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THE FLOW-CURVATURE OF PLANE PARAMETRIZED CURVES

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Abstract. We introduce and study a new frame and a new curvature function for a fixed parametrization of a plane curve. This new frame is called *flow* since it involves the time-dependent rotation of the usual Frenet flow; the angle of rotation is exactly the current parameter. The flow-curvature is calculated for several examples obtaining the logarithmic spirals (and the circle as limit case) and the Grim Reaper as flat-flow curves. A main result is that the scaling with $\frac{1}{\sqrt{2}}$ of both Frenet and flow-frame belong to the same fiber of the Hopf bundle. Moreover, the flow-Fermi-Walker derivative is defined and studied.

1. Introduction

The theory of geometric flows is a new and fascinating field of research in geometric analysis. The most simple of them is *the curve shortening flow* and already the excellent survey [3] is twenty years old. Recall that the main geometric tool in this last flow is the well-known curvature of plane curves. Hence, to give a re-start to this problem seams to search for variants of the curvature, or in terms of [9], *deformations* of the usual curvature. The goal of this short note is to propose such a deformation which in turn defines a Fermi-Walker type derivative.

Fix an open interval $I \subseteq \mathbb{R}$ and consider $C \subset \mathbb{R}^2$ a regular parametrized curve of equation:

$$C : r(t) = (x(t), y(t)) = x(t)\bar{i} + y(t)\bar{j}, \quad \|r'(t)\| > 0, \quad t \in I.'$$

The ambient setting $\mathbb{R}^2$ is an Euclidean vector space with respect to the canonical inner product:

$$\langle u, v\rangle = x^1y^1 + x^2y^2, u = (x^1, x^2) \in \mathbb{R}^2, v = (y^1, y^2) \in \mathbb{R}^2, \quad 0 \leq \|u\|^2 = \langle u, u\rangle.$$

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The infinitesimal generator of the rotations in $\mathbb{R}^2 = \mathbb{C}$ is the linear vector field, called *angular*:

$$\xi(u) := -x^2 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^2}, \quad \xi(u) = i \cdot u = i \cdot (x^1 + ix^2), \quad i = \sqrt{-1}.$$

It is a complete vector field with integral curves the circles $\mathcal{C}(O, r)$:

$$\begin{cases} \gamma^{\xi}_{u_0}(t) = (u^1_0 \cos t - u^2_0 \sin t, u^1_0 \sin t + u^2_0 \cos t) = R(t) \cdot \begin{pmatrix} u^1_0 \\ u^2_0 \end{pmatrix}, \quad t \in \mathbb{R}, \\ r = \|u_0\| = \|(u^1_0, u^2_0)\|, \quad R(t) := \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \in SO(2) = S^1 \end{cases}$$

and since the rotations $R(t)$ are isometries of the Riemannian metric $g_{can} = dx^2 + dy^2 = |dz|^2$ it follows that ξ is a Killing vector field of the Riemannian manifold $(\mathbb{R}^2, g_{can})$. The first integrals of ξ are the Gaussian functions i.e. multiples of the square norm: $f_\alpha(x, y) = \alpha(x^2 + y^2)$, $\alpha \in \mathbb{R}$. For an arbitrary vector field $X = A(x, y)\frac{\partial}{\partial x} + B(x, y)\frac{\partial}{\partial y}$ its Lie bracket with ξ is:

$$[X, \xi] = (yA_x - xA_y - B)\frac{\partial}{\partial x} + (A + yB_x - xB_y)\frac{\partial}{\partial y},$$

where the subscript denotes the variable corresponding to the partial derivative. For example, ξ commutes with *the radial* (or Euler) vector field $E(x, y) = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$, which is also a complete vector field having as integral curves the homotheties $\gamma^E_{u_0}(t) = e^t u_0$ for all $t \in \mathbb{R}$.

The Frenet apparatus of the curve C is provided by:

$$\begin{cases} T(t) = \frac{r'(t)}{\|r'(t)\|}, \quad N(t) = i \cdot T(t) = \frac{1}{\|r'(t)\|}(-y'(t), x'(t)), \\ k(t) = \frac{1}{\|r'(t)\|}\langle T'(t), N(t)\rangle = \frac{1}{\|r'(t)\|^3}\langle r''(t), ir'(t)\rangle = \frac{1}{\|r'(t)\|^3}[x'(t)y''(t) - y'(t)x''(t)]. \end{cases}$$

Hence, if C is naturally parametrized (or parametrized by arc-length) i.e. $\|r'(s)\| = 1$ for all $s \in I$ then $r''(s) = k(s)ir'(s)$. In a complex approach based on $z(t) = x(t) + iy(t) \in \mathbb{C} = \mathbb{R}^2$ we have:

$$\begin{cases} k(t) = \frac{1}{|z'(t)|^3} Im(\bar{z}'(t) \cdot z''(t)) = \frac{1}{|z'(t)|} Im\left(\frac{z''(t)}{z'(t)}\right) = \frac{1}{|z'(t)|} Im\left[\frac{d}{dt}(\ln z'(t))\right] \in \mathbb{R}, \\ Re(\bar{z}'(t) \cdot z''(t)) = \frac{1}{2}\frac{d}{dt}\|r'(t)\|^2, \quad f_\alpha(z) = \alpha|z|^2. \end{cases}$$

The multiplication with the complex unit i corresponds to the rotation $R\left(\frac{\pi}{2}\right)$; we have also:

$$\frac{d}{dt}R(t) = R\left(t + \frac{\pi}{2}\right) = R(t)R\left(\frac{\pi}{2}\right) = R\left(\frac{\pi}{2}\right)R(t),$$

and the Frenet equations can be unified by means of the column matrix $\mathcal{F}(t) = \begin{pmatrix} T \\ N \end{pmatrix}(t)$ as:

$$\frac{d}{dt}\mathcal{F}(t) = \|r'(t)\|k(t)R\left(-\frac{\pi}{2}\right)\mathcal{F}(t).$$

It is an amazing fact that if the general rotation $R(t)$ belongs to the Lie group $SO(2) = S^1$ its particular values $R\left(\pm\frac{\pi}{2}\right)$ are elements of its Lie algebra $so(2)$ of skew-symmetric 2×2 matrices. In fact, $\{R\left(\frac{\pi}{2}\right)\}$ is exactly the basis of $so(2)$.

2. Main Results

This short note defines a new frame and correspondingly a new curvature function for C:

Definition 1. *The flow-frame of C consists in the pair of unit vectors $(E^f_1(t), E^f_2(t)) \in T^2 := S^1 \times S^1$ given by:*

$$\mathcal{E}(t) := \begin{pmatrix} E^f_1 \\ E^f_2 \end{pmatrix}(t) = R(t)\mathcal{F}(t) = \begin{pmatrix} \cos tT(t) - \sin tN(t) \\ \sin tT(t) + \cos tN(t) \end{pmatrix} \tag{1}$$

the letter f being the initial of the word "flow". The flow-curvature of C is the smooth function $k_f : I \to \mathbb{R}$ given by the flow-equations:

$$\frac{d}{dt}\mathcal{E}(t) = \|r'(t)\| k_f(t) R\left(-\frac{\pi}{2}\right)\mathcal{E}(t). \tag{2}$$

Before starting its study we point out that this work is dedicated the memory of Academician Radu Miron (1927-2022). He was always interested in the geometry of curves and besides his theory of *Myller configurations* ([11]) he generalized also a type of curvature for space curves in [10]. We remark also that this note follows the idea of Bishop in his delightful note [2] and that the flow-curvature of spacelike parametrized curves in the Lorentz plane was introduced by the author in [4]. The hyperbolic curves are studied also by the author in [5].

Returning to our subject we note as a first main result:

Proposition 1. *The expression of the flow-curvature is:*

$$k_f(t) = k(t) - \frac{1}{\|r'(t)\|} < k(t). \tag{3}$$

As a consequence, the curve C and its trigonometrical rotation iC share the same flow-curvature.

Proof We have directly in the flow-frame:

$$\|r'(t)\| k_f(t) R\left(-\frac{\pi}{2}\right) = R\left(t + \frac{\pi}{2}\right) R(-t) + \|r'(t)\| k(t) R(t) R\left(-\frac{\pi}{2}\right) R(-t) \tag{4}$$

and the conclusion follows. Concerning the consequence it is obvious that C and $iC : t \to (-y(t), x(t))$ share the same curvature k and the same second term from (3). □

Example 1. *i) If C is the line $r_0 + tu, t \in \mathbb{R}$ with the vector $u \neq \bar{0} = (0,0)$ then k_f is constant:*

$$k_f(t) = -\frac{1}{\|u\|} = constant < 0. \tag{5}$$

In particular, if u is an unit vector then $k_f(t) = -1$.
ii) The circle $\mathcal{C}(O,R)$ with the usual parametrization $r(t) = Re^{it}$ is a flat-flow curve i.e. $k_f = 0$. Indeed, the flow-frame is constant and universal for the families of concentric circles i.e. it does not depend on the radius R (exactly as the Frenet frame):

$$E_1^f = (0,1) = \bar{j}, \quad E_2^f = (-1,0) = -\bar{i}. \tag{6}$$

More generally, if C is expressed in polar coordinates as $C : \rho = \rho(t)$ for $t \in I$ then C is a flat-flow curve if and only if C is a logarithmic spiral $\rho_{R,\alpha}(t) = Re^{\alpha t}$, $R, \alpha > 0$ and $t \in \mathbb{R}$. The limit case $\alpha \to 0$ gives the circle $\mathcal{C}(O,R)$ and the flow-frame of the logarithmic spiral is: $E_1^f = \frac{1}{\sqrt{\alpha^2+1}}(\alpha, 1)$, $E_2^f = \frac{1}{\sqrt{\alpha^2+1}}(-1, \alpha)$; if $\alpha = \cot\varphi$ then $E_1^f = e^{\varphi i}$, $E_2^f = e^{i\left(\frac{\pi}{2}+\varphi\right)}$.
iii) Fix $R \in (0,+\infty)$ and the plane curve $C : w = F(Re^{it})$ with t as an increasing parameter and $F = F(z)$ a holomorphic function. Then the curvatures are:

$$k(t) = \frac{1}{|zF'(z)|} Re\left(1 + \frac{zF''(z)}{F'(z)}\right), \quad k_f(t) = \frac{1}{|zF'(z)|} Re\left(\frac{zF''(z)}{F'(z)}\right). \tag{7}$$

For the circle example of $F(z) = z^2$ it results $k = \frac{1}{R^2} = constant$ and $k_f = \frac{1}{2R^2} = constant$ which proves the proper dependence of k_f on the parametrizations of C. □

Remark 1. *i) Suppose that I is symmetric with respect to $0 \in \mathbb{R}$ and that C is positively oriented in the terms of Definition 1.14 from [14, p. 17]. Suppose also the C is convex; then applying the Theorem 1.18 of page 19 from the same book it results for the usual curvature the inequality $k \geq 0$. Hence the opposite curve $C^- : t \in I \to r(-t)$ has the flow-curvature $k_f < 0$.*
ii) An important tool in dynamics is the Fermi-Walker derivative. Let $\mathcal{X}_C$ be the set of vector fields along the curve C. Then the Fermi-Walker derivative is the map ([6]) $\nabla_C^{FW} : \mathcal{X}_C \to \mathcal{X}_C$:

$$\nabla_C^{FW}(X) := \frac{d}{dt}X + \|r'(\cdot)\| k[\langle X, N\rangle T - \langle X, T\rangle N] = \frac{d}{dt}X + \|r'(\cdot)\| k[X^\flat(N)T - X^\flat(T)N] \tag{8}$$

with $X^\flat$ the differential 1-form dual to X with respect to the Euclidean metric. In a matrix form we can express this as follows:

$$\nabla_C^{FW} = \frac{d}{dt} - \|r'\| k \begin{vmatrix} (\cdot)^\flat(T) & (\cdot)^\flat(N) \\ T & N \end{vmatrix} = \frac{d}{dt} + \|r'\| k \begin{vmatrix} T & (\cdot)^\flat(T) \\ N & (\cdot)^\flat(N) \end{vmatrix}. \tag{9}$$

It is natural to make here a remark concerning rotation-minimizing fields $X \in \mathcal{X}_C$ i.e. fields satisfying:

$$\frac{d}{dt}X(t) = \lambda(t)T(t), \quad \langle X(t), T(t)\rangle = 0$$

for a smooth function $\lambda = \lambda(t)$. Then the Fermi-Walker derivative of such X is also parallel with the tangent T:

$$\nabla_C^{FW} X(t) = [\lambda(t) + \|r'(t)\| k(t)\langle X(t), N(t)\rangle]T(t).$$

Calculating the Fermi-Walker derivative on our frames we get:

$$\nabla_C^{FW}(T) = \nabla_C^{FW}(N) = 0, \quad \nabla_C^{FW}(E_1^f) = -E_2^f, \quad \nabla_C^{FW}(E_2^f) = E_1^f. \tag{10}$$

With the matrix notation we can express these relations as:

$$\nabla_C^{FW}(\mathcal{F}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \nabla_C^{FW}(\mathcal{E}) = R\left(\frac{\pi}{2}\right)\mathcal{E} \tag{11}$$

and the Fermi-Walker derivative can be expressed in terms of k_f as:

$$\nabla_C^{FW}(X) = \frac{d}{dt}X + (1 + \|r'\| k_f)[X^\flat(N)T - X^\flat(T)N]. \tag{12}$$

Also, we can define the flow-Fermi-Walker derivative as:

$$\nabla_C^{fFW}(X) := \frac{d}{dt}X + \|r'(\cdot)\| k_f[X^\flat(N)T - X^\flat(T)N] = \nabla_C^{FW}(X) + T \wedge N(X) \tag{13}$$

with the skew-symmetric endomorphism $\wedge \in so(2)$ defined by:

$$X \wedge Y := \langle X, \cdot\rangle Y - \langle Y, \cdot\rangle X = (X^1Y^2 - X^2Y^1)R\left(\frac{\pi}{2}\right), X = (X^1, X^2), Y = (Y^1, Y^2).$$

Then:

$$\nabla_C^{fFW}(\mathcal{F}) = R\left(-\frac{\pi}{2}\right)\mathcal{F}, \quad \nabla_C^{fFW}(\mathcal{E}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{14}$$

As in the usual case, if $V, W \in \mathcal{X}_C$ are flow-Fermi-Walker fields i.e. with zero flow-Fermi-Walker derivative then the value $< V, W > \in \mathbb{R}$ is constant along C.

iii) Remark that the 4-dimensional vectors $\frac{1}{\sqrt{2}}\mathcal{F}$ and $\frac{1}{\sqrt{2}}\mathcal{E}$ belong to the Clifford torus $\frac{1}{\sqrt{2}}T^2 \subset S^3$. A remarkable Riemannian submersion is the Hopf map $H : S^3 \subset \mathbb{C}^2 \to S^2(\frac{1}{2}) \subset \mathbb{R} \times \mathbb{C}$:

$$H(z, w) = \left(\frac{1}{2}(|z|^2 - |w|^2), z\bar{w}\right). \tag{15}$$

It follows:

$$H\left(\frac{1}{\sqrt{2}}\mathcal{F}(t)\right) = \left(0, \frac{1}{2}T(t)\bar{N}(t)\right) = \left(0, -\frac{i}{2}\right) = H\left(\frac{1}{\sqrt{2}}\mathcal{E}(t)\right). \tag{16}$$

Hence, considering H as a projection map of the S^1-principal bundle $S^3 \to S^2(\frac{1}{2})$ we have that $\frac{1}{\sqrt{2}}\mathcal{F}$ and $\frac{1}{\sqrt{2}}\mathcal{E}$ belong to the same fiber, namely that over the South *pole of the sphere* $S^2(\frac{1}{2})$.

iv) Suppose now that our curve C *belongs to the plane* xOz *of the physical space* $\mathbb{R}^3$ *as* $C: r(t) = (f(t), 0, F(t))$ *with* $f > 0$ *on* I *and consider the rotational surface generated by* C *as:*

$$\Sigma: \bar{r}(t, \varphi) := (f(t)\cos\varphi, f(t)\sin\varphi, F(t)), \quad \varphi \in S^1.$$

Its principal curvatures depend only on t, *[8, p. 85]:*

$$k_1 = k, \quad k_2 = \frac{F'}{\|r'\| f} \tag{17}$$

and then for $F' = f$ *we have that* k_f *of* C *is exactly the difference* $k_1 - k_2$ *of the principal curvatures of* Σ*; consequently the umbilic circles of* Σ *are provided by the zeros of* k_f *and are parametrized by* $\varphi \in S^1$.

For $F' = f$ *the curvatures of* C *are expressed only through the function* F *as:*

$$k(t) = \frac{[F''(t)]^2 - F'(t)F'''(t)}{[F'(t)^2 + F''(t)^2]^{\frac{3}{2}}}, \quad k_f(t) = \frac{-F'(t)F'''(t) - [F'(t)]^2}{[F'(t)^2 + F''(t)^2]^{\frac{3}{2}}} \tag{18}$$

and due to the presence of the third derivative of F *we recall its Schwarzian derivative:*

$$S_F = \frac{F'''}{F'} - \frac{3}{2}\left(\frac{F''}{F'}\right)^2 \tag{19}$$

which implies the new formulae:

$$k = \frac{(F'')^2 - 2(F')^2 S_F}{2[(F')^2 + (F'')^2]^{\frac{3}{2}}}, \quad k_f = \frac{-3(F'')^2 - 2(F')^2 S_F - 2(F')^2}{2[(F')^2 + (F'')^2]^{\frac{3}{2}}}. \tag{20}$$

In conclusion, a smooth F *with negative Schwarzian derivative will give a positive curvature* k *for* C *while a positive Schwarzian derivative* S_F *produces a negative flow-curvature* k_f.

v) The nature and the relationship between our frames can be put in the framework of moving frames of [8, p. 32]. Recall that the set of all orientation-preserving Euclidean isometries forms a Lie group, $E(2) := \mathbb{R}^2 \times SO(2)$, *with the standard projection* π_1 *on the first factor making* $E(2) \to \mathbb{R}^2$ *an* S^1*-principal bundle. A moving frame along* C *is a map* $F: I \to E(2)$ *such that* $\pi_1 \circ F = r$. *But* C *defines also a* 1*-parameter family of bijections of* $SO(2)$*:*

$L^C: I \to Bijections(SO(2)), t \to L^C(t): SO(2) \to SO(2), A \to R(t)A, (L^C(t))^{-1} = L^C(-t)$.

Then our frames are $\mathcal{F}: I \to E(2)$ *as* $\mathcal{F}(t) = (r(t), T(t), N(t))$ *and* $\mathcal{E}: I \to E(2)$ *as* $\mathcal{E}(t) = (r(t), (L^C(t) \circ \pi_2 \circ \mathcal{F})(t))$.

vi) Suppose now that the curve C *is in the space* $\mathbb{R}^3$ *and is bi-regular; hence it has the Frenet frame* (T, N, B) *and the pair (curvature, torsion)=*(k, τ). *We define its flow-frame as:*

$$\begin{pmatrix} T \\ E_2^f \\ E_3^f \end{pmatrix}(t) := \begin{pmatrix} 1 & 0_2(h) \\ 0_2(v) & R(t) \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \quad 0_2(h) := (0, 0), \quad 0_2(v) := \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and then, its matrix moving equation is:

$$\frac{d}{dt}\begin{pmatrix} T \\ E_2^f \\ E_3^f \end{pmatrix}(t) = \|r'(t)\| \begin{pmatrix} 0 & k_f^2(t) & k_f^3(t) \\ -k_f^2(t) & 0 & \tau_f(t) \\ -k_f^3(t) & -\tau_f(t) & 0 \end{pmatrix} \begin{pmatrix} T \\ E_2^f \\ E_3^f \end{pmatrix}(t).$$

A similar computation yields:

$$k_f^2(t) = k(t)\cos t, \quad k_f^3(t) = k(t)\sin t, \quad \tau_f(t) = \tau(t) - \frac{1}{\|r'(t)\|} < \tau(t).$$

We point out the formal similarity with the Darboux equations of a curve on a given surface and then a curve C with vanishing τ_f will be called flow-geodesic in $\mathbb{R}^3$. Hence, if C is naturally parametrized then C is a flow-geodesic if and only if its torsion has the constant value 1; for this class of space curves and examples see [1]. In order to express the above moving equation in the compact form as in the theory of space curves:

$$\omega_f(t) \times T(t) = T'(t), \quad \omega_f(t) \times E_2^f(t) = (E_2^f)'(t), \quad \omega_f(t) \times E_3^f(t) = (E_3^f)'(t)$$

we associate a vector field along C, called flow-Darboux:

$$\omega_f(t) := \|\gamma'(t)\|[\tau_f(t)T(t) - k_f^3(t)E_f^2(t) + k_f^2(t)E_3^f(t)].$$

Something similar but with the rotation with respect to an angle $\theta = \theta(s)$ appears in [13] under the name of quasi frame for C. Our choice corresponds to the angle $\theta(s) = -s$.

vii) Suppose that the curvature function $t \to k(t)$ is always strictly positive (or strictly negative). Then the evolute of C is the curve:

$$C_e : r_e(t) := r(t) + \frac{1}{k(t)}N(t).$$

With this model in mind, for a non-flat-flow curve we associate its flow-evolute as being the curve:

$$C_{fe} : r_{fe}(t) := r(t) + \frac{1}{k_f(t)}E_2^f(t).$$

We will obtain this curve for some examples below. So, the line C discussed in the example 1i has the flow-evolute

$$C_{fe} : r_{fe}(t) = r_0 + (t - \sin t)u - \cos t(iu)$$

and for $r_0 = (0,1) = iu$ this last curve is exactly the cycloid of radius $R = 1$ according to the example 3 below. $\square$

Returning to the plane curves let $J \subseteq \mathbb{R}$ be another open interval and fix the diffeomorphism $\varphi : s \in J \to t \in I$ with the smooth inverse $\varphi^{-1} : t \in I \to s \in J$. Since

$r'(s) = \varphi'(s)r'(t(s))$ we restrict our study to the class $Diff_+(J, I)$ of orientation-preserving diffeomorphisms: $\varphi'(s) > 0$, for all $s \in J$. The transformation of the flow-curvature under the action of φ is:

$$k_f(s) = k(t) - \frac{1}{\varphi'(s)\|r'(t)\|} \tag{21}$$

and then:

$$k_f(s) - k_f(t) = \frac{1}{\|r'(t)\|}\left[1 - \frac{1}{\varphi'(s)}\right]. \tag{22}$$

Proposition 2. *(the rigidity of the flow-curvature) The only orientation-preserving diffeomorphism φ which preserves also the flow-curvature of C is an interval shift on the real line $\varphi(s) = s + s_0, \quad s_0 \in (0, +\infty)$.*

A natural important problem is the class of curves with prescribed flow-curvature. For example, if we ask the vanishing of the flow-curvature for a graphic curve $C_F : r(t) = (t, F(t))$ then it follows the differential equation:

$$\frac{F''(t)}{[1 + (F'(t))^2]^{\frac{3}{2}}} = \frac{1}{[1 + (F'(t))^2]^{\frac{1}{2}}}. \tag{23}$$

Since this equation reads:

$$\frac{F''(t)}{1 + (F'(t))^2} = 1 \tag{24}$$

we have exactly the Grim Reaper solution, [3, p. 28], a famous solution of the curve shortening flow:

$$F_u(t) = u - \ln(\cos t), \quad t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad u \in \mathbb{R} \tag{25}$$

with the usual curvature $k(t) = \cos t$ and the frames:

$$\mathcal{F}(t) = \begin{pmatrix} e^{it} \\ e^{i\left(t + \frac{\pi}{2}\right)} \end{pmatrix}, \quad \mathcal{E} = \begin{pmatrix} (1,0) = \bar{i} \\ (0,1) = \bar{j} \end{pmatrix} = constant. \tag{26}$$

Another formalism is that of [15, p. 2] if $r : S^1 \simeq [0, 2\pi) \to \mathbb{R}^2$ is naturally parametrized then there exists the smooth function $\theta : S^1 \to \mathbb{R}$, called *normal angle*, such that:

$$N(s) = e^{i\theta(s)} = (\cos\theta(s), \sin\theta(s)), \quad T(s) = -iN(s) = -ie^{i\theta(s)} = e^{i(\theta(s) - \frac{\pi}{2})} \tag{27}$$

and then the Frenet equations yield:

$$\frac{d\theta}{ds}(s) = k(s). \tag{28}$$

In conclusion, the constant value $\beta \in \mathbb{R}$ of the flow-curvature of a closed convex curve means $\theta(s) = (\beta + 1)s + \alpha$ for all $s \in S^1$ with $\alpha \in \mathbb{R}$ an arbitrary constant. The flow-frame corresponding to the equations (27) is:

$$E_1^f(s) = (\sin(\theta(s)-t(s)), -\cos(\theta(s)-t(s))), E_2^f(s) = (\cos(\theta(s)-t(s)), \sin(\theta(s)-t(s))) \tag{29}$$

which, in turn, is the Frenet frame of a new curve with the same natural parameter s but having the normal angle $\tilde{\theta}(s) := \theta(s) - t(s)$.

The formula (28) can be replaced with $\frac{d(\theta - \pi/2)}{ds}(s) = k(s)$ which expresses the curvature k as the derivative of the angle between $T \in \mathcal{X}_C$ and the unit vector $\bar{i}$. Following this approach the paper [7] generalizes k to a curvature-type function k_V defined with respect to an arbitrary $V \in \mathcal{X}_C$. A main result of the cited work is that $k_V = k_W$ if and only if the angle between V and W is constant along C. Hence, we can apply the last statement of the Remark ii) and then two flow-Fermi-Walker unit vectors $V, W \in \mathcal{X}_C$ yield the same curvature-type function.

In the following we present a couple of examples in order to remark the computational aspects of our approach.

Example 2. *The involute of the unit circle S^1 is:*

$$C : r(t) = (\cos t + t \sin t, \sin t - t \cos t) = (1 - it)e^{it}, \quad t \in (0, +\infty). \tag{30}$$

A direct computation gives:

$$r'(t) = (t \cos t, t \sin t) = te^{it}, \quad \|r'(t)\| = t, \quad k(t) = \frac{1}{t} > 0, \tag{31}$$

and then this curve is also a flat-flow one and having the same flow-frame as the Grim Reaper. This example can be treated also with respect to a natural parameter $s \in (0, +\infty)$ which is provided by $t := \sqrt{2s}$. For example, the normal angle function is $\theta(s) = \frac{\pi}{2} + \sqrt{2s}$ since then $r'(s) = e^{i\sqrt{2s}}$. Comparing with the approach above it results the constants $\alpha = \frac{\pi}{2}$ and $\beta = \sqrt{2} - 1$. $\square$

Example 3. *Recall that for $R > 0$ the cycloid of radius R has the equation:*

$$C : r(t) = R(t - \sin t, 1 - \cos t) = R[(t, 1) - e^{i(\frac{\pi}{2} - t)}], \quad t \in \mathbb{R}. \tag{32}$$

Remark that here we have a twisted situation of the Remark iv) namely the derivative of the first component of the vector $r(t)$ is exactly the second component. The Schwarzian derivative is:

$$S_{t-\sin t}(t) = \frac{\cos t}{\sin t} - \frac{3}{2}\left(\frac{\cos \frac{t}{2}}{\sin \frac{t}{2}}\right)^2, \quad t \in \mathbb{R} \setminus \mathbb{Z}\pi. \tag{33}$$

We have immediately:

$$r'(t) = R(1-\cos t, \sin t) = R[(1,0) - e^{it}], \|r'(t)\| = 2R|\sin\frac{t}{2}|, k(t) = -\frac{1}{4R|\sin\frac{t}{2}|}, \tag{34}$$

and then we restrict our definition domain to $(0, \pi)$*. It follows:*

$$\begin{cases} k_f(t) = -\frac{3}{4R\sin\frac{t}{2}} < 0, \\ E_1^f(t) = (\sin\frac{3t}{2}, \cos\frac{3t}{2}) = e^{i\left(\frac{\pi}{2}-\frac{3t}{2}\right)}, E_2^f(t) = (-\cos\frac{3t}{2}, \sin\frac{3t}{2}) = e^{i\left(\pi-\frac{3t}{2}\right)}. \end{cases} \tag{35}$$

Again a natural parameter s *is provided by:* $t = 2\arccos\left(1 - \frac{s}{4R}\right)$ *and the flow-evolute of* C *is the curve:*

$$C_{fe} : r_{fe}(t) = R(t - \sin t, 1 - \cos t) + \frac{4}{3}R\sin\frac{t}{2}(\cos t, -\sin t), \quad t \in (0, \pi).$$

□

Example 4. *The derivative curve* r' *from* (31) *is an Archimedes' spiral. This spiral is given in polar coordinates as:*

$$A(spiral) : \rho(t) = Rt, \quad R > 0 \tag{36}$$

and hence:

$$k_f(t) = \frac{1}{R(t^2+1)^{\frac{3}{2}}} > 0 \tag{37}$$

while its flow-evolute is the curve:

$$C_{fe} : r_{fe}(t) = R(t\cos t, t\sin t) + R(1+t^2)(-\sin t - t\cos t, \cos t - t\sin t).$$

□

Example 5. *Fix* $\alpha \in \mathbb{R}^*$ *and the naturally parametrized curve* C*. Then the* α*-parallel curve of* C *is the new curve:*

$$C_\alpha : \tilde{r}(t) := r(t) + \alpha N(t), \quad t \in I \tag{38}$$

with:

$$\tilde{T}(t) = \frac{1-\alpha k(t)}{|1-\alpha k(t)|}T(t), \quad \tilde{N}(t) = \frac{1-\alpha k(t)}{|1-\alpha k(t)|}N(t), \quad \tilde{k}(t) = k(t). \tag{39}$$

Hence, we consider that α *does not belongs to the range of the function* $\frac{1}{k}$ *and the new flow-curvature is:*

$$\tilde{k}_f(t) = k(t) - \frac{1}{|1-\alpha k(t)|}. \tag{40}$$

□

We finish this note with the problem raised in the beginning, namely the possible variants of the curve shortening flow. Recall that the setting of this question consists in a 1-parameter family of plane curves $C_u : r = r_u(t) = r(t,u)$ satisfying:

$$\frac{\partial r(t,u)}{\partial u} = k(t,u)N(t,u). \tag{41}$$

It follows immediately an expression in terms of flow-apparatus:

$$\frac{\partial r(t,u)}{\partial u} = \left(k_f(t,u) + \frac{1}{\|r'(t,u)\|}\right)[-\sin t E_1^f(t,u) + \cos t E_2^f(t,u)]. \tag{42}$$

The first variant which we propose as an open problem is to study the flow-variant of (41):

$$\frac{\partial r(t,u)}{\partial u} = k_f(t,u)E_2^f(t,u). \tag{43}$$

The second variant is to generalize all this study through a general smooth function $\Omega \in C^\infty(\mathbb{R})$. More precisely, we use the equation (1) with R replaced by $R \circ \Omega$ to define the notion of Ω-frame for the plane curve C; we note that for a particular choice of Ω the 3-dimensional variant of the remark vi) is called *positional adapted frame* in [12]. Then the Ω-curvature of the plane curve C is:

$$k_\Omega(t) = k(t) - \frac{\Omega'(t)}{\|r'(t)\|} \tag{44}$$

and the curves in polar coordinates with vanishing Ω-curvature are provided by:

$$\rho(t) = Re^{\int_{t_0}^{t} \cot[\Omega(u)-u+C]du}, \quad R > 0, \quad C \in \mathbb{R}. \tag{45}$$

The flow-curvature corresponds to the identity map $\Omega = 1_{\mathbb{R}}$. Moreover, if C is naturally parametrized then $k_\Omega = (\theta - \Omega)'$ which means that the case $\Omega = \theta + constant$ provides a zero Ω-curvature.

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