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Dynamical Interpretation of Logistic Map using Euler's Algorithm

Sanjeev [*,1], **Anjali** [β,2], **Ashish** [α,3] **and A. K. Malik** [§,4]
[*]Department of Mathematics, Suraj Degree College, Mahendergarh, India, [β]Department of Mathematics, National Institute of Technology, Mizoram, India, [α]Department of Mathematics, Government College, Satnali-123024, India, [§]School of Mathematics, UP Rajarshi Tandon Open University, Prayagraj-211021, India.

ABSTRACT In the last two decades, the dynamics of difference and differential equations have found a celebrated place in science and engineering such as weather forecasting, secure communication, transportation problems, biology, the population of species, etc. In this article, we deal with the dynamical behavior of the logistic map using Euler's numerical algorithm. The dynamical properties of the Euler's type logistic system are derived analytically as well as experimentally using the bifurcation diagram. In the analytical section the dynamical properties such as fixed point, period-doubling, and irregularity are examined followed by a few theorems. Further, in the experimental section, the dynamical properties of the Euler's type logistic system are studied using the period-doubling bifurcation plot. Because the dynamics of the Euler's map depend on the Euler's control parameter h, therefore, the three major cases are discussed for $h = 0.1, 0.4$ and 0.7. The result shows that as the value of parameter h decreases from 1 to 0 the growth rate parameter r increases rapidly. Therefore, the improved chaotic regime in bifurcation plots may improve the chaos based applications in science and engineering such as secure communication.

KEYWORDS
Fixed point
Periodic orbit
Euler's Algorithm
Chaos

INTRODUCTION

In the last few decades, the term "Chaos" has become the subject matter of the study in mathematics which determines the fixed and periodic and irregularity in the nonlinear dynamical systems. This concept was described by Poincare (1899) when he examined the qualitative results in nonlinear systems and celestial mechanics. But in 1960's Lorenz (1963) again recalled it and examined it chaotic part which depends on the initial condition. Further, May (1976) and Lorenz (1963) researched much important arithmetic after that all the nonlinear dynamical system has been saturated with analytical and numerical results of difference and differential equations. The logistic map $rx(1-x)$, is the most researched difference map in the nonlinear dynamics which is also known as the model of population growth introduced by V. F. Verhulst. In 1978, Feigenbaum (1978) examined the generic dynamical properties of the logistic map using experimental and analytical simulations. Moreover, for a brief elementary analysis of about the nonlinear dynamical systems and their qualitative properties one may read the following published and unpublished research like Robinson (1995), Alligood *et al.* (1996), Ausloos *et al.* (2006), Devaney (1948), Holmgren (1994), and Ashish *et al.* (2019b).

Since 1930, the nonlinear dynamical systems have played a vital role in various applications of science and engineering such as cryptography, transportation problems, traffic signal control system, secure communication systems, neural network, switch technology, electronics and many other branches of science. In last two decades the discrete logistic map and its various generalized versions have been studied in the literature as a road map in the nonlinear dynamical systems. In 1996, Song *et al.* (1996) researched the dynamical behavior of logistic map using error valued feedback method for synchronization of the dynamics and Molina *et al.* (1996) examined the embedded dimension of various chaotic maps using time series methodology.

A communication system to develop irregular signals was developed using difference maps by Singh *et al.* (2010) in 2010. Further, in 2013 Radwan *et al.* (2013) introduced the various modulated

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[1] sanjeevrao.652@gmail.com
[2] anjaliyadav42877@gmail.com
[3] drashishkumar108@gmail.com (**Corresponding Author**)
[4] ajendermalik@gmail.com

discrete difference systems and described their dynamical properties such as fixed-point state, period-doubling interpretations and chaotic behavior. Parasad *et al.* (2014) described the stabilization in the fixed and periodic states of the logistic maps. In 2005, Rani *et al.* (2005) and Kumar*et al.* (2005), using the new technology examined the stabilization in the chaotic maps. They introduced a two-step feedback method, that is, Ishikawa iterates, that shows that the logistic map has fast convergence for the extra range of the control parameter r as compared to one-step feedback procedure and also presented a comparative study in Picard orbit in Agarwal and Rani Rani *et al.* (2009). In 1953, Mann (1953) introduced a novel three-step feedback procedure also known as Mann iterative method which give superior results in functional analysis and every branch of mathematics. Further, Chugh *et al.* (2012), in 2012 examined the stability and convergence of the logistic map using another four-step feedback procedure also known as Noor iterative method. Khamosh (2020) and Kumar (2020) studied the dynamics of the generalized logistic map in superior orbit (see also Renu *et al.* (2022)).

Recently, He *et al.* (2023) introduced an homotopy perturbation method which increases the effectiveness in nonlinear oscillator systems. It is also observed that the frequency accuracy may be improved the oscillators by increasing the iteration in the system. Ashish *et al.* (2019a) established the chaotic behavior of the logistic map using superior technique and examined the onset chaos properties like period-doubling to chaos, period-3 window a road map to chaos and maximum Lyapunov exponent. Later, they examined the dynamical properties using cobweb plot, time-series analysis and bifurcation plot in superior orbit Ashish *et al.* (2018).

In 2019, stabilization in fixed and periodic states was examined and its application in transportation system was examined Ashish *et al.* (2021a), Ashish *et al.* (2021b), and Ashish *et al.* (2021c). The article is divided into five major sections. Section 1 is introductory in nature and describes a brief literature review and Section 2 contains the basic definition of fixed point, periodic point and Picard feedback procedure. In Section 3, the analytical results are proved for the Euler's type logistic map and experimental analysis is carried out in section 4. Finally, the article is concluded in Section 5.

PRELIMINARIES

This section deals with the basic terminologies, notions and definition which are continuously used in the article.

Definition 1. . Let f be a one-dimensional map defined on non-empty sets X. Then the Picard orbit which is also known as orbit of function is the set of all iterates of an initial point x and defined as $x_{n+1} = f(x_n)$.

Definition 2. . Let f be a one-dimensional map defined on a set X, where X is a non-empty set. A point $x \in X$ is said to be periodic fixed point of period-p or cycle-p if it satisfies $f^p(x) = x$, where p is a positive integer.

Definition 3. . Let f be a one-dimensional map defined on a set X, where X is a non-empty set. A point $x \in X$ is said to be fixed if it satisfies the condition $f(x) = x$.

ANALYTICAL INTERPRETATION

This section deals with the analytical study of the logistic map $rz(1-z)$ where $r \in [0,4]$ and $z \in [0,1]$ using Euler's numerical algorithm. The Euler's numerical algorithm is given by

$$E_h(z,r) = z + hf_r(z). \tag{1}$$

This equation has two regular fixed points $z^* = 0$ and $z^* = 1$. Since the solutions for an initiator $z_0 \in [0,1]$ and $r > 0$, approaches to the regular fixed state $z^* = 1$ from the interval $[-z,z]$. But such type of system has not much importance in the dynamics of one-dimensional chaotic maps. Therefore, the given system (1) is modified in more simplified quadratic discrete system. For this let us consider the parameter $x = \frac{hr}{1+hr}z$, then the relation (1) is described by

$$E_h(x,r) = (1+hr)x(1-x) \tag{2}$$

where x belongs to the closed interval $[0,1]$ and the relation (2) is called as Euler's type novel logistic system. Now, let us determine the following result regarding fixed point, periodic point and the stability of this novel Euler's logistic map.

Theorem 1. *Let $f_r(x) = rx(1-x)$ be the one-dimensional logistic map defined on the closed interval $[0,1]$ and $r \in [0,4]$. Then, show that 0 and $\frac{hr}{1+hr}$ are the fixed points for the Euler's type logistic map.*

Proof. Since $f_r(x) = rx(1-x)$ and $E_h(x,r) = (1+hr)x(1-x)$, is the Euler's logistic system, then from the definition of the fixed point, we can say

$$\begin{aligned}
&E_h(x,r) = x,\\
&(1+hr)x(1-x) = x,\\
&(1+hr)x(1-x) - x = 0,\\
&x[(1+hr)(1-x) - 1] = 0,\\
\text{thus,}\quad &x = 0 \quad \text{and} \quad x = \frac{hr}{1+hr}.
\end{aligned}$$

Thus, the point $x = 0$ and $x = \frac{hr}{1+hr}$, where $h, r > 0$ is the Euler's fixed point for the Euler's Logistic map. This completes the proof of the theorem. While, Figure 1 shows the functional representation of the logistic map in Euler's numerical algorithm. Figure shows the fixed-point $x = 0$ and $x = \frac{hr}{1+hr}$ where the diagonal axis intersects the functional graph of the map. Similarly, the periodic fixed point of order two are also derived using the following theorem. □

Theorem 2. *Let $f_r(x) = rx(1-x)$ be the one-dimensional logistic map defined on the closed interval $[0,1]$ and $r \in [0,4]$. Then, show that $E_h^2(x,r)$ has four fixed points for the Euler's map.*

Proof. Since $f_r(x) = rx(1-x)$ and $E_h(x,r) = (1+hr)x(1-x)$, is the Euler's logistic system, then from the definition of fixed point, we can say

$$\begin{aligned}
&E_h^2(x,r) = E_h(E_h(x,r),r) = x,\\
&(1+hr)2x(1-x)(1-((1+hr)x(1-x))) = x,\\
&(1+hr)2x(1-x)(1-((1+hr)x(1-x))) - x = 0.
\end{aligned}$$

Then, solving the above relation, we obtain the following four roots:

$$x_1 = 0,$$
$$x_2 = \frac{hr}{1+hr},$$
$$x_3 = \frac{(1+hr) - \sqrt{(hr-2)(2+hr)}}{2(1+hr)},$$
$$x_4 = \frac{(1+hr) + \sqrt{(hr-2)(2+hr)}}{2(1+hr)}.$$

Thus, x_1, x_2, x_3, and x_4 are the four fixed points for the system $E_h^2(x,r)$. The fixed point x_1 and x_2 are the trivial point of order one as discussed in Theorem 1 and x_3 and x_4 are periodic point of order two for the given Euler's logistic system. But it is observed that the periodic roots are real if and only if $r > 2/h$. Further, the Figure 2 shows the graphical representation for the fixed points x_1, x_2, x_3, and x_4 which intersect the diagonal axis $y = x$ of the graph. Proceeding in this way the we can get the periodic points of higher orders, that is the periodic points of order 4, 8, 16, 32, and so on using the dynamical system $E_h^3(x,r)$, $E_h^4(x,r)$, $E_h^5(x,r)$, $E_h^6(x,r)$, and so on. But it is not so simple to solve the higher order equations using analytically. Therefore, they are determined using the numerical simulation in computational software Mathematica, and SPSS. □

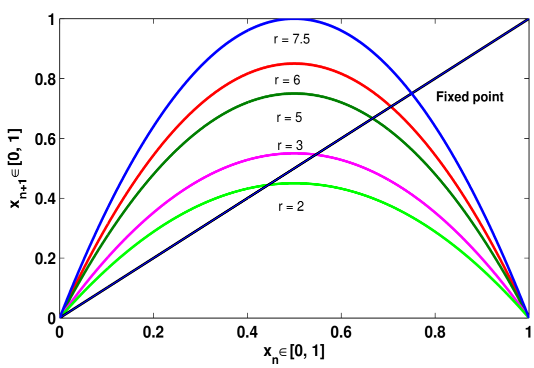

Figure 1 Functional plot for the Euler's Logistic Map $E_h(x,r)$ for a variety of r value at $h = 0.4$

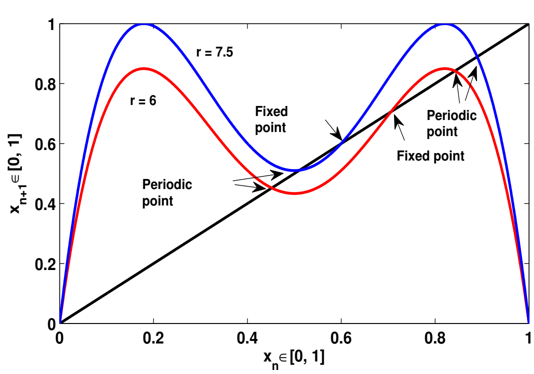

Figure 2 Functional plot for the Euler's Logistic Map $E_h^2(x,r)$ for $r = 7.5$ and 6 at $h = 0.4$

EXPERIMENTAL INTERPRETATION

This section deals with the experimental study of the one-dimensional logistic map $rx(1-x)$, where $r \in [0,4]$ and $x \in [0,1]$ using Euler's numerical algorithm. As studied in the above section the dynamics of the Euler's type logistic map depends on the two-control parameter, Euler's parameter h and the logistic parameter r. Therefore, the nature of the Euler's system $E_h(x,r)$ is examined for different parameter values of h and the regime and the dynamical behavior for the advanced range of parameter r is determined. Let us take the three cases for $h = 0.1, 0.4$ and 0.7 and examine the dynamical nature using bifurcation plot as follows:

Case-1: Dynamics for $E_h(x,r)$ at $h = 0.1$, and $0 \le r \le 30$

When $h = 0.1$, the Euler's map has stable fixed-point behavior up to value $r = 20$, after that the first bifurcation is seen at $r > 20$ at which the Euler's orbit is divided in to two period orbits x_3 and x_4 of order two as determined in above section and. The stability in the periodic fixed point of order 2 is then studied for $20 < r \le 24.899$ as shown in Figures 3 and 4. Further, for $r > 24.899$ the characteristics of the Euler's map is again noticed in which the periodicity of order two, that is, x_3 and x_4 are further divided in to the periodic fixed points of order four as shown in the Figure 4 for the range of parameter $24.899 < r \le 25.44$. But the parameter r increases through 24.899, the periodicity of order two becomes unstable and the periodic point of order four become stable for $24.899 < r \le 25.44$.

Proceeding in this way as the value of parameter r increases through 25.44 the Euler's orbit bifurcates in to higher order periodic fixed points, that is, in the order of $8, 16, 32, 64, \ldots$ and soon as shown in the Figure 4. But as the parameter r approaches to 25.6996 the dynamics of the Euler's logistic map tends to chaotic regime. The magnified Figure 4 shows the complete period-doubling regime, Figure 5 shows the magnified chaotic regime and Figure 6 represents the magnified period-3 window regime. Finally, the above analysis is summarized in the following proposition.

Proposition 1. *It is noticed that the dynamics of the Euler's type logistic system admits higher range of the control parameter r, that is, r lies between 0 and 30 at $h = 0.1$ as compared to the standard logistic system $rx(1-x)$, where r approaches from 0 to 4.*

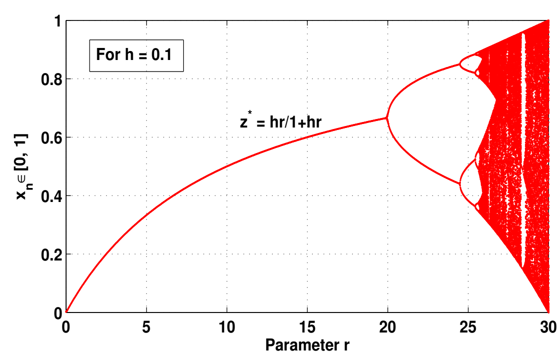

Figure 3 Bifurcation plot for the map $E_h(x,r)$ for $h = 0.1$ and $r \in [0,30]$

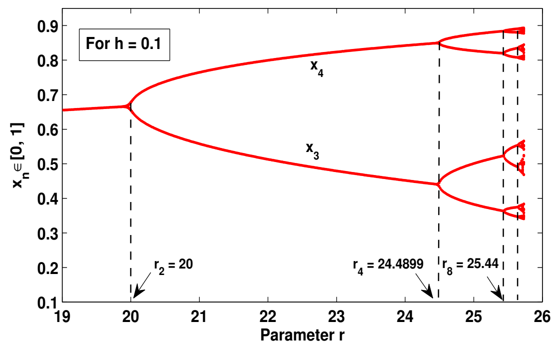

Figure 4 Magnified periodic regime for the map $E_h(x, r)$ for $h = 0.1$

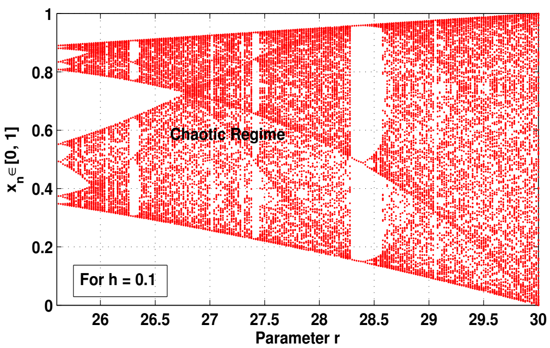

Figure 5 Magnified Chaotic regime for the map $E_h(x, r)$ for $h = 0.1$

Case-2: Dynamics for $E_h(x, r)$ at $h = 0.4$, and $0 \leq r \leq 7.5$

Further, as the range of parameter h is increased and is taken as $h = 0.4$, range the control parameter r decreases rapidly, that is, r approaches from 0 to 7.5 as shown in bifurcation plot Figure 4. While Figure 7 gives the complete dynamics of the Euler's logistic system which describes fixed-point state, periodic state and chaotic regime. At $h = 0.4$, the Euler's map has stable fixed-point behavior up to $r = 5$, after that the first bifurcation occurs at $r = 6.1255$ at which the Euler's orbit is divided in to two period orbits x_3 and x_4 of order two as determined in above the section. The stability in the periodic fixed points of order 2 is then studied for $6.1255 < r \leq 6.3573$ as shown in Figure 8. Further, for $r > 6.3573$ the orbit of the Euler's logistic map is again noticed in which the periodicity of order two, that is, x_3 and x_4 are further divided in to the periodic fixed points of order four as shown in the Figure 8 for the range of parameter $6.1255 < r \leq 6.3573$. But the parameter r increases through 6.1255, the periodicity of order two becomes unstable and the periodic point of order four become stable for $6.1255 < r \leq 6.3573$.

Proceeding in this way as the value of parameter r increases through 6.4299 the Euler's orbit bifurcates in to higher order periodic fixed points, that is, in the order of $8, 16, 32, 64, \ldots$ and soon as shown in the Figure 8. But as the parameter r approaches to 6.4299 the dynamics of the Euler's logistic map tends to chaotic regime as shown in the magnified Figure 9. The magnified Figure 8 shows the complete period-doubling regime while the magnified Figure 10 represent the complete chaotic regime with period-3 window.

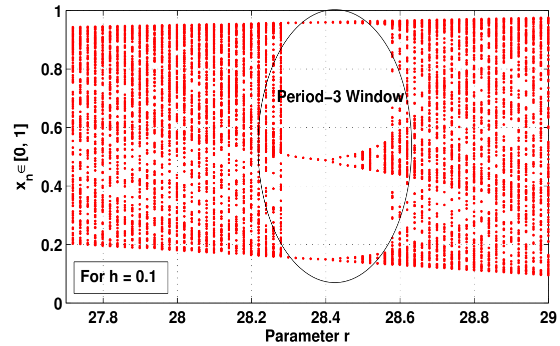

Figure 6 Magnified period-3 window for the map $E_h(x, r)$ for $h = 0.1$

Proposition 2. *It is noticed that the dynamics of the Euler's type logistic system again admits higher range of the control parameter r, that is, r lies between 0 and 7.5 at $h = 0.4$ as compared to the standard logistic system $rx(1 - x)$, where r approaches from 0 to 4.*

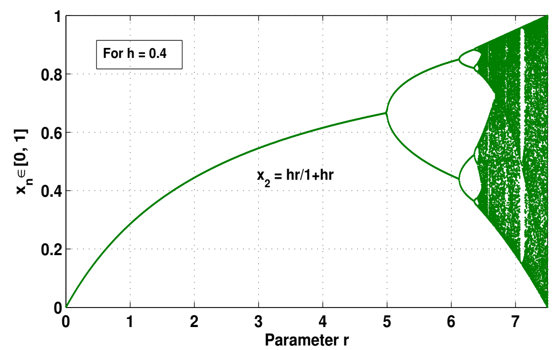

Figure 7 Bifurcation plot for the map $E_h(x, r)$ for $h = 0.4$ and $r \in [0, 7.5]$

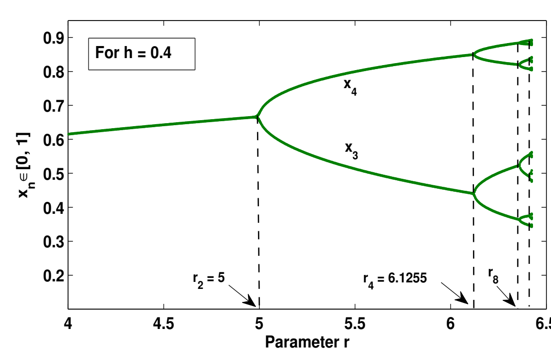

Figure 8 Magnified periodic regime for the map $E_h(x, r)$ for $h = 0.4$

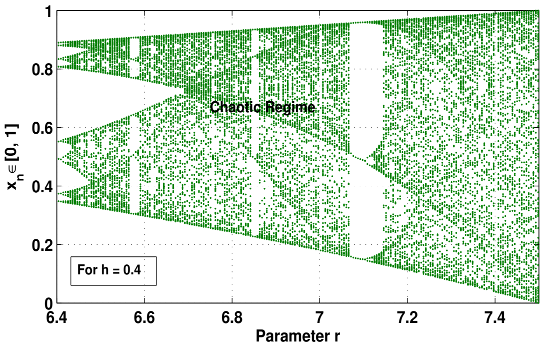

Figure 9 Magnified Chaotic regime for the map $E_h(x,r)$ for $h = 0.4$

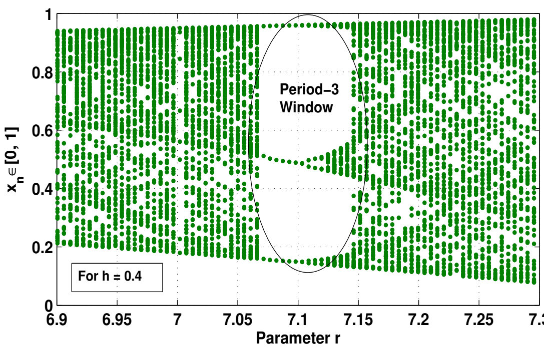

Figure 10 Magnified period-3 window regime for the map $E_h(x,r)$ for $h = 0.4$

Case-3: Dynamics for $E_h(x,r)$ at $h = 0.7$, and $0 \le r \le 4.25$

Further, as the range of parameter h is increased and is taken as $h = 0.4$, range the control parameter r decreases rapidly, that is, r approaches from 0 to 4.25 as shown in bifurcation plot Figure 11. While Figure 11 gives the complete dynamics of the Euler's logistic system which describes fixed-point state, periodic state and chaotic regime. At $h = 0.7$, the Euler's map has stable fixed-point behavior up to $r = 2.8339$, after that the first bifurcation occurs at $r = 3.4953$ at which the Euler's orbit is divided in to two period orbits x_3 and x_4 of order two as determined in above section and.

Proceeding in this way as the value of parameter r increases through 3.4953 the Euler's orbit bifurcates in to higher order periodic fixed points, that is, in the order of $8, 16, 32, 64, \ldots$ and soon as shown in the Figure 12. But as the parameter r approaches to 3.6696 the dynamics of the Euler's logistic map tends to chaotic regime. The magnified Figure 12 shows the complete period-doubling regime while the magnified Figure 13 represent the complete chaotic regime with period-3 window. While Figure 14 shows a comparative representation of the bifurcation plots for the parameter value $h = 0.1, 0.4$ and 0.7. Thus, we summarize the case 3 as follows:

Proposition 3. *It is observed that as the Euler's parameter range of the h is increased the range of the growth rate parameter r are decreases simultaneously. But for the lower range of Euler's parameter h the growth rate parameter range is higher than the standard logistic system.*

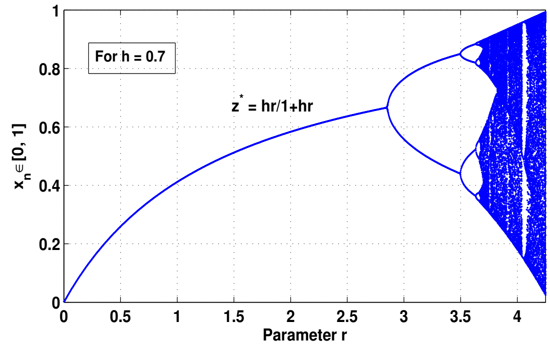

Figure 11 Bifurcation plot for the map $E_h(x,r)$ for $h = 0.7$ and $r \in [0, 4.25]$

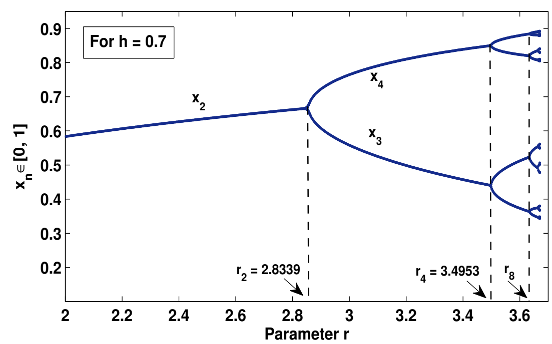

Figure 12 Magnified periodic regime for the map $E_h(x,r)$ for $h = 0.7$

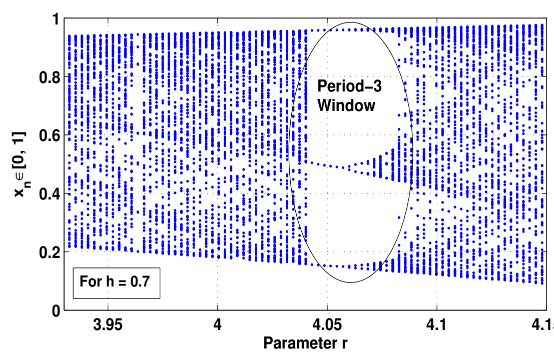

Figure 13 Magnified period-3 window region for the map $E_h(x,r)$ for $h = 0.7$

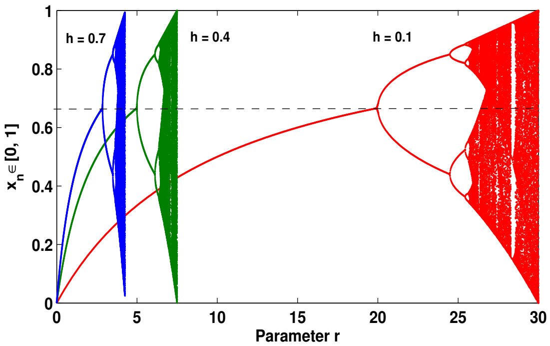

Figure 14 Comparative representation of bifurcation plots for $h = 0.1, 0.4$ and 0.7

CONCLUSION

In this article using some computational work on conventional logistic map in Euler's numerical algorithm is studied. The whole dynamics depends on the two control parameters h and r. Therefore, the following results are concluded from the main section:

- The dynamical properties of the Euler's type logistic map are determined analytically as well as experimentally.
- In the analytical section the Euler's logistic type map is derived and the fixed and periodic points are calculated followed by the Theorem 1 and Theorem 2.
- Further, in experimental section the dynamical properties of the Euler's logistic map are studied using period-doubling bifurcation plot. Because the dynamics of the Euler's map depends on the Euler's control parameter h, three cases are discussed for all the dynamical properties for $h = 0.1, 0.4$ and 0.7.
- It is also observed that the map exhibits its dynamical properties for a large range of parameter r, as compared to the existing methods. It is also observed that as compared to Picard iteration method which has growth rate $r \in [0, 4]$ and Mann iteration $r \in [0, 4.22]$, in this technique the growth rate parameter r approaches to 30. Hence it may improve the chaos-based application in engineering and science such as secure communication and cryptography.

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Conflicts of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Availability of data and material

Not applicable.

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