y_{n-2}x_{n-3}}{x_{n-1}\left(\alpha_n + \beta_n y_{n-2}x_{n-3}\right)},\ n \in \mathbb{N}_0, \tag{4}$$

where the initial conditions $x_{-j}, y_{-j}, j \in \{1,2,3\}$ and the sequences $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$, $(\alpha_n)_{n\in\mathbb{N}_0}$, $(\beta_n)_{n\in\mathbb{N}_0}$ are non-zero real numbers can be solved in closed-form. In addition, they acquired the forbidden set of the initial values x_{-j}, y_{-j}, $j = \overline{1,3}$ for system (4) and gave a study of the long-term behavior of its solutions when for every $n \in \mathbb{N}_0$, all the sequences $(a_n), (b_n), (\alpha_n), (\beta_n)$ are constant. They used the change of variables to acquire the solutions of system (4).

Recently, the authors of [11], obtained exact formulas for the solutions of the two-dimensional system of difference equations

$$x_{n+1} = \frac{x_{n-k+1}y_{n-k}}{y_n\left(a_n + b_n x_{n-k+1}y_{n-k}\right)},\ y_{n+1} = \frac{x_{n-k}y_{n-k+1}}{x_n\left(c_n + d_n y_{n-k}y_{n-k+1}\right)},\ n \in \mathbb{N}_0, \tag{5}$$

where $(a_n)_{n\in\mathbb{N}_0}, (b_n)_{n\in\mathbb{N}_0}, (c_n)_{n\in\mathbb{N}_0}$ and $(d_n)_{n\in\mathbb{N}_0}$ are non-zero real sequences. Note that, system (4) can be obtained by taking $k=2$ in system (5).

In addition, Kara and Yazlik showed that the following two-dimensional system of non-linear difference equations

$$x_n = \frac{x_{n-k}y_{n-k-l}}{y_{n-l}\left(a_n + b_n x_{n-k}y_{n-k-l}\right)},\ y_n = \frac{y_{n-k}x_{n-k-l}}{x_{n-l}\left(\alpha_n + \beta_n y_{n-k}x_{n-k-l}\right)},\ n\in\mathbb{N}_0, \quad (6)$$

where $k,l\in\mathbb{N}$, $(a_n)_{n\in\mathbb{N}_0}, (b_n)_{n\in\mathbb{N}_0}, (\alpha_n)_{n\in\mathbb{N}_0}$, $(\beta_n)_{n\in\mathbb{N}_0}$ and the initial values x_{-i}, y_{-i}, $i=\overline{1,k+l}$, are real numbers can be solved in [17]. Also, by using these obtained formulas, they investigated the asymptotic behavior of well-defined solutions of system (6) for the case $k=2$, $l=k$. They used the change of variables to obtain the solutions of system (6).

Quite recently, authors of [18] showed that three-dimensional system of difference equations

$$\begin{aligned} x_n =& \frac{x_{n-2}z_{n-3}}{z_{n-1}\left(a_n + b_n x_{n-2}z_{n-3}\right)}, \\ y_n =& \frac{y_{n-2}x_{n-3}}{x_{n-1}\left(\alpha_n + \beta_n y_{n-2}x_{n-3}\right)},\ n\in\mathbb{N}_0, \\ z_n =& \frac{z_{n-2}y_{n-3}}{y_{n-1}\left(A_n + B_n z_{n-2}y_{n-3}\right)}, \end{aligned} \quad (7)$$

where the initial values x_{-j}, y_{-j}, z_{-j}, $j\in\{1,2,3\}$ and the sequences $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$, $(\alpha_n)_{n\in\mathbb{N}_0}$, $(\beta_n)_{n\in\mathbb{N}_0}$, $(A_n)_{n\in\mathbb{N}_0}$, $(B_n)_{n\in\mathbb{N}_0}$ are non-zero real numbers, can be solved in closed form. They used the change of variables to acquire the solutions of system (7).

Finally, in [19], Kara et al. obtained explicit formulas for the well defined solutions of the following system of difference equations

$$\begin{aligned} x_{n+1} =& \frac{\prod_{j=0}^{k} z_{n-3j}}{\prod_{j=1}^{k} x_{n-(3j-1)}\left(a_n + b_n \prod_{j=0}^{k} z_{n-3j}\right)}, \\ y_{n+1} =& \frac{\prod_{j=0}^{k} x_{n-3j}}{\prod_{j=1}^{k} y_{n-(3j-1)}\left(c_n + d_n \prod_{j=0}^{k} x_{n-3j}\right)},\ n\in\mathbb{N}_0, \\ z_{n+1} =& \frac{\prod_{j=0}^{k} y_{n-3j}}{\prod_{j=1}^{k} z_{n-(3j-1)}\left(e_n + f_n \prod_{j=0}^{k} y_{n-3j}\right)}, \end{aligned} \quad (8)$$

where $k\in\mathbb{N}_0$, the initial conditions x_{-i}, y_{-i}, z_{-i}, $i=\overline{0,3k}$ and the sequences $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$, $(c_n)_{n\in\mathbb{N}_0}$, $(d_n)_{n\in\mathbb{N}_0}$, $(e_n)_{n\in\mathbb{N}_0}$, $(f_n)_{n\in\mathbb{N}_0}$ are real numbers. They used change of variables to obtain the solutions of system (8).

In this paper, we study the following three-dimensional system of difference equations

$$\begin{aligned} x_n &= \frac{x_{n-2}y_{n-3}}{y_{n-1}\left(a+bx_{n-2}y_{n-3}\right)}, \\ y_n &= \frac{y_{n-2}z_{n-3}}{z_{n-1}\left(c+dy_{n-2}z_{n-3}\right)}, n \in \mathbb{N}_0, \\ z_n &= \frac{z_{n-2}x_{n-3}}{x_{n-1}\left(e+fz_{n-2}x_{n-3}\right)}, \end{aligned} \tag{9}$$

where the initial values x_{-i}, y_{-i}, z_{-i}, $i = \overline{1,3}$ and the parameters a, b, c, d, e, f are non-zero real numbers. We solve system (9) in closed form by using convenient transformation. We obtain the solutions of system (9) in explicit form according to the parameters a, c and e are equal 1 or not equal 1. In addition, we get periodic solutions of system (9). Finally, we define the forbidden set of the initial conditions by using the obtained formulas. Note that system (9) is three-dimensional form of equation (2) and system (4).

Definition 1. (**Periodicity**) Let $(x_n, y_n, z_n)_{n\geq -3}$ be solution to difference equations system (9). The solution $(x_n, y_n, z_n)_{n\geq -3}$ is said to be eventually periodic p if $x_{n+p} = x_n$, $y_{n+p} = y_n$, $z_{n+p} = z_n$ for all $n \geq n_0$ where $n_0 \in \mathbb{Z}$, $p \in \mathbb{Z}^+$. If $n_0 = -3$ is said that the solution is periodic with period p.

Lemma 1. [24] Let $(\alpha_n)_{n\in\mathbb{N}_0}$ and $(\beta_n)_{n\in\mathbb{N}_0}$ be two sequences of real numbers and the sequences x_{2m+i}, $i \in \{0,1\}$, be solutions of the equations

$$x_{2m+i} = \alpha_{2m+i}x_{2(m-1)+i} + \beta_{2m+i},\ m \in \mathbb{N}_0. \tag{10}$$

Then, for each fixed $i \in \{0,1\}$ and $m \geq -1$, equation (10) has the general solution

$$x_{2m+i} = x_{i-2}\prod_{j=0}^{m}\alpha_{2j+i} + \sum_{l=0}^{m}\beta_{2l+i}\prod_{j=l+1}^{m}\alpha_{2j+i}.$$

Further, if $(\alpha_n)_{n\in\mathbb{N}_0}$ and $(\beta_n)_{n\in\mathbb{N}_0}$ are constant and $i \in \{0,1\}$, then

$$x_{2m+i} = \begin{cases} \alpha^{m+1}x_{i-2} + \beta\frac{1-\alpha^{m+1}}{1-\alpha}, & \text{if } \alpha \neq 1, \\ x_{i-2} + \beta\left(m+1\right), & \text{if } \alpha = 1. \end{cases}$$

2. The Solutions of System (9) in Closed Form

Let $\{(x_n, y_n, z_n)\}_{n\geq -3}$ be a solution of system (9). If at least one of the initial conditions x_{-j}, y_{-j}, z_{-j}, $j = \overline{1,3}$, is equal to zero, then the solution of system (9) is not defined. For example, if $x_{-3} = 0$, then $z_0 = 0$ and so y_1 is not defined. Similarly, if $y_{-3} = 0$ (or $z_{-3} = 0$), then $x_0 = 0$ (or $y_0 = 0$) and so z_1 (or x_1) is not defined. For $j = 1, 2$, the other cases are similar. On the other hand, if

$x_{n_0} = 0$ ($n_0 \in \mathbb{N}_0$), $x_n \neq 0$, for $-3 \leq n \leq n_0 - 1$, and x_k, y_k and z_k are defined for $-3 \leq k \leq n_0 - 1$, then according to the first equation in (9) we get that $y_{n_0-3} = 0$. If $n_0 - 3 \leq -1$, then $y_{-j_0} = 0$, for $j_0 \in \{1, 2, 3\}$. If $3 \leq n_0 \leq 5$ then from this and the second equation in (9) we have that $y_{n_0-5} = 0$ or $z_{n_0-6} = 0$. If $n_0 - 5 \leq 0$, then $z_{-j_0} = 0$, for $j_0 \in \{1, 2, 3\}$ and $y_{-j_1} = 0$, for $j_1 \in \{1, 2\}$. If $n_0 > 5$ from this and first equation in (9) we have that $y_{n_0-5} = 0$ or $z_{n_0-6} = 0$. If $n_0 > 5$ and $z_{n_0-6} = 0$ from this and third, second, first equations in (9) we have that $x_{n_0-2} = 0$, which is a contradiction. The other cases ($y_{n_1} = 0$ and $z_{n_2} = 0$) can be similarly proved. Thus, for every well-defined solution of system (9) we have that $x_n y_n z_n \neq 0$, $n \geq -3$, if and only if $x_{-i} y_{-i} z_{-i} \neq 0$, for $i = \overline{1, 3}$. Note that the system (9) can be written in the form

$$\begin{aligned} \frac{1}{x_n y_{n-1}} &= \frac{a + b x_{n-2} y_{n-3}}{x_{n-2} y_{n-3}}, \\ \frac{1}{y_n z_{n-1}} &= \frac{c + d y_{n-2} z_{n-3}}{y_{n-2} z_{n-3}}, \quad n \in \mathbb{N}_0, \\ \frac{1}{z_n x_{n-1}} &= \frac{e + f z_{n-2} x_{n-3}}{z_{n-2} x_{n-3}}. \end{aligned} \tag{11}$$

Using the following variables

$$u_n = \frac{1}{x_n y_{n-1}}, \ v_n = \frac{1}{y_n z_{n-1}}, \ w_n = \frac{1}{z_n x_{n-1}}, \ n \geq -2, \tag{12}$$

then system (11) transforms to the following linear difference equations

$$u_n = a u_{n-2} + b, \ v_n = c v_{n-2} + d, \ w_n = e w_{n-2} + f, \ n \in \mathbb{N}_0, \tag{13}$$

From Lemma 1, the solutions of equations in (13) are

$$\begin{aligned} u_{2m+i} &= \begin{cases} a^{m+1} u_{i-2} + \frac{1-a^{m+1}}{1-a} b, & \text{if } a \neq 1, \\ u_{i-2} + (m+1) b & \text{if } a = 1, \end{cases} \\ v_{2m+i} &= \begin{cases} c^{m+1} v_{i-2} + \frac{1-c^{m+1}}{1-c} d, & \text{if } c \neq 1, \\ v_{i-2} + (m+1) d, & \text{if } c = 1, \end{cases} \quad m \in \mathbb{N}_0, \\ w_{2m+i} &= \begin{cases} e^{m+1} w_{i-2} + \frac{1-e^{m+1}}{1-e} f, & \text{if } e \neq 1, \\ w_{i-2} + (m+1) f, & \text{if } e = 1, \end{cases} \end{aligned} \tag{14}$$

for $i \in \{0, 1\}$. From equations in (12) we get

$$\begin{aligned} x_{2m+i} &= \frac{v_{2m+i-1}}{u_{2m+i}} \frac{u_{2m+i-3}}{w_{2m+i-2}} \frac{w_{2m+i-5}}{v_{2m+i-4}} x_{2(m-3)+i}, \\ y_{2m+i} &= \frac{w_{2m+i-1}}{v_{2m+i}} \frac{v_{2m+i-3}}{u_{2m+i-2}} \frac{u_{2m+i-5}}{w_{2m+i-4}} y_{2(m-3)+i}, \ m \in \mathbb{N}, \\ z_{2m+i} &= \frac{u_{2m+i-1}}{w_{2m+i}} \frac{w_{2m+i-3}}{v_{2m+i-2}} \frac{v_{2m+i-5}}{u_{2m+i-4}} z_{2(m-3)+i}, \end{aligned}$$

where $i \in \{1, 2\}$, and consequently

$$\begin{aligned} x_{6m+l} &= \frac{v_{6m+l-1}}{u_{6m+l}} \frac{u_{6m+l-3}}{w_{6m+l-2}} \frac{w_{6m+l-5}}{v_{6m+l-4}} x_{6(m-1)+l}, \ m \in \mathbb{N}_0, \\ y_{6m+l} &= \frac{w_{6m+l-1}}{v_{6m+l}} \frac{v_{6m+l-3}}{u_{6m+l-2}} \frac{u_{6m+l-5}}{w_{6m+l-4}} y_{6(m-1)+l}, \ m \in \mathbb{N}_0, \\ z_{6m+l} &= \frac{u_{6m+l-1}}{w_{6m+l}} \frac{w_{6m+l-3}}{v_{6m+l-2}} \frac{v_{6m+l-5}}{u_{6m+l-4}} z_{6(m-1)+l}, \ m \in \mathbb{N}_0, \end{aligned} \tag{15}$$

where $l = \overline{3, 8}$, as far as $6m + l \geq 3$. From (15), we have that

$$\begin{aligned} x_{6m+l} &= x_{l-6} \prod_{s=0}^{m} \frac{v_{6s+l-1}}{u_{6s+l}} \frac{u_{6s+l-3}}{w_{6s+l-2}} \frac{w_{6s+l-5}}{v_{6s+l-4}}, \\ y_{6m+l} &= y_{l-6} \prod_{s=0}^{m} \frac{w_{6s+l-1}}{v_{6s+l}} \frac{v_{6s+l-3}}{u_{6s+l-2}} \frac{u_{6s+l-5}}{w_{6s+l-4}}, \\ z_{6m+l} &= z_{l-6} \prod_{s=0}^{m} \frac{u_{6s+l-1}}{w_{6s+l}} \frac{w_{6s+l-3}}{v_{6s+l-2}} \frac{v_{6s+l-5}}{u_{6s+l-4}}, \end{aligned} \tag{16}$$

where $m \geq -1$ and $l = \overline{3, 8}$. From (16), we get

$$\begin{aligned} x_{6m+2t+k} &= x_{2t+k-6} \prod_{s=0}^{m} \frac{v_{6s+2t+k-1}}{u_{6s+2t+k}} \frac{u_{6s+2t+k-3}}{w_{6s+2t+k-2}} \frac{w_{6s+2t+k-5}}{v_{6s+2t+k-4}}, \\ y_{6m+2t+k} &= y_{2t+k-6} \prod_{s=0}^{m} \frac{w_{6s+2t+k-1}}{v_{6s+2t+k}} \frac{v_{6s+2t+k-3}}{u_{6s+2t+k-2}} \frac{u_{6s+2t+k-5}}{w_{6s+2t+k-4}}, \\ z_{6m+2t+k} &= z_{2t+k-6} \prod_{s=0}^{m} \frac{u_{6s+2t+k-1}}{w_{6s+2t+k}} \frac{w_{6s+2t+k-3}}{v_{6s+2t+k-2}} \frac{v_{6s+2t+k-5}}{u_{6s+2t+k-4}}, \end{aligned} \tag{17}$$

for $t \in \{1, 2, 3\}$ and $k \in \{1, 2\}$. Employing (14) in (17), we get solutions of system (9).

3. Particular Cases of System (9)

Now, we will examine the solutions in 8 different cases depending on whether the parameters a, c and e are equal 1 or not equal 1.

3.1. Case $a \neq 1$, $c \neq 1$, $e \neq 1$

In this case, the solutions of system (9) can be written in the following form

$$\begin{aligned} x_{6m+2t+1} = x_{2t-5} \prod_{s=0}^{m} & \frac{x_{-1}y_{-1}z_{-1}}{x_{-3}y_{-3}z_{-3}} \frac{c^{3s+t+1}\left((1-c) - y_{-2}z_{-3}d\right) + y_{-2}z_{-3}d}{a^{3s+t+1}\left((1-a) - x_{-1}y_{-2}b\right) + x_{-1}y_{-2}b} \\ & \times \frac{a^{3s+t}\left((1-a) - x_{-2}y_{-3}b\right) + x_{-2}y_{-3}b}{e^{3s+t}\left((1-e) - x_{-2}z_{-1}f\right) + x_{-2}z_{-1}f} \\ & \times \frac{e^{3s+t-1}\left((1-e) - x_{-3}z_{-2}f\right) + x_{-3}z_{-2}f}{c^{3s+t-1}\left((1-c) - y_{-1}z_{-2}d\right) + y_{-1}z_{-2}d}, \end{aligned}$$

$$x_{6m+2t+2}=x_{2t-4}\prod_{s=0}^{m}\frac{x_{-3}y_{-3}z_{-3}}{x_{-1}y_{-1}z_{-1}}\frac{c^{3s+t+1}\left((1-c)-y_{-1}z_{-2}d\right)+y_{-1}z_{-2}d}{a^{3s+t+2}\left((1-a)-x_{-2}y_{-3}b\right)+x_{-2}y_{-3}b}$$
$$\times\frac{a^{3s+t}\left((1-a)-x_{-1}y_{-2}b\right)+x_{-1}y_{-2}b}{e^{3s+t+1}\left((1-e)-x_{-3}z_{-2}f\right)+x_{-3}z_{-2}f}$$
$$\times\frac{e^{3s+t-1}\left((1-e)-x_{-2}z_{-1}f\right)+x_{-2}z_{-1}f}{c^{3s+t}\left((1-c)-y_{-2}z_{-3}d\right)+y_{-2}z_{-3}d},$$

$$y_{6m+2t+1}=y_{2t-5}\prod_{s=0}^{m}\frac{x_{-1}y_{-1}z_{-1}}{x_{-3}y_{-3}z_{-3}}\frac{e^{3s+t+1}\left((1-e)-x_{-3}z_{-2}f\right)+x_{-3}z_{-2}f}{c^{3s+t+1}\left((1-c)-y_{-1}z_{-2}d\right)+y_{-1}z_{-2}d}$$
$$\times\frac{c^{3s+t}\left((1-c)-y_{-2}z_{-3}d\right)+y_{-2}z_{-3}d}{a^{3s+t}\left((1-a)-x_{-1}y_{-2}b\right)+x_{-1}y_{-2}b}$$
$$\times\frac{a^{3s+t-1}\left((1-a)-x_{-2}y_{-3}b\right)+x_{-2}y_{-3}b}{e^{3s+t-1}\left((1-e)-x_{-2}z_{-1}f\right)+x_{-2}z_{-1}f},$$

$$y_{6m+2t+2}=y_{2t-4}\prod_{s=0}^{m}\frac{x_{-3}y_{-3}z_{-3}}{x_{-1}y_{-1}z_{-1}}\frac{e^{3s+t+1}\left((1-e)-x_{-2}z_{-1}f\right)+x_{-2}z_{-1}f}{c^{3s+t+2}\left((1-c)-y_{-2}z_{-3}d\right)+y_{-2}z_{-3}d}$$
$$\times\frac{c^{3s+t}\left((1-c)-y_{-1}z_{-2}d\right)+y_{-1}z_{-2}d}{a^{3s+t+1}\left((1-a)-x_{-2}y_{-3}b\right)+x_{-2}y_{-3}b}$$
$$\times\frac{a^{3s+t-1}\left((1-a)-x_{-1}y_{-2}b\right)+x_{-1}y_{-2}b}{e^{3s+t}\left((1-e)-x_{-3}z_{-2}f\right)+x_{-3}z_{-2}f},$$

$$z_{6m+2t+1}=z_{2t-5}\prod_{s=0}^{m}\frac{x_{-1}y_{-1}z_{-1}}{x_{-3}y_{-3}z_{-3}}\frac{a^{3s+t+1}\left((1-a)-x_{-2}y_{-3}b\right)+x_{-2}y_{-3}b}{e^{3s+t+1}\left((1-e)-z_{-1}x_{-2}f\right)+z_{-1}x_{-2}f}$$
$$\times\frac{e^{3s+t}\left((1-e)-z_{-2}x_{-3}f\right)+z_{-2}x_{-3}f}{c^{3s+t}\left((1-c)-y_{-1}z_{-2}d\right)+y_{-1}z_{-2}d}$$
$$\times\frac{c^{3s+t-1}\left((1-c)-y_{-2}z_{-3}d\right)+y_{-2}z_{-3}d}{a^{3s+t-1}\left((1-a)-y_{-2}x_{-1}b\right)+y_{-2}x_{-1}b},$$

$$z_{6m+2t+2}=z_{2t-4}\prod_{s=0}^{m}\frac{x_{-3}y_{-3}z_{-3}}{x_{-1}y_{-1}z_{-1}}\frac{a^{3s+t+1}\left((1-a)-y_{-2}x_{-1}b\right)+y_{-2}x_{-1}b}{e^{3s+t+2}\left((1-e)-z_{-2}x_{-3}f\right)+z_{-2}x_{-3}f}$$
$$\times\frac{e^{3s+t}\left((1-e)-z_{-1}x_{-2}f\right)+z_{-1}x_{-2}f}{c^{3s+t+1}\left((1-c)-y_{-2}z_{-3}d\right)+y_{-2}z_{-3}d}$$
$$\times\frac{c^{3s+t-1}\left((1-c)-y_{-1}z_{-2}d\right)+y_{-1}z_{-2}d}{a^{3s+t}\left((1-a)-y_{-3}x_{-2}b\right)+y_{-3}x_{-2}b},$$

for $m \geq -1$ and $t \in \{1, 2, 3\}$.

3.2. Case $a = 1$**,** $c \neq 1$**,** $e \neq 1$

In this case, solutions of system (9) are as follows

$$x_{6m+2t+1} = x_{2t-5} \prod_{s=0}^{m} \frac{x_{-1}y_{-1}z_{-1}}{x_{-3}y_{-3}z_{-3}} \frac{c^{3s+t+1}\left((1-c) - y_{-2}z_{-3}d\right) + y_{-2}z_{-3}d}{1 + x_{-1}y_{-2}(3s+t+1)b} \times \frac{1 + x_{-2}y_{-3}(3s+t)b}{e^{3s+t}\left((1-e) - x_{-2}z_{-1}f\right) + x_{-2}z_{-1}f} \times \frac{e^{3s+t-1}\left((1-e) - x_{-3}z_{-2}f\right) + x_{-3}z_{-2}f}{c^{3s+t-1}\left((1-c) - y_{-1}z_{-2}d\right) + y_{-1}z_{-2}d},$$

$$x_{6m+2t+2} = x_{2t-4} \prod_{s=0}^{m} \frac{x_{-3}y_{-3}z_{-3}}{x_{-1}y_{-1}z_{-1}} \frac{c^{3s+t+1}\left((1-c) - y_{-1}z_{-2}d\right) + y_{-1}z_{-2}d}{1 + x_{-2}y_{-3}(3s+t+2)b} \times \frac{1 + x_{-1}y_{-2}(3s+t)b}{e^{3s+t+1}\left((1-e) - x_{-3}z_{-2}f\right) + x_{-3}z_{-2}f} \times \frac{e^{3s+t-1}\left((1-e) - x_{-2}z_{-1}f\right) + x_{-2}z_{-1}f}{c^{3s+t}\left((1-c) - y_{-2}z_{-3}d\right) + y_{-2}z_{-3}d},$$

$$y_{6m+2t+1} = y_{2t-5} \prod_{s=0}^{m} \frac{x_{-1}y_{-1}z_{-1}}{x_{-3}y_{-3}z_{-3}} \frac{e^{3s+t+1}\left((1-e) - x_{-3}z_{-2}f\right) + x_{-3}z_{-2}f}{c^{3s+t+1}\left((1-c) - y_{-1}z_{-2}d\right) + y_{-1}z_{-2}d} \times \frac{c^{3s+t}\left((1-c) - y_{-2}z_{-3}d\right) + y_{-2}z_{-3}d}{1 + x_{-1}y_{-2}(3s+t)b} \times \frac{1 + x_{-2}y_{-3}(3s+t-1)b}{e^{3s+t-1}\left((1-e) - x_{-2}z_{-1}f\right) + x_{-2}z_{-1}f},$$

$$y_{6m+2t+2} = y_{2t-4} \prod_{s=0}^{m} \frac{x_{-3}y_{-3}z_{-3}}{x_{-1}y_{-1}z_{-1}} \frac{e^{3s+t+1}\left((1-e) - x_{-2}z_{-1}f\right) + x_{-2}z_{-1}f}{c^{3s+t+2}\left((1-c) - y_{-2}z_{-3}d\right) + y_{-2}z_{-3}d} \times \frac{c^{3s+t}\left((1-c) - y_{-1}z_{-2}d\right) + y_{-1}z_{-2}d}{1 + x_{-2}y_{-3}(3s+t+1)b} \times \frac{1 + x_{-1}y_{-2}(3s+t-1)b}{e^{3s+t}\left((1-e) - x_{-3}z_{-2}f\right) + x_{-3}z_{-2}f},$$

$$z_{6m+2t+1} = z_{2t-5} \prod_{s=0}^{m} \frac{x_{-1}y_{-1}z_{-1}}{x_{-3}y_{-3}z_{-3}} \frac{1 + x_{-2}y_{-3}(3s+t+1)b}{e^{3s+t+1}\left((1-e) - x_{-2}z_{-1}f\right) + x_{-2}z_{-1}f} \times \frac{e^{3s+t}\left((1-e) - x_{-3}z_{-2}f\right) + x_{-3}z_{-2}f}{c^{3s+t}\left((1-c) - y_{-1}z_{-2}d\right) + y_{-1}z_{-2}d} \times \frac{c^{3s+t-1}\left((1-c) - y_{-2}z_{-3}d\right) + y_{-2}z_{-3}d}{1 + x_{-1}y_{-2}(3s+t-1)b},$$

$$z_{6m+2t+2} = z_{2t-4} \prod_{s=0}^{m} \frac{x_{-3}y_{-3}z_{-3}}{x_{-1}y_{-1}z_{-1}} \frac{1 + x_{-1}y_{-2}(3s+t+1)b}{e^{3s+t+2}\left((1-e) - x_{-3}z_{-2}f\right) + x_{-3}z_{-2}f}$$
$$\times \frac{e^{3s+t}\left((1-e) - x_{-2}z_{-1}f\right) + x_{-2}z_{-1}f}{c^{3s+t+1}\left((1-c) - y_{-2}z_{-3}d\right) + y_{-2}z_{-3}d}$$
$$\times \frac{c^{3s+t-1}\left((1-c) - y_{-1}z_{-2}d\right) + y_{-1}z_{-2}d}{1 + x_{-2}y_{-3}(3s+t)b},$$

for $m \geq -1$ and $t \in \{1, 2, 3\}$.

3.3. Case $a \neq 1$, $c = 1$, $e \neq 1$

In this case, the solutions of system (9) can be written in the following form

$$x_{6m+2t+1} = x_{2t-5} \prod_{s=0}^{m} \frac{x_{-1}y_{-1}z_{-1}}{x_{-3}y_{-3}z_{-3}} \frac{1 + y_{-2}z_{-3}(3s+t+1)d}{a^{3s+t+1}\left((1-a) - x_{-1}y_{-2}b\right) + x_{-1}y_{-2}b}$$
$$\times \frac{a^{3s+t}\left((1-a) - x_{-2}y_{-3}b\right) + x_{-2}y_{-3}b}{e^{3s+t}\left((1-e) - x_{-2}z_{-1}f\right) + x_{-2}z_{-1}f}$$
$$\times \frac{e^{3s+t-1}\left((1-e) - x_{-3}z_{-2}f\right) + x_{-3}z_{-2}f}{1 + y_{-1}z_{-2}(3s+t-1)d},$$

$$x_{6m+2t+2} = x_{2t-4} \prod_{s=0}^{m} \frac{x_{-3}y_{-3}z_{-3}}{x_{-1}y_{-1}z_{-1}} \frac{1 + y_{-1}z_{-2}(3s+t+1)d}{a^{3s+t+2}\left((1-a) - x_{-2}y_{-3}b\right) + x_{-2}y_{-3}b}$$
$$\times \frac{a^{3s+t}\left((1-a) - x_{-1}y_{-2}b\right) + x_{-1}y_{-2}b}{e^{3s+t+1}\left((1-e) - x_{-3}z_{-2}f\right) + x_{-3}z_{-2}f}$$
$$\times \frac{e^{3s+t-1}\left((1-e) - x_{-2}z_{-1}f\right) + x_{-2}z_{-1}f}{1 + y_{-2}z_{-3}(3s+t)d},$$

$$y_{6m+2t+1} = y_{2t-5} \prod_{s=0}^{m} \frac{x_{-1}y_{-1}z_{-1}}{x_{-3}y_{-3}z_{-3}} \frac{e^{3s+t+1}\left((1-e) - x_{-3}z_{-2}f\right) + x_{-3}z_{-2}f}{1 + y_{-1}z_{-2}(3s+t+1)d}$$
$$\times \frac{1 + y_{-2}z_{-3}(3s+t)d}{a^{3s+t}\left((1-a) - x_{-1}y_{-2}b\right) + x_{-1}y_{-2}b}$$
$$\times \frac{a^{3s+t-1}\left((1-a) - x_{-2}y_{-3}b\right) + x_{-2}y_{-3}b}{e^{3s+t-1}\left((1-e) - x_{-2}z_{-1}f\right) + x_{-2}z_{-1}f},$$

$$y_{6m+2t+2} = y_{2t-4} \prod_{s=0}^{m} \frac{x_{-3}y_{-3}z_{-3}}{x_{-1}y_{-1}z_{-1}} \frac{e^{3s+t+1}\left((1-e) - x_{-2}z_{-1}f\right) + x_{-2}z_{-1}f}{1 + y_{-2}z_{-3}(3s+t+2)d}$$
$$\times \frac{1 + y_{-1}z_{-2}(3s+t)d}{a^{3s+t+1}\left((1-a) - x_{-2}y_{-3}b\right) + x_{-2}y_{-3}b}$$

$$\times \frac{a^{3s+t-1}\left((1-a)-x_{-1}y_{-2}b\right)+x_{-1}y_{-2}b}{e^{3s+t}\left((1-e)-x_{-3}z_{-2}f\right)+x_{-3}z_{-2}f},$$

$$z_{6m+2t+1}=z_{2t-5}\prod_{s=0}^{m}\frac{x_{-1}y_{-1}z_{-1}}{x_{-3}y_{-3}z_{-3}}\frac{a^{3s+t+1}\left((1-a)-x_{-2}y_{-3}b\right)+x_{-2}y_{-3}b}{e^{3s+t+1}\left((1-e)-x_{-2}z_{-1}f\right)+x_{-2}z_{-1}f}$$
$$\times \frac{e^{3s+t}\left((1-e)-x_{-3}z_{-2}f\right)+x_{-3}z_{-2}f}{1+y_{-1}z_{-2}(3s+t)d}$$
$$\times \frac{1+y_{-2}z_{-3}(3s+t-1)d}{a^{3s+t-1}\left((1-a)-x_{-1}y_{-2}b\right)+x_{-1}y_{-2}b},$$

$$z_{6m+2t+2}=z_{2t-4}\prod_{s=0}^{m}\frac{x_{-3}y_{-3}z_{-3}}{x_{-1}y_{-1}z_{-1}}\frac{a^{3s+t+1}\left((1-a)-x_{-1}y_{-2}b\right)+x_{-1}y_{-2}b}{e^{3s+t+2}\left((1-e)-x_{-3}z_{-2}f\right)+x_{-3}z_{-2}f}$$
$$\times \frac{e^{3s+t}\left((1-e)-x_{-2}z_{-1}f\right)+x_{-2}z_{-1}f}{1+y_{-2}z_{-3}(3s+t+1)d}$$
$$\times \frac{1+y_{-1}z_{-2}(3s+t-1)d}{a^{3s+t}\left((1-a)-x_{-2}y_{-3}b\right)+x_{-2}y_{-3}b},$$

for $m\geq -1$ and $t\in\{1,2,3\}$.

3.4. Case $a\neq 1$, $c\neq 1$, $e=1$

In this case, solutions of system (9) are as follows

$$x_{6m+2t+1}=x_{2t-5}\prod_{s=0}^{m}\frac{x_{-1}y_{-1}z_{-1}}{x_{-3}y_{-3}z_{-3}}\frac{c^{3s+t+1}\left((1-c)-y_{-2}z_{-3}d\right)+y_{-2}z_{-3}d}{a^{3s+t+1}\left((1-a)-x_{-1}y_{-2}b\right)+x_{-1}y_{-2}b}$$
$$\times \frac{a^{3s+t}\left((1-a)-x_{-2}y_{-3}b\right)+x_{-2}y_{-3}b}{1+x_{-2}z_{-1}(3s+t)f}$$
$$\times \frac{1+x_{-3}z_{-2}(3s+t-1)f}{c^{3s+t-1}\left((1-c)-y_{-1}z_{-2}d\right)+y_{-1}z_{-2}d},$$

$$x_{6m+2t+2}=x_{2t-4}\prod_{s=0}^{m}\frac{x_{-3}y_{-3}z_{-3}}{x_{-1}y_{-1}z_{-1}}\frac{c^{3s+t+1}\left((1-c)-y_{-1}z_{-2}d\right)+y_{-1}z_{-2}d}{a^{3s+t+2}\left((1-a)-x_{-2}y_{-3}b\right)+x_{-2}y_{-3}b}$$
$$\times \frac{a^{3s+t}\left((1-a)-x_{-1}y_{-2}b\right)+x_{-1}y_{-2}b}{1+x_{-3}z_{-2}(3s+t+1)f}$$
$$\times \frac{1+x_{-2}z_{-1}(3s+t-1)f}{c^{3s+t}\left((1-c)-y_{-2}z_{-3}d\right)+y_{-2}z_{-3}d},$$

$$y_{6m+2t+1}=y_{2t-5}\prod_{s=0}^{m}\frac{x_{-1}y_{-1}z_{-1}}{x_{-3}y_{-3}z_{-3}}\frac{1+x_{-3}z_{-2}(3s+t+1)f}{c^{3s+t+1}\left((1-c)-y_{-1}z_{-2}d\right)+y_{-1}z_{-2}d}$$

$$\times \frac{c^{3s+t}\left((1-c)-y_{-2}z_{-3}d\right)+y_{-2}z_{-3}d}{a^{3s+t}\left((1-a)-x_{-1}y_{-2}b\right)+x_{-1}y_{-2}b} \times \frac{a^{3s+t-1}\left((1-a)-x_{-2}y_{-3}b\right)+x_{-2}y_{-3}b}{1+x_{-2}z_{-1}(3s+t-1)f},$$

$$y_{6m+2t+2}=y_{2t-4}\prod_{s=0}^{m}\frac{x_{-3}y_{-3}z_{-3}}{x_{-1}y_{-1}z_{-1}}\frac{1+x_{-2}z_{-1}(3s+t+1)f}{c^{3s+t+2}\left((1-c)-y_{-2}z_{-3}d\right)+y_{-2}z_{-3}d} \times \frac{c^{3s+t}\left((1-c)-y_{-1}z_{-2}d\right)+y_{-1}z_{-2}d}{a^{3s+t+1}\left((1-a)-x_{-2}y_{-3}b\right)+x_{-2}y_{-3}b} \times \frac{a^{3s+t-1}\left((1-a)-x_{-1}y_{-2}b\right)+x_{-1}y_{-2}b}{1+x_{-3}z_{-2}(3s+t)f},$$

$$z_{6m+2t+1}=z_{2t-5}\prod_{s=0}^{m}\frac{x_{-1}y_{-1}z_{-1}}{x_{-3}y_{-3}z_{-3}}\frac{a^{3s+t+1}\left((1-a)-x_{-2}y_{-3}b\right)+x_{-2}y_{-3}b}{1+x_{-2}z_{-1}(3s+t+1)f} \times \frac{1+x_{-3}z_{-2}(3s+t)f}{c^{3s+t}\left((1-c)-y_{-1}z_{-2}d\right)+y_{-1}z_{-2}d} \times \frac{c^{3s+t-1}\left((1-c)-y_{-2}z_{-3}d\right)+y_{-2}z_{-3}d}{a^{3s+t-1}\left((1-a)-x_{-1}y_{-2}b\right)+x_{-1}y_{-2}b},$$

$$z_{6m+2t+2}=z_{2t-4}\prod_{s=0}^{m}\frac{x_{-3}y_{-3}z_{-3}}{x_{-1}y_{-1}z_{-1}}\frac{a^{3s+t+1}\left((1-a)-x_{-1}y_{-2}b\right)+x_{-1}y_{-2}b}{1+x_{-3}z_{-2}(3s+t+2)f} \times \frac{1+x_{-2}z_{-1}(3s+t)f}{c^{3s+t+1}\left((1-c)-y_{-2}z_{-3}d\right)+y_{-2}z_{-3}d} \times \frac{c^{3s+t-1}\left((1-c)-y_{-1}z_{-2}d\right)+y_{-1}z_{-2}d}{a^{3s+t}\left((1-a)-x_{-2}y_{-3}b\right)+x_{-2}y_{-3}b},$$

for $m \geq -1$ and $t \in \{1,2,3\}$.

3.5. Case $a=1$, $c=1$, $e \neq 1$

In this case, the solution of system (9) can be written in the following form

$$x_{6m+2t+1}=x_{2t-5}\prod_{s=0}^{m}\frac{x_{-1}y_{-1}z_{-1}}{x_{-3}y_{-3}z_{-3}}\frac{1+y_{-2}z_{-3}(3s+t+1)d}{1+x_{-1}y_{-2}(3s+t+1)b} \times \frac{1+x_{-2}y_{-3}(3s+t)b}{e^{3s+t}\left((1-e)-x_{-2}z_{-1}f\right)+x_{-2}z_{-1}f} \times \frac{e^{3s+t-1}\left((1-e)-x_{-3}z_{-2}f\right)+x_{-3}z_{-2}f}{1+y_{-1}z_{-2}(3s+t-1)d},$$

$$x_{6m+2t+2} = x_{2t-4} \prod_{s=0}^{m} \frac{x_{-3}y_{-3}z_{-3}}{x_{-1}y_{-1}z_{-1}} \frac{1 + y_{-1}z_{-2}(3s+t+1)d}{1 + x_{-2}y_{-3}(3s+t+2)b} \times \frac{1 + x_{-1}y_{-2}(3s+t)b}{e^{3s+t+1}\left((1-e) - x_{-3}z_{-2}f\right) + x_{-3}z_{-2}f} \times \frac{e^{3s+t-1}\left((1-e) - x_{-2}z_{-1}f\right) + x_{-2}z_{-1}f}{1 + y_{-2}z_{-3}(3s+t)d},$$

$$y_{6m+2t+1} = y_{2t-5} \prod_{s=0}^{m} \frac{x_{-1}y_{-1}z_{-1}}{x_{-3}y_{-3}z_{-3}} \frac{e^{3s+t+1}\left((1-e) - x_{-3}z_{-2}f\right) + x_{-3}z_{-2}f}{1 + y_{-1}z_{-2}(3s+t+1)d} \times \frac{1 + y_{-2}z_{-3}(3s+t)d}{1 + x_{-1}y_{-2}(3s+t)b} \frac{1 + x_{-2}y_{-3}(3s+t-1)b}{e^{3s+t-1}\left((1-e) - x_{-2}z_{-1}f\right) + x_{-2}z_{-1}f},$$

$$y_{6m+2t+2} = y_{2t-4} \prod_{s=0}^{m} \frac{x_{-3}y_{-3}z_{-3}}{x_{-1}y_{-1}z_{-1}} \frac{e^{3s+t+1}\left((1-e) - x_{-2}z_{-1}f\right) + x_{-2}z_{-1}f}{1 + y_{-2}z_{-3}(3s+t+2)d} \times \frac{1 + y_{-1}z_{-2}(3s+t)d}{1 + x_{-2}y_{-3}(3s+t+1)b} \frac{1 + x_{-1}y_{-2}(3s+t-1)b}{e^{3s+t}\left((1-e) - x_{-3}z_{-2}f\right) + x_{-3}z_{-2}f},$$

$$z_{6m+2t+1} = z_{2t-5} \prod_{s=0}^{m} \frac{x_{-1}y_{-1}z_{-1}}{x_{-3}y_{-3}z_{-3}} \frac{1 + x_{-2}y_{-3}(3s+t+1)b}{e^{3s+t+1}\left((1-e) - x_{-2}z_{-1}f\right) + x_{-2}z_{-1}f} \times \frac{e^{3s+t}\left((1-e) - x_{-3}z_{-2}f\right) + x_{-3}z_{-2}f}{1 + y_{-1}z_{-2}(3s+t)d} \frac{1 + y_{-2}z_{-3}(3s+t-1)d}{1 + x_{-1}y_{-2}(3s+t-1)b},$$

$$z_{6m+2t+2} = z_{2t-4} \prod_{s=0}^{m} \frac{x_{-3}y_{-3}z_{-3}}{x_{-1}y_{-1}z_{-1}} \frac{1 + x_{-1}y_{-2}(3s+t+1)b}{e^{3s+t+2}\left((1-e) - x_{-3}z_{-2}f\right) + x_{-3}z_{-2}f} \times \frac{e^{3s+t}\left((1-e) - x_{-2}z_{-1}f\right) + x_{-2}z_{-1}f}{1 + y_{-2}z_{-3}(3s+t+1)d} \frac{1 + y_{-1}z_{-2}(3s+t-1)d}{1 + x_{-2}y_{-3}(3s+t)b},$$

for $m \geq -1$ and $t \in \{1, 2, 3\}$.

3.6. Case $a = 1$, $c \neq 1$, $e = 1$

In this case, solutions of system (9) are as follows

$$x_{6m+2t+1} = x_{2t-5} \prod_{s=0}^{m} \frac{x_{-1}y_{-1}z_{-1}}{x_{-3}y_{-3}z_{-3}} \frac{c^{3s+t+1}\left((1-c) - y_{-2}z_{-3}d\right) + y_{-2}z_{-3}d}{1 + x_{-1}y_{-2}(3s+t+1)b} \times \frac{1 + x_{-2}y_{-3}(3s+t)b}{1 + x_{-2}z_{-1}(3s+t)f} \frac{1 + x_{-3}z_{-2}(3s+t-1)f}{c^{3s+t-1}\left((1-c) - y_{-1}z_{-2}d\right) + y_{-1}z_{-2}d},$$

$$x_{6m+2t+2} = x_{2t-4} \prod_{s=0}^{m} \frac{x_{-3}y_{-3}z_{-3}}{x_{-1}y_{-1}z_{-1}} \frac{c^{3s+t+1}\left((1-c)-y_{-1}z_{-2}d\right)+y_{-1}z_{-2}d}{1+x_{-2}y_{-3}(3s+t+2)b} \times \frac{1+x_{-1}y_{-2}(3s+t)b}{1+x_{-3}z_{-2}(3s+t+1)f} \frac{1+x_{-2}z_{-1}(3s+t-1)f}{c^{3s+t}\left((1-c)-y_{-2}z_{-3}d\right)+y_{-2}z_{-3}d},$$

$$y_{6m+2t+1} = y_{2t-5} \prod_{s=0}^{m} \frac{x_{-1}y_{-1}z_{-1}}{x_{-3}y_{-3}z_{-3}} \frac{1+x_{-3}z_{-2}(3s+t+1)f}{c^{3s+t+1}\left((1-c)-y_{-1}z_{-2}d\right)+y_{-1}z_{-2}d} \times \frac{c^{3s+t}\left((1-c)-y_{-2}z_{-3}d\right)+y_{-2}z_{-3}d}{1+x_{-1}y_{-2}(3s+t)b} \frac{1+x_{-2}y_{-3}(3s+t-1)b}{1+x_{-2}z_{-1}(3s+t-1)f},$$

$$y_{6m+2t+2} = y_{2t-4} \prod_{s=0}^{m} \frac{x_{-3}y_{-3}z_{-3}}{x_{-1}y_{-1}z_{-1}} \frac{1+x_{-2}z_{-1}(3s+t+1)f}{c^{3s+t+2}\left((1-c)-y_{-2}z_{-3}d\right)+y_{-2}z_{-3}d} \times \frac{c^{3s+t}\left((1-c)-y_{-1}z_{-2}d\right)+y_{-1}z_{-2}d}{1+x_{-2}y_{-3}(3s+t+1)b} \frac{1+x_{-1}y_{-2}(3s+t-1)b}{1+x_{-3}z_{-2}(3s+t)f},$$

$$z_{6m+2t+1} = z_{2t-5} \prod_{s=0}^{m} \frac{x_{-1}y_{-1}z_{-1}}{x_{-3}y_{-3}z_{-3}} \frac{1+x_{-2}y_{-3}(3s+t+1)b}{1+x_{-2}z_{-1}(3s+t+1)f} \times \frac{1+x_{-3}z_{-2}(3s+t)f}{c^{3s+t}\left((1-c)-y_{-1}z_{-2}d\right)+y_{-1}z_{-2}d} \times \frac{c^{3s+t-1}\left((1-c)-y_{-2}z_{-3}d\right)+y_{-2}z_{-3}d}{1+x_{-1}y_{-2}(3s+t-1)b},$$

$$z_{6m+2t+2} = z_{2t-4} \prod_{s=0}^{m} \frac{x_{-3}y_{-3}z_{-3}}{x_{-1}y_{-1}z_{-1}} \frac{1+x_{-1}y_{-2}(3s+t+1)b}{1+x_{-3}z_{-2}(3s+t+2)f} \times \frac{1+x_{-2}z_{-1}(3s+t)f}{c^{3s+t+1}\left((1-c)-y_{-2}z_{-3}d\right)+y_{-2}z_{-3}d} \times \frac{c^{3s+t-1}\left((1-c)-y_{-1}z_{-2}d\right)+y_{-1}z_{-2}d}{1+x_{-2}y_{-3}(3s+t)b},$$

for $m \geq -1$ and $t \in \{1,2,3\}$.

3.7. Case $a \neq 1$, $c = 1$, $e = 1$

In this case, the solution of system (9) can be written in the following form

$$x_{6m+2t+1} = x_{2t-5} \prod_{s=0}^{m} \frac{x_{-1}y_{-1}z_{-1}}{x_{-3}y_{-3}z_{-3}} \frac{1+y_{-2}z_{-3}(3s+t+1)d}{a^{3s+t+1}\left((1-a)-x_{-1}y_{-2}b\right)+x_{-1}y_{-2}b}$$

$$\times \frac{a^{3s+t}\left((1-a)-x_{-2}y_{-3}b\right)+x_{-2}y_{-3}b}{1+x_{-2}z_{-1}(3s+t)f}\frac{1+x_{-3}z_{-2}(3s+t-1)f}{1+y_{-1}z_{-2}(3s+t-1)d},$$

$$x_{6m+2t+2}=x_{2t-4}\prod_{s=0}^{m}\frac{x_{-3}y_{-3}z_{-3}}{x_{-1}y_{-1}z_{-1}}\frac{1+y_{-1}z_{-2}(3s+t+1)d}{a^{3s+t+2}\left((1-a)-x_{-2}y_{-3}b\right)+x_{-2}y_{-3}b}$$
$$\times \frac{a^{3s+t}\left((1-a)-x_{-1}y_{-2}b\right)+x_{-1}y_{-2}b}{1+x_{-3}z_{-2}(3s+t+1)f}\frac{1+x_{-2}z_{-1}(3s+t-1)f}{1+y_{-2}z_{-3}(3s+t)d},$$

$$y_{6m+2t+1}=y_{2t-5}\prod_{s=0}^{m}\frac{x_{-1}y_{-1}z_{-1}}{x_{-3}y_{-3}z_{-3}}\frac{1+x_{-3}z_{-2}(3s+t+1)f}{1+y_{-1}z_{-2}(3s+t+1)d}$$
$$\times \frac{1+y_{-2}z_{-3}(3s+t)d}{a^{3s+t}\left((1-a)-x_{-1}y_{-2}b\right)+x_{-1}y_{-2}b}$$
$$\times \frac{a^{3s+t-1}\left((1-a)-x_{-2}y_{-3}b\right)+x_{-2}y_{-3}b}{1+x_{-2}z_{-1}(3s+t-1)f},$$

$$y_{6m+2t+2}=y_{2t-4}\prod_{s=0}^{m}\frac{x_{-3}y_{-3}z_{-3}}{x_{-1}y_{-1}z_{-1}}\frac{1+x_{-2}z_{-1}(3s+t+1)f}{1+y_{-2}z_{-3}(3s+t+2)d}$$
$$\times \frac{1+y_{-1}z_{-2}(3s+t)d}{a^{3s+t+1}\left((1-a)-x_{-2}y_{-3}b\right)+x_{-2}y_{-3}b}$$
$$\times \frac{a^{3s+t-1}\left((1-a)-x_{-1}y_{-2}b\right)+x_{-1}y_{-2}b}{1+x_{-3}z_{-2}(3s+t)f},$$

$$z_{6m+2t+1}=z_{2t-5}\prod_{s=0}^{m}\frac{x_{-1}y_{-1}z_{-1}}{x_{-3}y_{-3}z_{-3}}\frac{a^{3s+t+1}\left((1-a)-x_{-2}y_{-3}b\right)+x_{-2}y_{-3}b}{1+x_{-2}z_{-1}(3s+t+1)f}$$
$$\times \frac{1+x_{-3}z_{-2}(3s+t)f}{1+y_{-1}z_{-2}(3s+t)d}\frac{1+y_{-2}z_{-3}(3s+t-1)d}{a^{3s+t-1}\left((1-a)-x_{-1}y_{-2}b\right)+x_{-1}y_{-2}b},$$

$$z_{6m+2t+2}=z_{2t-4}\prod_{s=0}^{m}\frac{x_{-3}y_{-3}z_{-3}}{x_{-1}y_{-1}z_{-1}}\frac{a^{3s+t+1}\left((1-a)-x_{-1}y_{-2}b\right)+x_{-1}y_{-2}b}{1+x_{-3}z_{-2}(3s+t+2)f}$$
$$\times \frac{1+x_{-2}z_{-1}(3s+t)f}{1+y_{-2}z_{-3}(3s+t+1)d}\frac{1+y_{-1}z_{-2}(3s+t-1)d}{a^{3s+t}\left((1-a)-x_{-2}y_{-3}b\right)+x_{-2}y_{-3}b},$$

for $m\geq -1$ and $t\in\{1,2,3\}$.

3.8. Case $a=1$, $c=1$, $e=1$

In this case, solutions of system (9) are as follows

$$x_{6m+2t+1} = x_{2t-5} \prod_{s=0}^{m} \frac{x_{-1}y_{-1}z_{-1}}{x_{-3}y_{-3}z_{-3}} \frac{1+y_{-2}z_{-3}(3s+t+1)d}{1+x_{-1}y_{-2}(3s+t+1)b} \times \frac{1+x_{-2}y_{-3}(3s+t)b}{1+x_{-2}z_{-1}(3s+t)f} \frac{1+x_{-3}z_{-2}(3s+t-1)f}{1+y_{-1}z_{-2}(3s+t-1)d},$$

$$x_{6m+2t+2} = x_{2t-4} \prod_{s=0}^{m} \frac{x_{-3}y_{-3}z_{-3}}{x_{-1}y_{-1}z_{-1}} \frac{1+y_{-1}z_{-2}(3s+t+1)d}{1+x_{-2}y_{-3}(3s+t+2)b} \times \frac{1+x_{-1}y_{-2}(3s+t)b}{1+x_{-3}z_{-2}(3s+t+1)f} \frac{1+x_{-2}z_{-1}(3s+t-1)f}{1+y_{-2}z_{-3}(3s+t)d},$$

$$y_{6m+2t+1} = y_{2t-5} \prod_{s=0}^{m} \frac{x_{-1}y_{-1}z_{-1}}{x_{-3}y_{-3}z_{-3}} \frac{1+x_{-3}z_{-2}(3s+t+1)f}{1+y_{-1}z_{-2}(3s+t+1)d} \times \frac{1+y_{-2}z_{-3}(3s+t)d}{1+x_{-1}y_{-2}(3s+t)b} \frac{1+x_{-2}y_{-3}(3s+t-1)b}{1+x_{-2}z_{-1}(3s+t-1)f},$$

$$y_{6m+2t+2} = y_{2t-4} \prod_{s=0}^{m} \frac{x_{-3}y_{-3}z_{-3}}{x_{-1}y_{-1}z_{-1}} \frac{1+x_{-2}z_{-1}(3s+t+1)f}{1+y_{-2}z_{-3}(3s+t+2)d} \times \frac{1+y_{-1}z_{-2}(3s+t)d}{1+x_{-2}y_{-3}(3s+t+1)b} \frac{1+x_{-1}y_{-2}(3s+t-1)b}{1+x_{-3}z_{-2}(3s+t)f}$$

$$z_{6m+2t+1} = z_{2t-5} \prod_{s=0}^{m} \frac{x_{-1}y_{-1}z_{-1}}{x_{-3}y_{-3}z_{-3}} \frac{1+x_{-2}y_{-3}(3s+t+1)b}{1+x_{-2}z_{-1}(3s+t+1)f} \times \frac{1+x_{-3}z_{-2}(3s+t)f}{1+y_{-1}z_{-2}(3s+t)d} \frac{1+y_{-2}z_{-3}(3s+t-1)d}{1+x_{-1}y_{-2}(3s+t-1)b},$$

$$z_{6m+2t+2} = z_{2t-4} \prod_{s=0}^{m} \frac{x_{-3}y_{-3}z_{-3}}{x_{-1}y_{-1}z_{-1}} \frac{1+x_{-1}y_{-2}(3s+t+1)b}{1+x_{-3}z_{-2}(3s+t+2)f} \times \frac{1+x_{-2}z_{-1}(3s+t)f}{1+y_{-2}z_{-3}(3s+t+1)d} \frac{1+y_{-1}z_{-2}(3s+t-1)d}{1+x_{-2}y_{-3}(3s+t)b},$$

for $m \geq -1$ and $t \in \{1, 2, 3\}$.

Lemma 2. *If $a \neq 1$, $c \neq 1$, $e \neq 1$, $b \neq 0$, $d \neq 0$ and $f \neq 0$, then the system* (9) *has 6−periodic solutions.*

Proof. Let

$$\alpha_n = x_{n-2}y_{n-3}, \quad \beta_n = y_{n-2}z_{n-3} \quad \text{and} \quad \gamma_n = z_{n-2}x_{n-3},\ n \in \mathbb{N}_0.$$

Then from (9) we get

$$\alpha_{n+2} = \frac{\alpha_n}{a + b\alpha_n}, \quad \beta_{n+2} = \frac{\beta_n}{c + d\beta_n} \quad \text{and} \quad \gamma_{n+2} = \frac{\gamma_n}{e + f\gamma_n},\ n \in \mathbb{N}_0. \tag{18}$$

If $b \neq 0$, $d \neq 0$ and $f \neq 0$, then system (18) has a unique equilibrium solution which $(\overline{\alpha}, \overline{\beta}, \overline{\gamma})$ is different from $(0,0,0)$, that is,

$$\alpha_n = \overline{\alpha} = \frac{1-a}{b} \neq 0, \quad \beta_n = \overline{\beta} = \frac{1-c}{d} \neq 0, \quad \gamma_n = \overline{\gamma} = \frac{1-e}{f} \neq 0, \quad n \in \mathbb{N}_0.$$

If $\overline{\alpha} = 0$ or $\overline{\beta} = 0$ or $\overline{\gamma} = 0$, then system (9) has not well-defined solutions. From (18), we have

$$\begin{aligned}
x_{n-2} &= \frac{1-a}{by_{n-3}} = \frac{(1-a)d}{b(1-c)}z_{n-4} = \frac{(1-a)d(1-e)}{b(1-c)fx_{n-5}}, \\
&= \frac{d(1-e)}{(1-c)f}y_{n-6} = \frac{1-e}{fz_{n-7}} = x_{n-8},\ n \geq 5, \\
y_{n-2} &= \frac{1-c}{dz_{n-3}} = \frac{(1-c)f}{d(1-e)}x_{n-4} = \frac{(1-c)f(1-a)}{d(1-e)by_{n-5}} \\
&= \frac{f(1-a)}{(1-e)b}z_{n-6} = \frac{1-a}{bx_{n-7}} = y_{n-8},\ n \geq 5, \\
z_{n-2} &= \frac{1-e}{fx_{n-3}} = \frac{(1-e)b}{f(1-a)}y_{n-4} = \frac{(1-e)b(1-c)}{f(1-a)dz_{n-5}} \\
&= \frac{b(1-c)}{(1-a)d}x_{n-6} = \frac{1-c}{dy_{n-7}} = z_{n-8},\ n \geq 5,
\end{aligned}$$

from which along with the assumptions in Lemma 2, the results can be easily seen. □

The following theorem give the forbidden set of the initial conditions for system (9).

Theorem 1. *Assume that $a \neq 0$, $b \neq 0$, $c \neq 0$, $d \neq 0$, $e \neq 0$, $f \neq 0$. The forbidden set of the initial values for system (9) is given by the set*

$$\begin{aligned}
\mathbb{F} = \bigcup_{m \in \mathbb{N}_0} \bigcup_{i=0}^{1} \Big\{ &\frac{1}{x_{i-2}y_{i-3}} = \widehat{f}^{-m-1}\left(-\frac{b}{a}\right), \quad \frac{1}{y_{i-2}z_{i-3}} = g^{-m-1}\left(-\frac{d}{c}\right), \\
&\frac{1}{z_{i-2}x_{i-3}} = h^{-m-1}\left(-\frac{f}{e}\right) \Big\} \bigcup \bigcup_{j=1}^{3} \Big\{ \left(\vec{x}_{-(3,1)}, \vec{y}_{-(3,1)}, \vec{z}_{-(3,1)}\right) \in \mathbb{R}^9 : \\
&x_{-j} = 0 \ \mathit{or}\ y_{-j} = 0 \ \mathit{or}\ z_{-j} = 0 \Big\},
\end{aligned} \tag{19}$$

where $\vec{x}_{-(3,1)} = (x_{-3}, x_{-2}, x_{-1})$, $\vec{y}_{-(3,1)} = (y_{-3}, y_{-2}, y_{-1})$, $\vec{z}_{-(3,1)} = (z_{-3}, z_{-2}, z_{-1})$.

Proof. We have obtained that the set

$$\bigcup_{j=1}^{3} \Big\{ \left(\vec{x}_{-(3,1)}, \vec{y}_{-(3,1)}, \vec{z}_{-(3,1)}\right) \in \mathbb{R}^9 : \ x_{-j} = 0 \text{ or } y_{-j} = 0 \text{ or } z_{-j} = 0 \Big\},$$

where $\vec{x}_{-(3,1)} = (x_{-3}, x_{-2}, x_{-1})$, $\vec{y}_{-(3,1)} = (y_{-3}, y_{-2}, y_{-1})$, $\vec{z}_{-(3,1)} = (z_{-3}, z_{-2}, z_{-1})$, belongs to the forbidden set of the initial values for system (9) at the beginning of Section 2. If $x_{-j} \neq 0$, $y_{-j} \neq 0$ and $z_{-j} \neq 0$, $j \in \{1, 2, 3\}$, then system (9) is undefined if and only if

$$a + bx_{n-2}y_{n-3} = 0, \ c + dy_{n-2}z_{n-3} = 0, \ e + fz_{n-2}x_{n-3} = 0, \ n \in \mathbb{N}_0.$$

By taking into account the change of variables (12), we can write the corresponding conditions

$$u_{n-2} = -\frac{b}{a}, \ v_{n-2} = -\frac{d}{c} \text{ and } \ w_{n-2} = -\frac{f}{e}, \ n \in \mathbb{N}_0. \tag{20}$$

Therefore, we can determine the forbidden set of the initial values for system (9) by using system (13). We know that the statements

$$u_{2m+i} = \widehat{f}^{m+1}\left(u_{i-2}\right), \tag{21}$$

$$v_{2m+i} = g^{m+1}\left(v_{i-2}\right), \tag{22}$$

$$w_{2m+i} = h^{m+1}\left(w_{i-2}\right), \tag{23}$$

where $m \in \mathbb{N}_0$, $i \in \{0, 1\}$, $\widehat{f}(x) = ax + b$, $g(x) = cx + d$ and $h(x) = ex + f$, characterize the solutions of system (9). By using the conditions (20) and the statements (21)-(23), we have

$$u_{i-2} = \widehat{f}^{-m-1}\left(-\frac{b}{a}\right), \tag{24}$$

$$v_{i-2} = g^{-m-1}\left(-\frac{d}{c}\right), \tag{25}$$

$$w_{i-2} = h^{-m-1}\left(-\frac{f}{e}\right), \tag{26}$$

where $m \in \mathbb{N}_0$, $i \in \{0, 1\}$ and $abcdef \neq 0$. This means that if one of the conditions in (24)-(26) holds, then $m-$th iteration or $(m+1)-$th iteration in system (9) can not be calculated. Consequently, we obtain the result in (19). □

4. Conclusion

In this paper, we have solved the following three-dimensional system of difference equations

$$x_n = \frac{x_{n-2}y_{n-3}}{y_{n-1}\left(a + bx_{n-2}y_{n-3}\right)},$$

$$y_n = \frac{y_{n-2}z_{n-3}}{z_{n-1}\left(c + dy_{n-2}z_{n-3}\right)}, n \in \mathbb{N}_0,$$
$$z_n = \frac{z_{n-2}x_{n-3}}{x_{n-1}\left(e + fz_{n-2}x_{n-3}\right)},$$

where the initial values x_{-i}, y_{-i}, z_{-i}, $i = \overline{1,3}$ and the parameters a, b, c, d, e, f are non-zero real numbers. In addition, we have obtained the solutions of above system in explicit form according to the parameters a, c and e are equal 1 or not equal 1. Moreover, we have got periodic solutions of aforementioned system. Finally, we have identified the forbidden set of the initial conditions by using the acquired formulas.

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