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## On Parametric Surfaces with Constant Mean Curvature Along Given Smarandache Curves in Lie Group

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**Abstract** — This paper finds sufficient conditions to determine a surface whose mean curvature along a given Smarandache curve is constant in a three-dimensional Lie group. This is accomplished by using the Frenet frames of the specified curve to express surfaces that span the  $TN$ ,  $NB$ , and  $TB$  Smarandache curves parametrically. In terms of the curvatures of given Smarandache curves, marching scale functions, and their partial derivatives, the mean curvatures of these surfaces along the given  $TN$ ,  $NB$ , and  $TB$  Smarandache curves are determined. Sufficient conditions are found to maintain the provided mean curvatures of the resulting surfaces at a constant value. Finally, some examples are provided.

**Keywords** – Smarandache curve, mean curvature, Lie group

**Mathematics Subject Classification (2020)** – 53A04, 53A05

## 1. Introduction

The theory of curves is one of the most crucial research areas in classical differential geometry. For a very long time, and even now, special curves and their characterizations have been investigated. The use of special curves can be observed in nature, mechanical devices, computer-aided design, and other things. The Smarandache curve, one of the special curves, has a position vector made up of Frenet frame vectors on another regular curve. Ahmad first presented a few special Smarandache curves to the Euclidean space in [1]. Additionally, some researchers investigated Smarandache curves in the Lie group and Minkowski space [2,3], respectively.

In two ways, Lie groups are made of algebra and geometry, two significant branches of mathematics: first, Lie groups are groups, and second, they are smooth manifolds. As a result, there must be some kind of coherence between the Lie groups' geometric and algebraic structure. The current approach to geometry as a whole is based on the geometry of Lie groups. Additionally, numerous research findings on curves and surfaces in the 3-dimensional Lie group have been published in [4-8].

On the other hand, in differential geometry, surfaces can have a variety of remarkable effects and properties. Researchers later turned their focus to the construction surfaces along a special curve such as a geodesic, an asymptotic, or a line of curvature. Some recently research on these topics was done in [9-12]. The process in these papers is as follows: conditions for that curve to be a geodesic, asymptotic, and line of curvature have been given, and the parametric surface has been constructed as a linear combination of an

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isoparametric curve and its Frenet frame. A new study on construction surfaces with constant curvatures along a given curve was recently proposed by Bayram et al. [13,14].

We organized our paper as follows: we give some basic information regarding the Smarandache curve and surface theory in the 3-dimensional Lie group in Section 2. We build surfaces along the Smarandache curves of the specified curve in Section 3, and then we derive sufficient conditions for each case where the surfaces have constant mean curvature along the  $TN$ ,  $NB$ , and  $TB$  Smarandache curves. This study was derived from the second author’s master’s thesis under the supervision of the first author.

## 2. Preliminaries

The Frenet formulas for a unit speed curve  $\alpha(s)$  in the Lie group are expressed as follows:

$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 \\ -\kappa_1 & 0 & (\kappa_2 - \bar{\kappa}_2) \\ 0 & -(\kappa_2 - \bar{\kappa}_2) & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix} \tag{1}$$

where  $\kappa_1$  and  $\kappa_2$  are the curvature functions of  $\alpha(s)$  and  $\bar{\kappa}_2 = \frac{1}{2} \langle [T, N], B \rangle$  which was introduced [4-7], is the Lie group torsion of  $\alpha(s)$ . Here,  $T = \alpha'(s)$ ,  $\kappa_1(s) = \|D_T T\| = \|T'\|$ ,  $\kappa_2 = \|D_T B\| - \bar{\kappa}_2$ , and  $D_T X = X' + \frac{1}{2} [T, X]$ .

**Definition 2.1.** [5]  $\tilde{h} = \frac{\kappa_2 - \bar{\kappa}_2}{\kappa_1}$  is denoted the harmonic curvature function of  $\alpha(s)$ .

**Theorem 2.2.** [4,5] The curve is a general helix in Lie Group  $G$  if and only if its harmonic function is a constant function.

**Definition 2.3.** [15] Let  $\varphi = \varphi(s, t)$  be a surface in the 3-dimensional Lie Group, then the mean curvature of the ruled surface  $\varphi$  in three-dimensional Lie group is given by

$$H = \frac{En - 2Fm + lG}{EG - F^2} \tag{2}$$

where the surface normal  $N = \frac{\varphi_s \times \varphi_t}{\|\varphi_s \times \varphi_t\|}$ ,  $E = \langle \varphi_s, \varphi_s \rangle$ ,  $F = \langle \varphi_s, \varphi_t \rangle$ ,  $G = \langle \varphi_t, \varphi_t \rangle$ ,  $l = \langle \varphi_{ss}, N \rangle$ ,  $m = \langle \varphi_{st}, N \rangle$ , and  $n = \langle \varphi_{tt}, N \rangle$ .

**Definition 2.4.** [3] Smarandache curves are defined as regular curves whose position vectors are composed of Frenet frame vectors. This leads us

$TN$ -Smarandache curve is defined as  $\alpha_{TN}(s) = \frac{1}{\sqrt{2}} (T(s) + N(s))$ ,

$NB$ -Smarandache curve is defined as  $\alpha_{NB}(s) = \frac{1}{\sqrt{2}} (N(s) + B(s))$ ,

$TB$ -Smarandache curve is defined as  $\alpha_{TB}(s) = \frac{1}{\sqrt{2}} (T(s) + B(s))$ ,

and

$TNB$ -Smarandache curve is defined as  $\alpha_{TNB}(s) = \frac{1}{\sqrt{3}} (T(s) + N(s) + B(s))$ .

## 3. Surfaces with Constant Mean Curvature along Given Smarandache Curves

One of the special curves is the Smarandache curve, whose position vector is composed of Frenet frame vectors on another regular curve. Ahmad first presented a few unique Smarandache curves to the Euclidean space in [1]. Then, Değirmen et al. presented a few unique Smarandache curves to the Lie group in [3]. In this section, we will characterize the surfaces whose mean curvatures are constant in the three-dimensional Lie Group.

Consider  $\alpha(s)$  to be an arc-length parametrized curve on a surface  $P(s, t)$  in  $G$ . Then the curve  $\alpha$  is called an isoparametric curve if it is a parameter curve, that is, there exists a parameter  $t_0$  such that  $\alpha(s) = P(s, t_0)$ .

Since  $\alpha(s)$  is an isoparametric curve on this surface, there exists a parameter  $t = t_0 \in [0, T]$  such that  $\alpha(s) = P(s, t_0)$  that leads us

$$f(s, t_0) = g(s, t_0) = h(s, t_0) = 0 \text{ such that } s \in [0, L] \text{ and } t_0 \in [0, T] \tag{3}$$

Hence, if the  $TN$ ,  $NB$ , and  $TB$  Smarandache curves are isoparametric curves on this surface, then there exists a parameter  $t = t_0 \in [0, T]$  such that  $\alpha_{TN}(s) = P(s, t_0)$ ,  $\alpha_{NB}(s) = P(s, t_0)$ ,  $\alpha_{TB}(s) = P(s, t_0)$ , respectively.

$P(s, t)$  is defined based on the Smarandache curves of the curve  $\alpha(s)$  and using the Frenet frame of the curve in Lie group  $G$ , respectively, as follows

$$P(s, t) = \alpha_{TN}(s) + f(s, t)T(s) + g(s, t)N(s) + h(s, t)B(s) \tag{4}$$

$$P(s, t) = \alpha_{NB}(s) + f(s, t)T(s) + g(s, t)N(s) + h(s, t)B(s) \tag{5}$$

$$P(s, t) = \alpha_{TB}(s) + f(s, t)T(s) + g(s, t)N(s) + h(s, t)B(s) \tag{6}$$

Using the Formulation (2) to calculate mean curvatures, one can easily get the mean curvature of the surface provided in Equation (4) as follows:

$$H = \frac{p_0 p_1 + p_2 - p_3 p_4}{p_5 p_6} \tag{7}$$

where

$$p_0 = \frac{1}{\sqrt{2}}(f_t^2 + g_t^2 + h_t^2)$$

$$p_1 = \left( (\kappa_1 h_t - \tilde{h} \kappa_1 g_t) \left( -\frac{\partial}{\partial s} \kappa_1 - \kappa_1^2 \right) + (\kappa_1 h_t + \tilde{h} \kappa_1 f_t) \left( -\kappa_1^2 + \frac{\partial}{\partial s} \kappa_1 - (\tilde{h} \kappa_1)^2 \right) - (\kappa_1 g_t + \kappa_1 f_t) \left( \kappa_1 (\tilde{h} \kappa_1) - \frac{\partial}{\partial s} (\tilde{h} \kappa_1) \right) \right)$$

$$p_2 = \left( (\kappa_1 h_t - \tilde{h} \kappa_1 g_t) f_{tt} + (\kappa_1 h_t + \tilde{h} \kappa_1 f_t) g_{tt} - (\kappa_1 g_t + \kappa_1 f_t) h_{tt} \right) \left( \kappa_1^2 + \frac{(\tilde{h} \kappa_1)^2}{2} \right)$$

$$p_3 = 2 \left( (\kappa_1 h_t - \tilde{h} \kappa_1 g_t) (f_{ts} - g_t \kappa_1) + (\kappa_1 h_t + \tilde{h} \kappa_1 f_t) (f_t \kappa_1 + g_{ts} - h_t \tilde{h} \kappa_1) - (\kappa_1 g_t + \kappa_1 f_t) (g_t \tilde{h} \kappa_1 + h_{ts}) \right)$$

$$p_4 = \left( -\frac{1}{\sqrt{2}} \kappa_1 f_t + \frac{1}{\sqrt{2}} \kappa_1 g_t + \frac{1}{\sqrt{2}} \tilde{h} \kappa_1 h_t \right)$$

$$p_5 = 2 \left( \left( \kappa_1^2 + \frac{(\tilde{h} \kappa_1)^2}{2} \right) (f_t^2 + g_t^2 + h_t^2) - \frac{1}{2} (-\kappa_1 f_t + \kappa_1 g_t + \tilde{h} \kappa_1 h_t)^2 \right)$$

and

$$p_6 = \sqrt{(\kappa_1 h_t - \tilde{h} \kappa_1 g_t)^2 + (\kappa_1 h_t + \tilde{h} \kappa_1 f_t)^2 + (\kappa_1 g_t + \kappa_1 f_t)^2}$$

The mean curvature of the surface provided in Equation (5) is given as follows:

$$H = \frac{q_0 q_1 + q_2 - q_3 q_4}{q_5 q_6} \tag{8}$$

where

$$q_0 = (f_t^2 + g_t^2 + h_t^2)$$

$$q_1 = \left( (-\tilde{h}\kappa_1)(h_t + g_t) \left( -\frac{\partial}{\partial s}\kappa_1 + \kappa_1(\tilde{h}\kappa_1) \right) + (\kappa_1 h_t + \tilde{h}\kappa_1 f_t) \left( -\kappa_1^2 - \frac{\partial}{\partial s}(\tilde{h}\kappa_1) - (\tilde{h}\kappa_1)^2 \right) + (-\kappa_1 g_t + \tilde{h}\kappa_1 f_t) \left( -(\tilde{h}\kappa_1)^2 + \frac{\partial}{\partial s}(\tilde{h}\kappa_1) \right) \right)$$

$$q_2 = \left( (-\tilde{h}\kappa_1)(h_t + g_t) f_{tt} + (\kappa_1 h_t + \tilde{h}\kappa_1 f_t) g_{tt} + (-\kappa_1 g_t + \tilde{h}\kappa_1 f_t) h_{tt} \right) \left( \frac{\kappa_1^2}{2} + \frac{(\tilde{h}\kappa_1)^2}{2} \right)$$

$$q_3 = \frac{2}{\sqrt{2}} \left( (-\tilde{h}\kappa_1)(h_t + g_t) (f_{ts} - g_t \kappa_1) + (\kappa_1 h_t + \tilde{h}\kappa_1 f_t) (f_t \kappa_1 + g_{ts} - h_t \tilde{h}\kappa_1) + (-\kappa_1 g_t + \tilde{h}\kappa_1 f_t) (g_t \tilde{h}\kappa_1 + h_{ts}) \right)$$

$$q_4 = (-\kappa_1 f_t - (\tilde{h}\kappa_1)(h_t + g_t))$$

$$q_5 = \left( (\kappa_1^2 + (\tilde{h}\kappa_1)^2) (f_t^2 + g_t^2 + h_t^2) - (-\kappa_1 f_t - \tilde{h}\kappa_1 g_t + \tilde{h}\kappa_1 h_t)^2 \right)$$

and

$$q_6 = \sqrt{(-\tilde{h}\kappa_1)(g_t + h_t)^2 + (\kappa_1 h_t + \tilde{h}\kappa_1 f_t)^2 + (-\kappa_1 g_t + \tilde{h}\kappa_1 f_t)^2}$$

Then, the mean curvature of the surface provided in Equation (6) is given as follows:

$$H = \frac{r_0 r_1 - r_2}{r_3} \tag{9}$$

where

$$r_0 = (f_t^2 + g_t^2 + h_t^2)$$

$$r_1 = \left( (h_t) \left( -\frac{1}{\sqrt{2}}\kappa_1^2 + \frac{1}{\sqrt{2}}\kappa_1 \tilde{h}\kappa_1 \right) - (f_t) \left( \frac{1}{\sqrt{2}}\kappa_1 \tilde{h}\kappa_1 - \frac{1}{\sqrt{2}}(\tilde{h}\kappa_1)^2 \right) \right) + \frac{1}{2} (h_t f_{tt} - f_t h_{tt}) (\kappa_1 - \tilde{h}\kappa_1)^2$$

$$r_2 = 2 \left( \frac{1}{\sqrt{2}}(\kappa_1 - \tilde{h}\kappa_1) g_t \right) \left( (h_t) (f_{ts} - g_t \kappa_1) - (f_t) (g_t \tilde{h}\kappa_1 + h_{ts}) \right)$$

and

$$r_3 = \left( ((\kappa_1 - \tilde{h}\kappa_1)^2) (f_t^2 + g_t^2 + h_t^2) - ((\kappa_1 - \tilde{h}\kappa_1) g_t)^2 \right) \sqrt{(h_t^2 - f_t^2)}$$

Therefore, we can give the following main theorems for all the Smarandache curves of the curve  $\alpha(s)$ :

**Theorem 3.1.** Consider that the surface  $P(s, t)$  is determined by Equation (4). One of the following six conditions is satisfied if the mean curvature in Equation (7) along the isoparametric Smarandache curve  $TN$  of the curve  $\alpha(s)$  is constant:

**a)**  $f = g = h = f_t = f_{tt}(s, t_0) = 0$ ,  $g_t(s, t_0) = \text{constant} \neq 0$ ,  $h_t(s, t_0) = \text{constant} \neq 0$ ,  $\tilde{h}(s) = \text{constant}$ , and  $\kappa_1(s) = \text{constant}$ .

**b)**  $f = g = h = g_t = g_{tt}(s, t_0) = 0$ ,  $f_t(s, t_0) = \text{constant} \neq 0$ ,  $h_t(s, t_0) = \text{constant} \neq 0$ ,  $\tilde{h}(s) = \text{constant}$ , and  $\kappa_1(s) = \text{constant}$ .

**c)**  $f = g = h = h_t = h_{tt}(s, t_0) = 0$ ,  $f_t(s, t_0) = \text{constant} \neq 0$ ,  $g_t(s, t_0) = \text{constant} \neq 0$ ,  $\tilde{h}(s) = \text{constant}$ , and  $\kappa_1(s) = \text{constant}$ .

**d)**  $f = g = h = f_t = g_t = f_{tt} = g_{tt}(s, t_0) = 0$ ,  $h_t(s, t_0) \neq 0$ ,  $\tilde{h}(s) = \text{constant}$ , and  $\kappa_1(s) = \text{constant}$ .

**e)**  $f = g = h = f_t = h_t = f_{tt} = h_{tt}(s, t_0) = 0$ ,  $g_t(s, t_0) = \text{constant} \neq 0$ ,  $\tilde{h}(s) = \text{constant}$ , and  $\kappa_1(s) = \text{constant}$ .

f)  $f = g = h = h_t = g_t = h_{tt} = g_{tt}(s, t_0) = 0, f_t(s, t_0) \neq 0, \tilde{h}(s) = \text{constant},$  and  $\kappa_1(s) = \text{constant}.$

**Theorem 3.2.** Consider that the surface  $P(s, t)$  is determined by Equation (5). One of the following six conditions is satisfied if the mean curvature in Equation (8) along the isoparametric Smarandache curve  $NB$  of the curve  $\alpha(s)$  is constant:

a)  $f = g = h = f_t = f_{tt}(s, t_0) = 0, g_t(s, t_0) = \text{constant} \neq 0, h_t(s, t_0) = \text{constant} \neq 0, \tilde{h}(s) = \text{constant},$  and  $\kappa_1(s) = \text{constant}.$

b)  $f = g = h = g_t = g_{tt}(s, t_0) = 0, f_t(s, t_0) = \text{constant} \neq 0, h_t(s, t_0) = \text{constant} \neq 0, \tilde{h}(s) = \text{constant},$  and  $\kappa_1(s) = \text{constant}.$

c)  $f = g = h = h_t = h_{tt}(s, t_0) = 0, f_t(s, t_0) = \text{constant} \neq 0, g_t(s, t_0) = \text{constant} \neq 0, \tilde{h}(s) = \text{constant},$  and  $\kappa_1(s) = \text{constant}.$

d)  $f = g = h = f_t = g_t = f_{tt} = g_{tt}(s, t_0) = 0, h_t(s, t_0) \neq 0, \tilde{h}(s) = \text{constant},$  and  $\kappa_1(s) = \text{constant}.$

e)  $f = g = h = f_t = h_t = f_{tt} = h_{tt}(s, t_0) = 0, g_t(s, t_0) \neq 0, \tilde{h}(s) = \text{constant},$  and  $\kappa_1(s) = \text{constant}.$

f)  $f = g = h = g_t = h_t = g_{tt} = h_{tt}(s, t_0) = 0, f_t(s, t_0) \neq 0, \tilde{h}(s) = \text{constant},$  and  $\kappa_1(s) = \text{constant}.$

**Theorem 3.3.** Consider that the surface  $P(s, t)$  is determined by Equation (6). One of the following six conditions is satisfied if the mean curvature in Equation (9) along the isoparametric Smarandache curve  $TB$  of the curve  $\alpha(s)$  is constant:

a)  $f = g = h = f_t = f_{tt}(s, t_0) = 0, g_t(s, t_0) = \text{constant} \neq 0, h_t(s, t_0) = \text{constant} \neq 0, \tilde{h}(s) = \text{constant} \neq 1,$  and  $\kappa_1(s) = \text{constant}.$

b)  $f = g = h = g_t = g_{tt}(s, t_0) = 0, f_t(s, t_0) = \text{constant} \neq 0, h_t(s, t_0) = \text{constant} \neq 0, \tilde{h}(s) = \text{constant},$  and  $\kappa_1(s) = \text{constant}.$

c)  $f = g = h = h_t = h_{tt}(s, t_0) = 0, f_t(s, t_0) = \text{constant} \neq 0, g_t(s, t_0) = \text{constant} \neq 0, \tilde{h}(s) = \text{constant} \neq 1,$  and  $\kappa_1(s) = \text{constant}.$

d)  $f = g = h = f_t = g_t = f_{tt} = g_{tt}(s, t_0) = 0, h_t(s, t_0) \neq 0, \tilde{h}(s) = \text{constant} \neq 1,$  and  $\kappa_1(s) = \text{constant}.$

e)  $f = g = h = f_t = h_t = f_{tt} = h_{tt}(s, t_0) = 0, g_t(s, t_0) \neq 0, \tilde{h}(s) = \text{constant},$  and  $\kappa_1(s) = \text{constant}.$

f)  $f = g = h = g_t = h_t = h_{tt} = g_{tt}(s, t_0) = 0, f_t(s, t_0) \neq 0, \tilde{h}(s) = \text{constant} \neq 1,$  and  $\kappa_1(s) = \text{constant}.$

**Example 3.4.** Let  $\alpha(s)$  be a parametrized by  $\alpha(s) = \left(\frac{1}{\sqrt{2}} \cos s, \frac{1}{\sqrt{2}} \sin s, \frac{s}{\sqrt{2}}\right), 0 \leq s \leq 2\pi.$  Then, the Frenet vectors in the three dimensional Lie Group are given as

$$T(s) = \left(-\frac{1}{\sqrt{2}} \sin s, \frac{1}{\sqrt{2}} \cos s, \frac{1}{\sqrt{2}}\right)$$

$$N(s) = (-\cos s, -\sin s, 0)$$

and

$$B(s) = \frac{1}{\sqrt{2}}(\sin s, -\cos s, 1)$$

where  $\kappa_1 = \frac{1}{\sqrt{2}}, \bar{\kappa}_2 = 0, \kappa_2 = \frac{1}{\sqrt{2}},$  and  $\tilde{h}(s) = 1.$

Then, we can give the following cases:

**Case 1.** We can select  $f(s, t) = 0, g(s, t) = t^3, h(s, t) = s \sin t$ , and  $t_0 = 0$  while taking into account the (d) condition of Theorem 3.1. Consequently, the surface  $P_1(s, t)$  of the Lie group is provided by

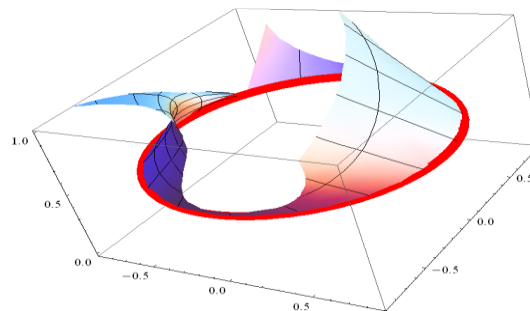
$$P_1(s, t) = \alpha_{TN}(s) + f(s, t)T(s) + g(s, t)N(s) + h(s, t)B(s)$$

$$P_1(s, t) = \frac{1}{\sqrt{2}}(T(s) + N(s)) + f(s, t)T(s) + g(s, t)N(s) + h(s, t)B(s)$$

and so

$$P_1(s, t) = \left( -\frac{\sin s}{2} - \frac{\cos s}{\sqrt{2}} - t^3 \cos s + \frac{1}{\sqrt{2}}s \sin t \sin s, \frac{\cos s}{2} - \frac{\sin s}{\sqrt{2}} - t^3 \sin s - s \sin t \frac{\cos s}{\sqrt{2}}, \frac{1}{2} + \frac{1}{\sqrt{2}}s \sin t \right)$$

which is plotted in Fig. 1, where  $0 \leq s \leq 2\pi$  and  $0 \leq t \leq 1$  with constant mean curvature  $H(s, t_0) = -\frac{1}{4}$ .



**Fig. 1.** The surface  $P_1(s, t)$  with constant mean curvature along the  $TN$  Smarandache curve of the curve  $\alpha(s)$

**Case 2.** We can select  $f(s, t) = e^{st}, g(s, t) = t^3, h(s, t) = 0$ , and  $t_0 = 0$  while taking into account the (f) condition of Theorem 3.2. Consequently, the surface  $P_2(s, t)$  of the Lie group is provided by

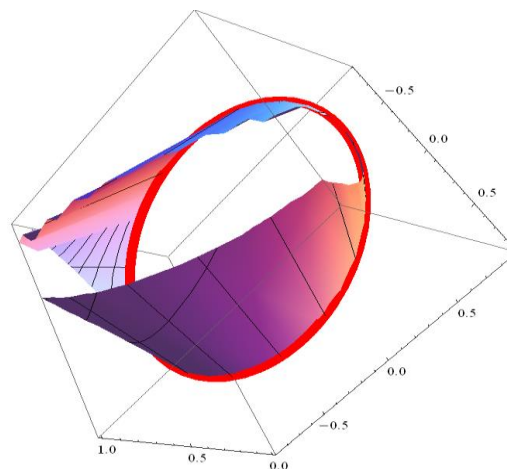
$$P_2(s, t) = \alpha_{NB}(s) + f(s, t)T(s) + g(s, t)N(s) + h(s, t)B(s)$$

$$P_2(s, t) = \frac{1}{\sqrt{2}}(N(s) + B(s)) + f(s, t)T(s) + g(s, t)N(s) + h(s, t)B(s)$$

and so

$$P_2(s, t) = \left( -\frac{1}{\sqrt{2}}\cos s + \frac{1}{2}\sin s - e^{st}\frac{1}{\sqrt{2}}\sin s - t^3 \cos s, -\frac{1}{\sqrt{2}}\sin s - \frac{1}{2}\cos s + e^{st}\frac{1}{\sqrt{2}}\cos s - t^3 \sin s, \frac{1}{2} + \frac{1}{\sqrt{2}}e^{st} \right)$$

which is plotted in Fig. 2, where  $0 \leq s \leq 2\pi$  and  $0 \leq t \leq 1$  with constant mean curvature  $H(s, t_0) = -\frac{1}{2}$ .



**Fig. 2.** The surface  $P_2(s, t)$  with constant mean curvature along the  $NB$  Smarandache curve of the curve  $\alpha(s)$

**Case 3.** We can select  $f(s, t) = st^3, g(s, t) = s \sin t, h(s, t) = 0$ , and  $t_0 = 0$  while taking into account the (e) condition of Theorem 3.3. Consequently, the surface  $P_3(s, t)$  of the Lie group is provided by

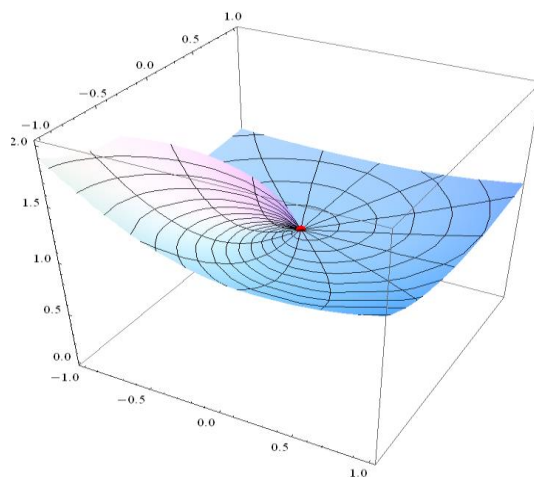
$$P_3(s, t) = \alpha_{TB}(s) + f(s, t)T(s) + g(s, t)N(s) + h(s, t)B(s)$$

$$P_3(s, t) = \frac{1}{\sqrt{2}}(T(s) + B(s)) + f(s, t)T(s) + g(s, t)N(s) + h(s, t)B(s)$$

and so

$$P_3(s, t) = \left( -st^3 \frac{1}{\sqrt{2}} \sin s - s \sin t \cos s, st^3 \frac{1}{\sqrt{2}} \cos s - s \sin t \sin s, 1 + \frac{1}{\sqrt{2}} st^3 \right)$$

which is plotted in Fig. 3, where  $0 \leq s \leq 2\pi$  and  $0 \leq t \leq 1$  with constant mean curvature  $H(s, t_0) = 0$ .



**Fig. 3.** The surface  $P_3(s, t)$  with constant mean curvature along the  $TB$  Smarandache curve of the curve  $\alpha(s)$

#### 4. Conclusion

In this study, we constructed surfaces along the given  $TN$ ,  $NB$ , and  $TB$  Smarandache curves of the curve  $\alpha(s)$ , and in each case, calculated the mean curvature of the given surfaces. Thus, sufficient conditions were derived to obtain surfaces with constant mean curvature along the  $TN$ ,  $NB$ , and  $TB$  Smarandache curves of the curve  $\alpha(s)$ , respectively. According to the given theorems, we constructed the surfaces  $P_i(s, t)$ , for  $1 \leq i \leq 3$  with constant mean curvature and illustrated them in Figs. 1-3 for the parameters  $0 \leq s \leq 2\pi$  and  $0 \leq t \leq 1$  by using Mathematica, respectively. In addition to the results shown in the manuscript, the work has also brought up a number of open questions for future studies, such as how to construct surfaces with constant Gauss curvatures along the given  $TN$ ,  $NB$ , and  $TB$  Smarandache curves.

#### Author Contributions

The first author provided the idea and wrote the first draft, and the second author made the calculations and wrote the first draft. They all read and approved the last version of the manuscript.

#### Conflict of Interest

All the authors declare no conflict of interest.

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