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Research Article

Evaluation formulas for the Tornheim and Euler-type double series

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Abstract

We give closed-form evaluation formulas for the real and imaginary parts of the series $\sum_{m,n=1}^{\infty} \frac{e^{2\pi i(mx-ny)}}{m^p n^r (mc+n)^q}$, $c \in \mathbb{N}$, in terms of certain zeta values. Particular choices of x and y lead to evaluation formulas for some Tornheim-type $\sum_{m,n=1}^{\infty} \frac{1}{m^p n^r (mc+n)^q}$ and Euler-type $\sum_{m,n=1}^{\infty} \frac{1}{n^p (mc+n)^q}$ double series and their alternating analogues.

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1. Introduction

The Tornheim function is defined by

$$T(r,s,t) = \sum_{m,n=1}^{\infty} \frac{1}{m^r n^s (m+n)^t},$$

where $r,s,t \in \mathbb{C}$, $\operatorname{Re}(r+t) > 1$, $\operatorname{Re}(s+t) > 1$ and $\operatorname{Re}(r+s+t) > 2$. The values $T(r,s,t)$ for $r,s,t \in \mathbb{N}$ were investigated by Tornheim [31] and some explicit formulas were obtained. Moreover, when $r+s+t$ is odd, it was proved that $T(r,s,t)$ is a polynomial in $\zeta(2), \ldots, \zeta(r+s+t)$ with rational coefficients without giving the polynomial, where $\zeta(s)$ denotes the usual Riemann zeta function. In 1958, Mordell [20] showed that $T(2r,2r,2r) = \pi^{6r} k_r$, where k_r is a rational number, $r \in \mathbb{N}$, and posed the problem of evaluating $T(2r+1, 2r+1, 2r+1)$. Afterwards, Subbarao and Sitaramachandrarao, in [30, Theorem 4.1] for $r,s,t \in \mathbb{N}$, gave a functional relation for $T(2r,2s,2t)$, which implies an evaluation formula for $T(2r,2r,2r)$. Later, Huard et al. [15, Theorem 2] gave the following explicit formula for $T(r,s,N-r-s)$ when $r,s,N \in \mathbb{N}$, $N \geq 3$ is odd, $1 \leq r+s \leq N-1$, $r \leq N-2$ and $s \leq N-2$:

$$T(r,s,N-r-s) = E_{N(r,s)} + E_{N(s,r)}, \tag{1.1}$$

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where

$$E_{N(r,s)} = (-1)^r \sum_{i=0}^{(N-r-s-1)/2} \binom{N-2i-s-1}{r-1} \zeta(2i)\zeta(N-2i) + (-1)^r \sum_{i=0}^{r/2} \binom{N-2i-s-1}{N-r-s-1} \zeta(2i)\zeta(N-2i).$$

Here and in the sequel we use $\sum_{j=0}^{\alpha}$ for $\sum_{j=0}^{\lfloor\alpha\rfloor}$, $0 \leq \alpha \in \mathbb{R}$, where $\lfloor\alpha\rfloor$ stands for the largest integer $\leq \alpha$. Afterwards, Tsumura [36, Theorem 4.5] and Nakamura [21, Theorem 1.2] gave functional relations for the Tornheim double series with different expressions, which is shown in [19] that these two expressions are the same. In [22], Nakamura showed that the function

$$T(s_1, s_2, s_3; x, y, z) = \sum_{m,n=1}^{\infty} \frac{e^{2\pi i(mx+ny+(m+n)z)}}{m^{s_1} n^{s_2} (m+n)^{s_3}}$$

can be meromorphically continued to $\mathbb{C}^3$ and gave an integral representation. Moreover, Nakamura gave two functional relations [22, Theorem 3.2] and [23, Theorem 3.1] which lead to evaluation formulas for some Tornheim double series.

In the study of evaluations of special values of certain zeta values, Okamoto [27] introduced

$$\zeta_{\mathbf{a_r},\mathbf{b_r}}(s_1, \ldots, s_{r+2}) = \sum_{m,n=1}^{\infty} \frac{1}{m^{s_1} n^{s_2}} \prod_{j=1}^{r} (a_j m + b_j n)^{-s_{j+2}},$$

where $\mathbf{a_r} = (a_1, \ldots, a_r) \in \mathbb{N}^r$, $\mathbf{b_r} = (b_1, \ldots, b_r) \in \mathbb{N}^r$. The author showed that for positive integers s_j, the values $\zeta_{\mathbf{a}_r,\mathbf{b}_r}(s_1, \ldots, s_{r+2})$ are reduced to the values

$$\zeta_{a,b}(r, u, v) = \sum_{m,n=1}^{\infty} \frac{1}{m^r n^u (am+bn)^v},$$

(cf. [27, Theorem 2.3]) and gave an evaluation formula for $\zeta_{a,b}(r, u, v)$ (cf. [27, Theorem 4.5]). Recently, in [16, Theorem 1.1], Kadota, Okamoto and Tasaka showed that the value $\zeta_{a,b}(r, u, v)$ can be expressed as $\mathbb{Q}$-linear combinations of $\pi^{2n} C_{p-2n}(d/K)$ and $\pi^{2n} S_{p-2n-1}(d/K)$ for odd $p = r+u+v$ and $0 \leq n \leq (p-3)/2$. Here $K = \mathrm{lcm}(a, b)$, $d \in \mathbb{Z}/K\mathbb{Z}$, $C_p(d/K) = \mathrm{Re}\, L(p; d/K)$, $S_p(d/K) = \mathrm{Im}\, L(p; d/K)$ and

$$L(s; x) = \sum_{m=1}^{\infty} \frac{e^{2\pi i m x}}{m^s}, \ x \in \mathbb{R}.$$

The authors also mentioned that from their related results an explicit formula for $\zeta_{a,b}(r, u, v)$ could be deduced, however, they admitted that this formula might be much complicated.

The Tornheim double series $T(r, 0, v)$ for $r, v \in \mathbb{N}$

$$T(r, 0, v) = \sum_{m,n=1}^{\infty} \frac{1}{m^r (m+n)^v} = \sum_{n=1}^{\infty} \sum_{m=1}^{n-1} \frac{1}{m^r n^v} = \sum_{n=1}^{\infty} \frac{H_{n-1}^{(r)}}{n^v}$$

is called the Euler double series or Euler sum. Here $H_n^{(r)}$ is the nth generalized harmonic number defined by $H_n^{(r)} = 1 + 1/2^r + 1/3^r + \cdots + 1/n^r$, $n, r \in \mathbb{N}$, with $H_0^{(r)} = 0$. The famous Euler's identity for the sum $T(1, 0, v)$ is [26]

$$2T(1, 0, v) = 2\sum_{n=1}^{\infty} \frac{H_{n-1}^{(1)}}{n^v} = v\zeta(v+1) - \sum_{j=2}^{v-1} \zeta(v+1-j)\zeta(j), \ v \in \mathbb{N} \backslash \{1\}. \tag{1.2}$$

There is a very comprehensive literature on the Tornheim double series (see, for example [3, 6, 13, 17–24, 32–38, 41]), and on the Euler double series and their extensions (see, for example [1, 4, 5, 7, 9–12, 14, 25, 28, 39, 40]). Most of these studies are on the evaluation of the Tornheim and Euler (type) double series in terms of the zeta values as in (1.1) and (1.2).

The aim of this study is to give closed-form evaluation formulas for the Tornheim-type double series

$$\sum_{m,n=1}^{\infty} \frac{\cos\left(2\pi\left(mz-nx\right)\right)}{m^r n^u \left(mc+n\right)^v}, \sum_{m,n=1}^{\infty} \frac{\sin\left(2\pi\left(mz-nx\right)\right)}{m^r n^u \left(mc+n\right)^v} \tag{1.3}$$

and the Euler-type double series

$$\sum_{m,n=1}^{\infty} \frac{\cos\left(2\pi\left(mx+ny\right)\right)}{m^p (nc+m)^q}, \sum_{m,n=1}^{\infty} \frac{\sin\left(2\pi\left(mx+ny\right)\right)}{m^p (nc+m)^q} \tag{1.4}$$

in terms of certain zeta values. For this purpose, inspired by [22, 27], we consider the following Tornheim-type double series

$$T\left(s_1, s_2, s_3; x, y, z; c\right) = \sum_{m,n=1}^{\infty} \frac{e^{2\pi i(mx+ny+(mc+n)z)}}{m^{s_1} n^{s_2} \left(mc+n\right)^{s_3}}. \tag{1.5}$$

It is clear that $T(s_1, s_2, s_3; x, y, z; 1) = T(s_1, s_2, s_3;x, y, z)$ is Nakamura's function and $T\left(s_1, s_2, s_3; 0, 0, 0; c\right) = \zeta_{c,1}\left(s_1, s_2, s_3\right)$ is Okamoto's function. Then we deduce two functional equations for the series $T(s_1, s_2, s_3; x, y, z; c)$. These equations lead explicit calculation formulas for the series given in (1.3) and (1.4) (see Theorem 3.3 and Corollary 3.8). The formulas given in these theorems give rise to the closed-form evaluation formulas for the Tornheim and Euler-type double series, some of which are listed in Table 1 and Table 2.

In particular, when $x = z = 0$, (3.8) of Theorem 3.3 reduces to the evaluation formula for $T(r, u, v; 0, 0, 0; c) = \zeta_{c,1}\left(r, u, v\right)$ which is considerably simpler than Okamoto's formula [27, Theorem 4.5] (cf. Remark 3.4). Moreover, for $c = 1$, equation (3.8) of Theorem 3.3 reduces to (for odd $r + u + v$)

$$2\sum_{m,n=1}^{\infty} \frac{\cos\left(2\pi\left(mz-nx\right)\right)}{m^r n^u \left(m+n\right)^v} = (-1)^u C(u, v, r; x, z) + (-1)^r C(r, v, u; x, z) \tag{1.6}$$

and equation (3.12) of Corollary 3.8 reduces to (for odd $p + q$)

$$\begin{aligned} 2\sum_{m,n=1}^{\infty} \frac{\cos\left(2\pi\left(mx+nz\right)\right)}{m^p (n+m)^q} &= L(p; x-z)\left\{L(q; z) + (-1)^q L(q; -z)\right\} \\ &\quad - L(p+q; x) - (-1)^q C(p, q, 0; x, -z). \end{aligned} \tag{1.7}$$

Here

$$\begin{aligned} C(u, v, r; x, z) = &-\sum_{j=0}^{u} \binom{u+v-j-1}{v-1} \frac{\mathcal{B}_j(x)}{j!/(2\pi i)^j} L(u+v+r-j; z) \\ &-\sum_{j=0}^{v} \binom{u+v-j-1}{u-1} (-1)^j \frac{\mathcal{B}_j(x)}{j!/(2\pi i)^j} L(u+v+r-j; x+z), \end{aligned}$$

where $\mathcal{B}_n(x)$ is the nth Bernoulli function (see p. 930). It is worthwhile to mention that not only the formulas (3.8) and (3.12) but also (1.6) and (1.7) cover many of the evaluation formulas previously-obtained: All the formulas on page 197 of [3], Theorem 2 of [15] (or (1.1)), Propositions 2.4 and 2.6 of [19], formulas on page 262 of [21], Corollary 4.3 of [22], Proposition 3.3 of [23], formulas in Section 4 of [18], Corollary 3 of [32], Theorem 3.6 of [33], Theorem 3.4 of [35], Theorem 4.1 of [37], and formulas on page 2695 of [41] are consequences of (1.6). All the formulas on page 278 of [4], formulas on page 395 of [5], Theorem 7.2 of [14], Theorem 1 of [15], Proposition 4.4 of [22], Proposition 3 of [25] are consequences of (1.7).

The organization of the paper is as follows: In Section 2 we prove a decomposition formula for products of two Bernoulli functions. Section 3 is devoted to obtain functional equations and evaluation formulas for the Tornheim-type series. The final section is on the Euler-type series.

Remark 1.1. We list some Tornheim and Euler-type double series for which we obtain closed-form evaluation formulas in the following tables.

Table 1. Some Tornheim series for which Theorem 3.3 yields evaluation formula

(a) if $r+u+v$ is odd

$$\sum_{m,n=1}^{\infty}\frac{1}{m^r n^u (mc+n)^v},\quad \sum_{m,n=1}^{\infty}\frac{(-1)^m}{m^r n^u (mc+n)^v},\quad \sum_{m,n=1}^{\infty}\frac{(-1)^n}{m^r n^u (mc+n)^v}$$

$$\sum_{m,n=1}^{\infty}\frac{(-1)^{m+n}}{m^r n^u (mc+n)^v},\quad \sum_{m,n=1}^{\infty}\frac{(-1)^n}{m^r (2n)^u (mc+2n)^v},\quad \sum_{m,n=1}^{\infty}\frac{(-1)^{m+n}}{m^r (2n)^u (mc+2n)^v}$$

$$\sum_{m,n=1}^{\infty}\frac{(-1)^m}{(2m)^r n^u (2mc+n)^v},\quad \sum_{m,n=1}^{\infty}\frac{(-1)^{m+n}}{(2m)^r n^u (2mc+n)^v}$$

$$\sum_{m,n=1}^{\infty}\left\{\frac{(-1)^{m+n}}{(2m)^r (2n)^u (2mc+2n)^v}+\frac{(-1)^{m+n}}{(2m-1)^r (2n-1)^u ((2m-1)c+(2n-1))^v}\right\}$$

(b) if $r+u+v$ is even

$$\sum_{m,n=1}^{\infty}\frac{(-1)^n}{m^r (2n-1)^u (mc+2n-1)^v},\quad \sum_{m,n=1}^{\infty}\frac{(-1)^m}{(2m-1)^r n^u ((2m-1)c+n)^v}$$

$$\sum_{m,n=1}^{\infty}\frac{(-1)^{m+n}}{m^r (2n-1)^u (mc+2n-1)^v},\quad \sum_{m,n=1}^{\infty}\frac{(-1)^{m+n}}{(2m-1)^r n^u ((2m-1)c+n)^v}$$

$$\sum_{m,n=1}^{\infty}\left\{\frac{(-1)^{m+n}}{(2m)^r (2n-1)^u (2mc+2n-1)^v}+\frac{(-1)^{m+n}}{(2m-1)^r (2n)^u ((2m-1)c+2n)^v}\right\}$$

Table 2. Some Euler series for which Corollary 3.8 yields evaluation formula

(a) if $p+q$ is odd

$$\sum_{m,n=1}^{\infty}\frac{1}{m^p(nc+m)^q},\quad \sum_{m,n=1}^{\infty}\frac{(-1)^n}{m^p(2nc+m)^q},\quad \sum_{m,n=1}^{\infty}\frac{(-1)^n}{m^p(nc+m)^q},\quad \sum_{m,n=1}^{\infty}\frac{(-1)^m}{(2m)^p (nc+2m)^q}$$

$$\sum_{m,n=1}^{\infty}\frac{(-1)^m}{m^p(nc+m)^q},\quad \sum_{m,n=1}^{\infty}\frac{(-1)^{m+n}}{m^p(2nc+m)^q},\quad \sum_{m,n=1}^{\infty}\frac{(-1)^{m+n}}{m^p(nc+m)^q},\quad \sum_{m,n=1}^{\infty}\frac{(-1)^{m+n}}{(2m)^p (nc+2m)^q}$$

$$\sum_{m,n=1}^{\infty}\left\{\frac{(-1)^{m+n}}{(2m)^p (2nc+2m)^q}+\frac{(-1)^{m+n}}{(2m-1)^p ((2n-1)c+2m-1)^q}\right\}$$

(b) if $p+q$ is even

$$\sum_{m,n=1}^{\infty}\frac{(-1)^{m+n}}{m^p((2n-1)c+m)^q},\sum_{m,n=1}^{\infty}\frac{(-1)^m}{(2m-1)^p (nc+2m-1)^q},\sum_{m,n=1}^{\infty}\frac{(-1)^{m+n}}{(2m-1)^p (nc+2m-1)^q}$$

$$\sum_{m,n=1}^{\infty}\frac{(-1)^n}{m^p((2n-1)c+m)^q},\sum_{m,n=1}^{\infty}\left\{\frac{(-1)^{m+n}}{(2m)^p ((2n-1)c+2m)^q}+\frac{(-1)^{m+n}}{(2m-1)^p (2nc+2m-1)^q}\right\}$$

2. Lemmas

The nth Bernoulli polynomial $B_n(x)$ is defined by the generating function (see [29])

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},\quad |t|<2\pi,$$

and the case $x=0$, i.e., $B_n=B_n(0)$ is called the nth Bernoulli number. We note that $B_0=1$, $B_1=-1/2$, $B_2=1/6$ and $B_{2n+1}=B_{2n-1}(1/2)=0$, $n\geq 1$.

The nth Bernoulli periodic function $\mathcal{B}_n(x)$, with period 1, is defined by

$$\mathcal{B}_n(x)=B_n(x-\lfloor x\rfloor),\ x\in\mathbb{R},$$

and has the following Fourier series representation:

$$\mathcal{B}_p(x)=-\frac{p!}{(2\pi i)^p}\sum_{n\neq 0}\frac{e^{2\pi inx}}{n^p}. \tag{2.1}$$

Here $x\in\mathbb{R}$ if $p>1$, $x\in\mathbb{R}\backslash\mathbb{Z}$ if $p=1$ and $\sum\limits_{n\neq 0}=\sum\limits_{\substack{n=-\infty\\ n\neq 0}}^{\infty}$. The function $\mathcal{B}_n(x)$ satisfies the following multiplication formula:

$$c^{p-1}\sum_{\mu=0}^{c-1}\mathcal{B}_p\left(x+\frac{\mu}{c}\right)=\mathcal{B}_p(cx). \tag{2.2}$$

While deriving functional equations for the series $T(s_1,s_2,s_3;x,y,z;c)$, we need the following decomposition formulas for $\mathcal{B}_p(X+Y)\mathcal{B}_q(Y)$. The first one is due to Can [8].

Lemma 2.1. *([8, Theorem 2.1]) For $X,Y\in\mathbb{R}$ and $p,q\geq 1$, we have*

$$\begin{aligned}\mathcal{B}_p(X+Y)\mathcal{B}_q(Y)&=\sum_{j=0}^{p}\binom{p}{j}\frac{q}{p+q-j}\mathcal{B}_{p+q-j}(Y)\mathcal{B}_j(X)+\frac{(-1)^{q-1}}{\binom{p+q}{q}}\mathcal{B}_{p+q}(X)\\&+\sum_{j=0}^{q}\binom{q}{j}\frac{p}{p+q-j}(-1)^j\mathcal{B}_{p+q-j}(X+Y)\mathcal{B}_j(X).\end{aligned} \tag{2.3}$$

The second is as follows:

Lemma 2.2. *Let $X,Y\in\mathbb{R}$, $A_p=-p!/(2\pi i)^p$ and $f(x)=e^{2\pi ix}+(-1)^{p+q}e^{-2\pi ix}$. For $p,q\geq 2$, we have*

$$\begin{aligned}&\frac{\mathcal{B}_p(X+Y)\mathcal{B}_q(Y)}{A_pA_q}-\frac{(-1)^q}{A_{p+q}}\mathcal{B}_{p+q}(X)\\&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{f(-mX+nY)}{(-m)^p(m+n)^q}+\frac{f((m+n)X+nY)}{(-m)^q(m+n)^p}+\frac{f(mX+(m+n)Y)}{m^pn^q}\right).\end{aligned} \tag{2.4}$$

Proof. It is seen from (2.1) that

$$\begin{aligned}\frac{\mathcal{B}_p(X+Y)\mathcal{B}_q(Y)}{A_pA_q}&=\sum_{n\neq 0}\sum_{m\neq 0}\frac{e^{2\pi i[nX+(m+n)Y]}}{n^pm^q}\\&=\sum_{\substack{n,m\neq 0\\ n+m=0}}\frac{e^{2\pi i[nX+(m+n)Y]}}{n^pm^q}+\sum_{\substack{n,m\neq 0\\ n+m\neq 0}}\frac{e^{2\pi i[nX+(m+n)Y]}}{n^pm^q}.\end{aligned} \tag{2.5}$$

It is clear that

$$\sum_{\substack{n,m\neq 0\\ n+m=0}}\frac{e^{2\pi i[nX+(m+n)Y]}}{n^pm^q}=(-1)^q\sum_{\substack{n\neq 0\\ n=-\infty}}^{\infty}\frac{e^{2\pi inX}}{n^{p+q}}=\frac{(-1)^q}{A_{p+q}}\mathcal{B}_{p+q}(X).$$

Now we consider the series with condition $n+m\neq 0$ in (2.5). We make the substitution $n=r-m$ and find that

$$\sum_{\substack{r,m\neq 0\\ r-m\neq 0}} \frac{e^{2\pi i[(r-m)X+rY]}}{(r-m)^p m^q} = (-1)^q \sum_{\substack{r,m\neq 0\\ r+m\neq 0}} \frac{e^{2\pi i[(r+m)X+rY]}}{(r+m)^p m^q}$$

$$= (-1)^q \left\{ \sum_{r,m>0} + \sum_{\substack{r>0,m<0\\ r>-m}} + \sum_{\substack{r>0,m<0\\ r<-m}} + \sum_{r,m<0} + \sum_{\substack{r<0,m>0\\ -r>m}} + \sum_{\substack{r<0,m>0\\ -r<m}} \right\}$$

$$\times \frac{e^{2\pi i[(r+m)X+rY]}}{(r+m)^p m^q}$$

$$= T_1+T_2+T_3+T_4+T_5+T_6.$$

Here, the series T_j can be written as

$$T_1 = (-1)^q \sum_{r,m>0} \frac{e^{2\pi i[(r+m)X+rY]}}{(r+m)^p m^q}, \qquad T_2 = \sum_{m,k>0} \frac{e^{2\pi i[kX+(m+k)Y]}}{k^p m^q},$$

$$T_3 = (-1)^p \sum_{r,k>0} \frac{e^{2\pi i(-kX+rY)}}{k^p (r+k)^q}, \qquad T_4 = (-1)^p \sum_{r,m>0} \frac{e^{-2\pi i[(r+m)X+rY]}}{(r+m)^p m^q},$$

$$T_5 = (-1)^{p+q} \sum_{m,k>0} \frac{e^{-2\pi i[kX+(m+k)Y]}}{k^p m^q}, \; T_6 = (-1)^q \sum_{r,k>0} \frac{e^{2\pi i(kX-rY)}}{k^p (r+k)^q}.$$

Therefore, we deduce that

$$\sum_{\substack{n,m\neq 0\\ n+m\neq 0}} \frac{e^{2\pi i[nX+(m+n)Y]}}{n^p m^q} = \sum_{n,m>0} \frac{e^{2\pi i[(n+m)X+nY]} + (-1)^{p+q} e^{2\pi i[-(n+m)X-nY]}}{(-1)^q m^q (n+m)^p}$$

$$+ (-1)^p \sum_{n,m>0} \frac{e^{2\pi i(-mX+nY)} + (-1)^{p+q} e^{2\pi i(mX-nY)}}{m^p (n+m)^q}$$

$$+ \sum_{n,m>0} \frac{e^{2\pi i[mX+(n+m)Y]} + (-1)^{p+q} e^{2\pi i[-mX-(n+m)Y]}}{m^p n^q}.$$

This completes the proof. □

3. Tornheim-type double series

In this section we first obtain two functional equations for the Tornheim-type double series via (2.3) and (2.4). Then, using these equations we give closed-form evaluation formulas.

The first functional equation is as follows:

Theorem 3.1. *For all $p,q,c\in\mathbb{N}$ and $s\in\mathbb{C}$, except singular points, we have*

$$\frac{1}{c^{p+q}} T(p,q,s;xc,0,z;1) + (-1)^p T(s,p,q;z,-x,0;c) + (-1)^q T(s,q,p;z,0,x;c)$$

$$+ \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{j=1}^{c-1} \frac{e^{2\pi i[(mc+j)x+(n+m)z]}}{(mc+j)^p (nc-j)^q (n+m)^s} = C(p,q,s;x,z;c), \tag{3.1}$$

where

$$C(p,q,s;x,z;c)=\sum_{j=0}^{p}\binom{p+q-j-1}{q-1}\frac{\mathcal{B}_j(x)L(p+q+s-j;z)}{A_jc^{p+q-j}}+\sum_{j=0}^{q}\binom{p+q-j-1}{p-1}(-1)^j\frac{\mathcal{B}_j(x)L(p+q+s-j;cx+z)}{A_jc^{p+q-j}} \tag{3.2}$$

and $A_r=-r!/(2\pi i)^r$.

Proof. We first assume that $p,q\geq 2$ and $\operatorname{Re}(s)>1$. Set $X=x$, $Y=(\mu+y)/c$ in (2.3) and (2.4). Then summing over $\mu=0,1,\ldots,c-1$, with the use of (2.2), we see that

$$\begin{aligned}&\sum_{m,n=1}^{\infty}\frac{(-1)^pe^{2\pi i(-mx+\frac{ny}{c})}+(-1)^qe^{2\pi i\left(mx-\frac{ny}{c}\right)}}{m^p(n+m)^q}\sum_{\mu=0}^{c-1}e^{2\pi in\frac{\mu}{c}}\\&+\sum_{m,n=1}^{\infty}\frac{(-1)^qe^{2\pi i[(m+n)x+\frac{ny}{c}]}+(-1)^pe^{2\pi i[-(m+n)x-\frac{ny}{c}]}}{m^q(n+m)^p}\sum_{\mu=0}^{c-1}e^{2\pi in\frac{\mu}{c}}\\&+\sum_{m,n=1}^{\infty}\frac{e^{2\pi i[mx+(m+n)\frac{y}{c}]}+(-1)^{p+q}e^{2\pi i[-mx-(m+n)\frac{y}{c}]}}{m^pn^q}\sum_{\mu=0}^{c-1}e^{-2\pi i(m+n)\frac{\mu}{c}}\\&=\frac{(2\pi i)^{p+q}}{p!q!}\sum_{j=0}^{p}\binom{p}{j}\frac{q}{p+q-j}\frac{\mathcal{B}_j(x)\mathcal{B}_{p+q-j}(y)}{c^{p+q-j-1}}\\&+\frac{(2\pi i)^{p+q}}{p!q!}\sum_{j=0}^{q}\binom{q}{j}\frac{p(-1)^j}{p+q-j}\frac{\mathcal{B}_j(x)\mathcal{B}_{p+q-j}(cx+y)}{c^{p+q-j-1}}.\end{aligned}\tag{3.3}$$

From the basic identity

$$\sum_{\mu=0}^{c-1}e^{2\pi in\frac{\mu}{c}}=\begin{cases}c, & \text{if } c\mid n,\\ 0, & \text{if } c\nmid n,\end{cases}$$

the left-hand side of (3.3) becomes

$$\begin{aligned}&c\sum_{m,n=1}^{\infty}\frac{(-1)^pe^{2\pi i(-mx+ny)}+(-1)^qe^{2\pi i(mx-ny)}}{m^p(nc+m)^q}\\&+c\sum_{m,n=1}^{\infty}\frac{(-1)^qe^{2\pi i[(m+nc)x+ny]}+(-1)^pe^{2\pi i[-(m+nc)x-ny]}}{m^q(nc+m)^p}\\&+c\sum_{\substack{m=1\\c\nmid n,c\nmid m\\c\mid(m+n)}}^{\infty}\sum_{n=1}^{\infty}\frac{e^{2\pi i[mx+(m+n)\frac{y}{c}]}+(-1)^{p+q}e^{2\pi i[-mx-(m+n)\frac{y}{c}]}}{m^pn^q}\\&+c\sum_{\substack{m=1\\c\mid n,c\mid m}}^{\infty}\sum_{n=1}^{\infty}\frac{e^{2\pi i[mx+(m+n)\frac{y}{c}]}+(-1)^{p+q}e^{2\pi i[-mx-(m+n)\frac{y}{c}]}}{m^pn^q}.\end{aligned}$$

We make the substitutions $m\to mc+j$ and $n\to nc-j$, $1\leq j\leq c-1$ for the series with conditions $c\nmid n$, $c\nmid m$ and $c\mid(m+n)$, and $m\to mc$ and $n\to nc$ for the series with conditions $c|n$ and $c|m$, respectively. We then find that

$$\begin{aligned}
&\sum_{m,n=1}^{\infty} \frac{(-1)^p e^{2\pi i(-mx+ny)} + (-1)^q e^{2\pi i(mx-ny)}}{m^p(nc+m)^q} \\
&+ c\sum_{m,n=1}^{\infty} \frac{(-1)^q e^{2\pi i[(m+nc)x+ny]} + (-1)^p e^{-2\pi i[(m+nc)x+ny]}}{m^q(nc+m)^p} \\
&+ c\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{j=1}^{c-1} \frac{e^{2\pi i[(mc+j)x+(m+n)y]} + (-1)^{p+q} e^{-2\pi i[(mc+j)x+(m+n)y]}}{(mc+j)^p(nc-j)^q} \\
&+ c\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{e^{2\pi i[mcx+(m+n)y]} + (-1)^{p+q} e^{-2\pi i[mcx+(m+n)y]}}{c^{p+q}m^p n^q}. \qquad (3.4)
\end{aligned}$$

We use (2.1) to write the right-hand side of (3.3) as

$$\begin{aligned}
&\sum_{j=0}^{p} \binom{p+q-j-1}{q-1} \frac{\mathcal{B}_j(x)}{c^{p+q-j-1}A_j} \sum_{k=1}^{\infty} \frac{(-1)^{p+q-j}e^{-2\pi iky} + e^{2\pi iky}}{k^{p+q-j}} \qquad (3.5) \\
&+ \sum_{j=0}^{q} \binom{p+q-j-1}{p-1} \frac{(-1)^j\mathcal{B}_j(x)}{c^{p+q-j-1}A_j} \sum_{k=1}^{\infty} \frac{(-1)^{p+q-j}e^{-2\pi ik(cx+y)} + e^{2\pi ik(cx+y)}}{k^{p+q-j}}.
\end{aligned}$$

We now multiply (3.4) and (3.5) with

$$\sum_{l=1}^{\infty} e^{2\pi il(z-y)} l^{-s}$$

and then integrate from 0 to 1 with respect to y. Then the resulting equations become

$$\begin{aligned}
&c\sum_{m,n=1}^{\infty} \left[\frac{(-1)^p e^{2\pi i(-mx+nz)}}{m^p(nc+m)^q n^s} + \frac{(-1)^q e^{2\pi i[(m+nc)x+nz]}}{m^q(nc+m)^p n^s} + \frac{e^{2\pi i[mcx+(n+m)z]}}{c^{p+q}m^p n^q (n+m)^s} \right] \\
&\quad + c\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{j=1}^{c-1} \frac{e^{2\pi i[(mc+j)x+(n+m)z]}}{(mc+j)^p(nc-j)^q(n+m)^s} \\
&= c(-1)^p T(s,p,q;z,-x,0;c) + c(-1)^q T(s,q,p;z,0,x;c) + \frac{c}{c^{p+q}} T(p,q,s;xc,0,z;1) \\
&\quad + c\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{j=1}^{c-1} \frac{e^{2\pi i[(mc+j)x+(n+m)z]}}{(mc+j)^p(nc-j)^q(n+m)^s}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{j=0}^{p} \binom{p+q-j-1}{q-1} \frac{\mathcal{B}_j(x)L(p+q+s-j;z)}{c^{p+q-j-1}A_j} \\
&\quad + \sum_{j=0}^{q} \binom{p+q-j-1}{p-1} (-1)^j \frac{\mathcal{B}_j(x)L(p+q+s-j;cx+z)}{c^{p+q-j-1}A_j} = cC(p,q,s;x,z;c),
\end{aligned}$$

respectively. These give (3.1) for $p, q \geq 2$ and $\mathrm{Re}\,(s) > 1$. For the remainder cases we need the analogues of [15, Eq. (1.5)]

$$T(r,t-1,s+1) + T(r-1,t,s+1) = T(r,t,s).$$

These are

$$T(p,q-1,s+1;z,x,y;c)+cT(p-1,q,s+1;z,x,y;c)=T(p,q,s;z,x,y;c),$$
$$T(p,q+1,s-1;z,x,y;c)-cT(p-1,q+1,s;z,x,y;c)=T(p,q,s;z,x,y;c),$$
$$T(p+1,q,s-1;z,x,y;c)-T(p+1,q-1,s;z,x,y;c)=cT(p,q,s;z,x,y;c),$$
$$T^{*}(p,q-1,s+1;x,0,z;c)+T^{*}(p-1,q,s+1;x,0,z;c)=cT^{*}(p,q,s;x,0,z;c),$$

where

$$T^{*}(p,q,s;x,0,z;c)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{j=1}^{c-1}\frac{e^{2\pi i[(mc+j)x+(n+m)z]}}{(mc+j)^{p}(nc-j)^{q}(n+m)^{s}}.$$

Using these identities in (3.1) for $p=2$ and $q\geq 2$ we deduce that

$$\begin{aligned}cC(2,q,s;x,z;c)=&\frac{1}{c^{1+q}}T(1,q,s+1;xc,0,z;1)+(-1)^{1}T(s+1,1,q;z,-x,0;c)\\&+(-1)^{q}T(s+1,q,1;z,0,x;c)+T^{*}(1,q,s+1;x,0,z;c)\\&+C(2,q-1,s+1;x,z;c).\end{aligned}$$

It can be seen from (3.2) that

$$cC(2,q,s;x,z;c)=C(2,q-1,s+1;x,z;c)+C(1,q,s+1;x,z;c),$$

hence (3.1) is obtained for $p=1$ and $q\geq 2$. The proof of (3.1) for the cases $p\geq 2$, $q=1$ and $p=q=1$ are similar, so we omit them. From [22, Theorem 2.1] $T(p,q,s;z,x,y;c)$ can be meromorphically continued to $\mathbb{C}^3$. Thus, by analytic continuation (3.1) is valid for $p,q\in\mathbb{N}$ and $s\in\mathbb{C}$, except singular points. □

In the next result we state the second functional equation. Its proof is similar to the that of Theorem 3.1, except with the substitutions $X=(\mu+z)/c$ and $Y=y$ in (2.3) and (2.4), so we omit it.

Theorem 3.2. *For all $p,q,c\in\mathbb{N}$ and $s\in\mathbb{C}$, except singular points, we have*

$$\begin{aligned}&\frac{(-1)^{q}}{c^{s+q}}T(s,q,p;cx,0,z;1)+(-1)^{p}T(p,s,q;-z,x,0;c)+T(p,q,s;z,0,x,c)\\&\quad+(-1)^{q}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\sum_{j=1}^{c-1}\frac{e^{2\pi i[(nc+j)x+(n+m)z]}}{(nc+j)^{s}(mc-j)^{q}(n+m)^{p}}=D(p,q,s;x,z;c),\end{aligned}\tag{3.6}$$

where

$$\begin{aligned}D(p,q,s;x,z;c)=&\sum_{j=0}^{p}\binom{p+q-j-1}{q-1}\frac{c^{p-j}}{A_j}\mathcal{B}_j(z)L(p+q+s-j;x)\\&+c^{p-1}\sum_{j=0}^{q}\binom{p+q-j-1}{p-1}\frac{(-1)^{j}}{A_j}\sum_{\mu=0}^{c-1}\mathcal{B}_j\left(\frac{\mu+z}{c}\right)L\left(p+q+s-j;\frac{\mu+z}{c}+x\right)\end{aligned}\tag{3.7}$$

and $A_r=-r!/(2\pi i)^r$.

Using the functional equations given in Theorems 3.1 and 3.2 we may deduce the following closed-form evaluation formulas.

Theorem 3.3. *Let $r,u,v\in\mathbb{N}$. If $r+u+v$ is odd, then*

$$2\sum_{m,n=1}^{\infty}\frac{\cos\left(2\pi\left(mz-nx\right)\right)}{m^{r}n^{u}\left(mc+n\right)^{v}}=(-1)^{u}C(u,v,r;x,z;c)+(-1)^{r}D(r,v,u;z,x;c)\tag{3.8}$$

and if $r+u+v$ is even, then

$$2i\sum_{m,n=1}^{\infty}\frac{\sin\left(2\pi\left(mz-nx\right)\right)}{m^{r}n^{u}\left(mc+n\right)^{v}}=(-1)^{u}C(u,v,r;x,z;c)-(-1)^{r}D(r,v,u;z,x;c), \tag{3.9}$$

where $C(u,v,r;x,z;c)$ and $D(r,v,u;z,x;c)$ are given by (3.2) and (3.7), respectively.

Proof. We take $(p,q,s)\to(u,v,r)$ in (3.1) and $(p,q,s)\to(r,v,u)$ in (3.6). Then multiplying by $(-1)^{u}$ and $(-1)^{u+v}$, respectively, and substracting the resulting equations give

$$\begin{aligned}T(r,u,v;z,-x,0;c)&-(-1)^{r+u+v}T(r,u,v;-z,x,0;c)\\&=(-1)^{u}C(u,v,r;x,z;c)-(-1)^{u+v}D(r,v,u;z,x;c).\end{aligned} \tag{3.10}$$

This yields the desired results. □

It should be mentioned that the formula (3.8) leads to the closed-form evaluation formulas for the series given in Table 1a) for the pairs $(z,x)\in\{(0,0),\ (0,1/2),\ (1/2,0),\ (1/2,1/2),\ (0,1/4),\ (1/4,0),\ (1/2,1/4),\ (1/4,1/2),\ (1/4,1/4)\}$. Closed-form evaluation formulas for the series given in Table 1b) follow from (3.9) for the pairs $(z,x)\in\{(0,-1/4),\ (1/4,0),\ (1/2,-1/4),\ (1/4,-1/2),\ (1/4,-1/4)\}$. Furthermore, from (3.8) and (3.9) we have the following results: If r is odd, then

$$\begin{aligned}T(r,r,r;0,0,0;c)=&-\sum_{j=0}^{(r-1)/2}\left(\frac{2}{c^{2r-2j}}+c^{r-2j}\right)\binom{2r-2j-1}{r-1}\zeta\left(2j\right)\zeta(3r-2j)\\&-\frac{c^{r-1}}{2}\sum_{j=0}^{(r-1)/2}\binom{2r-2j-1}{r-1}\frac{(-1)^{j}}{A_{j}}\sum_{\mu=0}^{c-1}\mathcal{B}_{j}\left(\frac{\mu}{c}\right)L\left(3r-j;\frac{\mu}{c}\right).\end{aligned}$$

If $r+u+v$ is even, then

$$C(u,v,r;x,z;c)=(-1)^{v}D(r,v,u;z,x;c)$$

for the pairs $(z,x)\in\{(0,0),\ (0,1/2),\ (1/2,0),\ (1/2,1/2)\}$.

Remark 3.4. When $x=z=0$ (3.8) reduces to

$$\begin{aligned}&T(r,u,v;0,0,0;c)=\zeta_{c,1}\left(r,u,v\right)\\&=(-1)^{u}\sum_{j=0}^{\max(u/2,v/2)}\left(\binom{u+v-2j-1}{v-1}+\binom{u+v-2j-1}{u-1}\right)\frac{\zeta\left(2j\right)\zeta(u+v+r-2j)}{c^{u+v-2j}}\\&+(-1)^{r}\sum_{j=0}^{r/2}\binom{r+v-2j-1}{v-1}c^{r-2j}\zeta(2j)\zeta(r+v+u-2j)\\&+\frac{(-1)^{r}}{c^{v+u}}\sum_{j=0}^{v/2}\binom{r+v-2j-1}{r-1}\zeta(2j)\zeta(r+v+u-2j)\\&+\frac{(-1)^{r}}{c^{v+u}2}\sum_{j=2}^{v}\binom{r+v-j-1}{r-1}\sum_{l=1}^{c-1}\zeta(r+v+u-j,l/c)\left(\zeta(j,l/c)+(-1)^{j}\zeta(j,1-l/c)\right)\\&+\pi i(-1)^{r}c^{r-1}\binom{r+v-2}{r-1}\sum_{\mu=1}^{c-1}\mathcal{B}_{1}\left(\frac{\mu}{c}\right)L\left(r+v+u-1;\frac{\mu}{c}\right)\end{aligned}$$

which is considerably simpler than Okamoto's formula [27, Theorem 4.5].

While considering special cases we need the following lemma.

Lemma 3.5. *Let $q \geq 1$, $r \geq 2$ be integers and $x, z \in \mathbb{R}$ with $z \notin \mathbb{Z}$ if $q = 1$. Then*

$$\begin{aligned}
&\frac{(-1)^q}{cA_q}\sum_{\mu=0}^{c-1}\mathcal{B}_q\left(\frac{\mu+z}{c}\right)L\left(r;\frac{\mu+z}{c}+x\right)\\
&=\frac{1}{c^{q+r}}\left\{L(q;-z)+(-1)^qL(q;z)\right\}L(r;cx+z)\\
&+\frac{1}{c^{q+r}}\sum_{l=1}^{c-1}\left\{\mathcal{L}(q;l/c;-z)+(-1)^q\mathcal{L}(q;1-l/c;z)e^{2\pi iz}\right\}\mathcal{L}(r;l/c;cx+z)e^{2\pi ilx}, \qquad (3.11)
\end{aligned}$$

where $A_r = -r!/(2\pi i)^r$ and

$$\mathcal{L}(s;z;x)=\sum_{m=0}^{\infty}\frac{e^{2\pi imx}}{(m+z)^s},$$

provided that the series converges.

Proof. It can be seen from (2.1) and definition of $L(p,x)$ that

$$\frac{1}{cA_q}\sum_{\mu=0}^{c-1}\mathcal{B}_q\left(\frac{\mu+z}{c}\right)L\left(r;\frac{\mu+z}{c}+x\right)=\sum_{\substack{h\neq 0\\ c\nmid h, c\nmid j\\ c|(j+h)}}\sum_{j=1}^{\infty}\frac{e^{2\pi i(j+h)\frac{z}{c}}e^{2\pi ijx}}{h^qj^r}+\sum_{\substack{h\neq 0\\ c|h,\ c|j}}\sum_{j=1}^{\infty}\frac{e^{2\pi i(j+h)\frac{z}{c}}e^{2\pi ijx}}{h^qj^r}.$$

We take $j = mc + l$ and $h = nc - l$, $1 \leq l \leq c - 1$ for the first series and $j = mc$ and $h = nc$ for the second. Then the right-hand side becomes

$$\begin{aligned}
&\sum_{n\neq 0}\sum_{m=0}^{\infty}\sum_{l=1}^{c-1}\frac{e^{2\pi inz}e^{2\pi im(cx+z)}}{(nc-l)^q(mc+l)^r}e^{2\pi ilx}+\sum_{n\neq 0}\sum_{m=1}^{\infty}\frac{e^{2\pi inz}e^{2\pi im(cx+z)}}{(nc)^q(mc)^r}\\
&=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{l=1}^{c-1}\frac{e^{2\pi inz}e^{2\pi im(cx+z)}}{(nc-l)^q(mc+l)^r}e^{2\pi ilx}+(-1)^q\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=1}^{c-1}\frac{e^{-2\pi inz}e^{2\pi im(cx+z)}}{(nc+l)^q(mc+l)^r}e^{2\pi ilx}\\
&+\sum_{n,m=1}^{\infty}\frac{e^{2\pi inz}e^{2\pi im(cx+z)}}{(nc)^q(mc)^r}+(-1)^q\sum_{n,m=1}^{\infty}\frac{e^{-2\pi inz}e^{2\pi im(cx+z)}}{(nc)^q(mc)^r},
\end{aligned}$$

from which the proof follows. □

Remark 3.6. We give several examples as demonstrations of the evaluation formula (3.8) (with the use of (3.11)). We first recall the Hurwitz zeta $\zeta(s,z)$ and the generalized eta (or alternating Hurwitz zeta) $\eta(s,z)$ functions:

$$\zeta(s,z)=\sum_{m=0}^{\infty}\frac{1}{(m+z)^s},\ \ \text{Re}(s)>1,\ \text{and}\ \eta(s,z)=\sum_{m=0}^{\infty}\frac{(-1)^m}{(m+z)^s},\ \ \text{Re}(s)>0,$$

where $z \in \mathbb{R}$ with $z \neq 0, -1, -2, \ldots$ (for more details, see for instance [29]). For particular choices of z and x we use the facts $\mathcal{L}(s;1;x) = e^{-2\pi ix}L(s;x)$, $\mathcal{L}(s;z;0) = \zeta(s,z)$, $\mathcal{L}(s;z;1/2) = \eta(s,z)$, $\zeta(s) = \zeta(s,1) = L(s;0)$ and $\eta(s) = \eta(s,1) = -L(s;1/2)$, the Dirichlet eta function (or alternating Riemann zeta function). The even values of the Riemann zeta and the Dirichlet eta functions are given by

$$\zeta(2n)=-\frac{(2\pi i)^{2n}}{2(2n)!}B_{2n}\ \text{and}\ \eta(2n)=\frac{(2\pi i)^{2n}}{2(2n)!}B_{2n}\left(\frac{1}{2}\right),\ n\geq 0.$$

Now, we list some examples: For $r = 1, u = v = 2$, $z = x = 0$ and $c = 3$,

$$\sum_{m,n=1}^{\infty}\frac{1}{mn^2(3m+n)^2}=\frac{481}{162}\zeta(5)-\frac{29\pi^2}{162}\zeta(3)-\frac{\pi}{3}Cl_4\left(\frac{2\pi}{3}\right),$$

where $Cl_{2r}(x)$ is the Clausen function

$$Cl_{2r}(x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^{2r}}, \ r \in \mathbb{N}.$$

For $r = u = v = 1$, $z = x = 0$ and $c = 3$,

$$\sum_{m,n=1}^{\infty} \frac{1}{mn(3m+n)} = \frac{5}{3}\zeta(3) - \frac{\pi}{3} Cl_2(2\pi/3).$$

For $r = 1, u = v = 3$, $z = x = 0$ and $c = 3$,

$$\sum_{m,n=1}^{\infty} \frac{1}{mn^3(3m+n)^3} = \frac{1097}{243}\zeta(7) - \frac{341}{1458}\pi^2\zeta(5) - \frac{\pi}{3} Cl_6\left(\frac{2\pi}{3}\right)$$
$$- \frac{1}{1458}\left\{\zeta(3,1/3) - \zeta(3,2/3)\right\}\left\{\zeta(4,1/3) - \zeta(4,2/3)\right\},$$

for $r = u = v = 1$, $z = 1/2$, $x = 0$ and $c = 2$,

$$\sum_{m,n=1}^{\infty} \frac{(-1)^m}{mn(2m+n)} = \zeta(3) - \frac{3}{8}\eta(3) - \frac{1}{4}\eta(2,1/2)\eta(1,1/2),$$

for $r = u = 3$, $v = 1$, $z = 1/2$, $x = 0$ and $c = 2$,

$$\sum_{m,n=1}^{\infty} \frac{(-1)^m}{m^3n^3(2m+n)} = 4\zeta(7) - \frac{7}{32}\eta(7) + \frac{\pi^2}{6}\zeta(5) + \frac{\pi^2}{24}\eta(5) - \frac{1}{16}\eta(6,1/2)\eta(1,1/2),$$

for $r = 4$, $u = 2$, $v = 1$, $z = 1/2$, $x = 0$ and $c = 3$,

$$\sum_{m,n=1}^{\infty} \frac{(-1)^m}{m^4n^2(3m+n)} = -\frac{81}{2}\zeta(7) + \frac{7}{54}\eta(7) - \frac{3\pi^2}{4}\zeta(5) - \frac{\pi^2}{18}\eta(5) - \frac{7\pi^4}{720}\zeta(3)$$
$$+ \frac{1}{54}\left\{\eta(6,1/3) + \eta(6,2/3)\right\}\left\{\eta(1,1/3) + \eta(1,2/3)\right\},$$

for $r = u = v = 1$, $z = x = 1/2$ and $c = 2$,

$$\sum_{m,n=1}^{\infty} \frac{(-1)^{m+n}}{mn(2m+n)} = -\frac{11}{8}\eta(3) + \frac{1}{4}\eta(2,1/2)\eta(1,1/2),$$

for $r = 2$, $u = 1$, $v = 2$, $z = x = 1/2$ and $c = 3$,

$$\sum_{m,n=1}^{\infty} \frac{(-1)^{m+n}}{m^2n(3m+n)^2} = -\frac{1}{27}\zeta(5) + \frac{727}{54}\eta(5) - \frac{8}{27}\pi^2\zeta(3) + \frac{\pi^2}{12}\eta(3)$$
$$+ \frac{1}{27}\left\{\zeta(4,2/3) - \zeta(4,1/3)\right\}\left\{\eta(1,2/3) + \eta(1,1/3)\right\}.$$

To present evaluation formulas for the Euler-type sums we need the following lemma which is an analogue of the so called the reflection or harmonic product formula [2, p. 972].

Lemma 3.7. *Let $p, q, c \in \mathbb{N}$. For $p, q \geq 2$, we have*

$$T(0,p,q;y,x,0;c) + T(0,q,p;cx-y,x,0;c)$$
$$= \frac{1}{c^{p+q}} \sum_{j=1}^{c} \mathcal{L}(p;j/c;cx-y)\mathcal{L}(q;j/c;y)e^{2\pi ijx} - L(p+q;x).$$

In particular,

$$T(0,p,q;0,0,0;c)+T(0,q,p;0,0,0;c)=\frac{1}{c^{p+q}}\sum_{j=1}^{c}\zeta(p,j/c)\zeta(q,j/c)-\zeta(p+q),$$

$$T(0,p,q;1/2,0,0;c)+T(0,q,p;1/2,0,0;c)=\sum_{j=1}^{c}\frac{\eta(p,j/c)\eta(q,j/c)}{c^{p+q}}-\zeta(p+q).$$

Proof. We set $nc+m=k'$ and $m=l'$ in the series

$$T(0,p,q;y,x,0;c)=\sum_{m,n=1}^{\infty}\frac{e^{2\pi iny}e^{2\pi imx}}{m^p(nc+m)^q}.$$

Since $0<nc=k'-l'$ we deduce that $k'=kc+j$ and $l'=lc+j$, $1\le j\le c$, and $0<n=k-l$. Thus

$$T(0,p,q;y,x,0;c)=\sum_{j=1}^{c}\sum_{k>l\ge 0}\sum\frac{e^{2\pi il(cx-y)}e^{2\pi iky}}{(lc+j)^p(kc+j)^q}e^{2\pi ijx}.$$

Now, we set $m=k'$ and $nc+m=l'$ in the series

$$T(0,q,p;cx-y,x,0;c)=\sum_{m,n=1}^{\infty}\frac{e^{2\pi in(cx-y)}e^{2\pi imx}}{m^q(nc+m)^p}.$$

Since $0<nc=l'-k'$ we infer $k'=kc+j$ and $l'=lc+j$, $1\le j\le c$, and $n=l-k$. Then we find that

$$\begin{aligned}
&T(0,p,q;y,x,0;c)+T(0,q,p;cx-y,x,0;c)\\
&=\sum_{j=1}^{c}\left\{\sum_{k>l\ge 0}\sum\frac{e^{2\pi il(cx-y)}e^{2\pi iky}}{(lc+j)^p(kc+j)^q}e^{2\pi ijx}+\sum_{l>k\ge 0}\sum\frac{e^{2\pi il(cx-y)}e^{2\pi iky}}{(lc+j)^p(kc+j)^q}e^{2\pi ijx}\right\}\\
&=\sum_{j=1}^{c}\left\{\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{e^{2\pi il(cx-y)}e^{2\pi iky}}{(lc+j)^p(kc+j)^q}e^{2\pi ijx}-\sum_{k=0}^{\infty}\frac{e^{2\pi ix(kc+j)}}{(kc+j)^{p+q}}\right\}\\
&=\frac{1}{c^{p+q}}\sum_{j=1}^{c}\sum_{k=0}^{\infty}\frac{e^{2\pi iky}}{(k+j/c)^q}\sum_{l=0}^{\infty}\frac{e^{2\pi il(cx-y)}}{(l+j/c)^p}e^{2\pi ijx}-\sum_{m=1}^{\infty}\frac{e^{2\pi ixm}}{m^{p+q}},
\end{aligned}$$

which is the desired result. □

Previously we have proved (3.6) for $p\in\mathbb{N}$ (cf. Theorem 3.2). However, we infer from Lemmas 3.5 and 3.7 that (3.6) is also valid for $p=0$. This shows that the closed-form evaluation formulas given in Theorem 3.3 are also valid for $r=0$.

The next corollary, a consequence of Theorem 3.3, presents closed-form evaluation formulas for the Euler double series.

Corollary 3.8. *Let $c\in\mathbb{N}$ and $p,q\in\mathbb{N}\backslash\{1\}$. If $p+q$ is odd, then*

$$2\sum_{m,n=1}^{\infty}\frac{\cos\left(2\pi\left(my+nx\right)\right)}{n^p\left(mc+n\right)^q}=D(0,q,p;-y,x;c)-(-1)^qC(p,q,0;x,-y;c),\tag{3.12}$$

and if $p+q$ is even, then

$$2i\sum_{m,n=1}^{\infty}\frac{\sin\left(2\pi\left(my+nx\right)\right)}{n^p\left(mc+n\right)^q}=D(0,q,p;-y,x;c)-(-1)^qC(p,q,0;x,-y;c),\tag{3.13}$$

where $C(p,q,0;x,y;c)$ and $D(0,q,p;y,x;c)$ are given by (3.2) and (3.7), respectively.

It is obvious that the closed-form evaluation formulas for the Euler sums given in Table 2a) are consequences of (3.12) for the pairs $(x,y) \in \{(0,0),\ (0,1/2),\ (1/2,0),\ (1/2,1/2),\ (0,1/4),\ (1/4,0),\ (1/2,1/4),\ (1/4,1/2),\ (1/4,1/4)\}$, and the formulas for the series given in Table 2b) follow from (3.13) for the pairs $(x,y) \in \{(0,1/4),\ (1/4,0),\ (1/2,1/4),\ (1/4,1/2),\ (1/4,1/4)\}$.

Remark 3.9. We list several examples deduced from (3.12). We have, for $p=2$, $q=5$, $x=y=0$ and $c=2$,

$$\sum_{m,n=1}^{\infty} \frac{1}{m^2(2n+m)^5} = -\frac{149}{256}\zeta(7) + \frac{5\pi^2}{192}\zeta(5) + \frac{\pi^4}{360}\zeta(3),$$

for $p=3$, $q=2$, $x=y=0$ and $c=3$,

$$\sum_{m,n=1}^{\infty} \frac{1}{m^3(3n+m)^2} = -\frac{233}{486}\zeta(5) + \frac{13}{243}\pi^2\zeta(3),$$

for $p=4$, $q=3$, $x=y=0$ and $c=4$,

$$\sum_{m,n=1}^{\infty} \frac{1}{m^4(4n+m)^3} = -\frac{16419}{32768}\zeta(7) + \frac{5\pi^2}{3072}\zeta(5) + \frac{\pi^4}{5760}\zeta(3) + \frac{1}{32768}\{\zeta(4,1/4) - \zeta(4,3/4)\}\{\zeta(3,1/4) - \zeta(3,3/4)\},$$

for $p=3, q=4$, $x=0$, $y=1/2$ and $c=5$,

$$\sum_{m,n=1}^{\infty} \frac{(-1)^n}{m^3(5n+m)^4} = -\frac{1}{2}\zeta(7) - \frac{7}{31\,250}\eta(7) + \frac{\pi^2}{1875}\eta(5) + \frac{1669\pi^4}{18750000}\eta(3) + \frac{1}{156250}\{\eta(3,4/5) - \eta(3,1/5)\}\{\eta(4,4/5) - \eta(4,1/5)\} + \frac{1}{156250}\{\eta(3,2/5) - \eta(3,3/5)\}\{\eta(4,2/5) - \eta(4,3/5)\},$$

for $p=3, q=2$, $x=1/2$, $y=0$, and $c=3$,

$$\sum_{m,n=1}^{\infty} \frac{(-1)^m}{m^3(3n+m)^2} = \frac{2}{243}\zeta(5) + \frac{79}{162}\eta(5) + \frac{\pi^2}{162}\zeta(3) - \frac{73\pi^2}{972}\eta(3),$$

for $p=5, q=4$, $x=y=1/2$ and $c=3$,

$$\sum_{m,n=1}^{\infty} \frac{(-1)^{m+n}}{m^5(3n+m)^4} = \frac{35}{19\,683}\zeta(9) + \frac{19\,627}{39\,366}\eta(9) + \frac{5\pi^2}{8748}\zeta(7) - \frac{5\pi^2}{6561}\eta(7) - \frac{3200}{6561}\zeta(5)\,\eta(4) - \frac{4}{243}\eta(4)\eta(5).$$

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