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Araştırma Makalesi - Research Article

Warped f-product Finsler Metrics

Bükülmüş f-Çarpımlı Finsler Metrikleri

Semail Ülgen[1*]

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ABSTRACT

This paper shows the existence of some Ricci-flat warped f-Product Finsler metrics. We investigate the general structure of this newly defined warped f-Product Finsler metrics, indeed we identify the metric form, spray coefficients of geodesics of the metric, and also the Ricci curvature in regards to the α_1 and α_2 Riemanian metrics.

Keywords-Warped f-Product Finsler Metrics, Ricci Curvature, Sprays, Riemanian Curvature

ÖZ

Bu makalede bazı Ricci-Düz bükülmüş f-Çarpımlı Finsler metriklerinin varlıkları gösterilmektedir. Bu yeni tanımlanan bükülmüş f-Çarpımlı Finsler metriklerinin genel yapısını araştırarak, metrik formunu, metriğin geodesiklerinin sprey katsayıları ve α_1 ve α_2 Riemanian metriklerine bağlı Ricci eğriliklerini belirledik.

Anahtar Kelimeler- Bükülmüş f-Çarpımlı Finsler Metrikleri, Ricci Eğrilikleri, Spreyler, Riemann Eğriliği

1*Corresponding Author Contact: sulgen@antalya.edu.tr (https://orcid.org/0000-0003-1381-1577)

Department of Industrial Engineering, Faculty of Engineering and Natural Sciences, Antalya Bilim University, Çıplaklı, Akdeniz Blv. No:290/A, 07190 Döşemealtı, Antalya, Turkey

I. INTRODUCTION

In geometry examples of semi-Riemannian manifolds with interesting curvature properties could often be obtained by using warped product structures, [1–3]. The warped product of (semi-)Riemannian manifolds is used in constructing new geometric models in theoretical physics in (semi-)Riemannian geometry. While, Robertson-Walker space-time, for example, is the relativistic model of the flow of perfect fluid, Schwarzchild geometry is the simplest relativistic model of universe with a single star. A model for the solar system could be better given by this than any Newtonian model. The simplest model for the black hole could also be given, [3].

In this paper we construct a newly defined warped product, namely the warped f-product of Finsler manifolds by the help of a warping function f and also the warped f-product Finsler metrics. We describe the geometry of the warped product Finsler manifold $\left(M \times_f N, F\right)$ by using the geometry on M and N, their Cartan torsion, and the properties of the warping function. We use this construction and we obtain a new warped product Finsler metric space as defined below.

Definition: Let (M_i, α_i) be arbitrary Riemannian manifolds for $i = 1, 2$, and let $M = M_1 \times M_2$ be the product of two manifolds M_1 and M_2. We assume that f is an arbitrary C^∞ warping function defined below

$$f : [0, \infty) \times [0, \infty) \longrightarrow [0, \infty)$$

satisfying the following three conditions:

$$i. \quad f(\lambda s, \lambda t) = \lambda f(s,t), \quad (\lambda > 0), \tag{1}$$

$$i. \quad f(s,t) \neq 0, \text{ if } (s,t) \neq 0, \tag{2}$$

$$iii. \; f(s,t) = s\phi(\rho), \;\; \rho = \frac{t}{s} \text{ for some } \phi. \tag{3}$$

For $x \in M$, and $y \in T_x M$, we define the warped f-product Finsler metric F,

$$F = \alpha_1(x_1, y_1) \sqrt{\phi\left(\varphi(x_1)^2 \left[\frac{\alpha_2(x_2, y_2)}{\alpha_1(x_1, y_1)}\right]^2\right)}$$

where $x = (x_1, x_2)$ $y = y_1 \oplus y_2$, and $T_x M = T_{(x_1, x_2)} [M_1 \times M_2] \cong T_{x_1} M_1 \oplus T_{x_2} M_2$.

Clearly, F has the following properties:

(a) $F(x,y) \geq 0$, and $F(x,y) = 0$ iff $y = 0$, (4)

(b) $F(x, \lambda y) = \lambda F(x,y)$, $\lambda > 0$, (5)

(c) $F(x,y)$ is C^∞ on $TM \setminus \{0\}$. (6)

In this paper in Lemma 3.1 we compute the positive definite warped f- product Finsler metric and its inverse and we find additional conditions on $f = f(s,t)$ under which the matrix $g_{ij} = \frac{1}{2} [F^2]_{y^i y^j}$ is positive definite. We take standard local coordinate systems (x^a, y^a) in TM_1 and $\left(x^\alpha, y^\beta\right)$ in TM_2. We express

$$\alpha_1(x_1, y_1) = \sqrt{\bar{g}_{ab}(x_1) y^a y^b}, \quad \text{and} \quad \alpha_2(x_2, y_2) = \sqrt{\bar{g}_{\alpha\beta}(x_2) y^\alpha y^\beta}, \text{ where } y_1 = y^a \frac{\partial}{\partial x^a}, \;\; y_2 = y^\alpha \frac{\partial}{\partial x^\alpha}$$

and $\rho = \frac{\varphi^2 \alpha_2^2}{\alpha_1^2}$ for a standard local coordinate system $(x^i, y^i) = \left(x^a, x^\alpha, y^a, y^\beta\right)$ in TM.

Then in Lemma 3.2 we characterize the geodesic coefficients of the spray of warped f-product Finsler metrics and in Theorem 1.1 we compute the Ricci curvature of the warped f- product Finsler metrics.

Riemannian metrics and Finsler metrics differ since Riemannian metrics are quadratic metrics, whereas Finsler metrics have no restriction on the quadratic property. Fortunately, one can naturally extend the Ricci curvature Ric in Riemannian geometry to Finsler geometry and study Finsler metrics $F = F(x,y)$ with isotropic Ricci curvature $Ric = Ric(x,y)$, also called Einstein metrics, i.e., $Ric = (n-1)\sigma F^2$, where σ is a scalar function in x on an n-dimensional manifold. There are Einstein metrics in a certain form that are Ricci-flat. It is still an open problem how to characterize warped f-product Finsler metrics as Einstein. Next theorem is useful in studying such an open problem. In this paper we first investigate the general structure of this newly defined warped f-product Finsler metrics.

Theorem 1.1 Let (M_i, α_i) be arbitrary Riemannian manifolds for $i = 1,2$, and $M = M_1 \times M_2$, product of two manifolds M_1 and M_2. A warped f-product Finsler metric F

$$F = \alpha_1(x_1, y_1)\sqrt{\phi\left(\varphi(x_1)^2\left[\frac{\alpha_2(x_2, y_2)}{\alpha_1(x_1, y_1)}\right]^2\right)}$$

is Ricci flat, but not flat, for $x = (x_1, x_2) \in M$, $y = y_1 \oplus y_2 \in T_x M$, if and only if Ric_1 and Ric_2 satisfy the following equations:

$$\begin{aligned}\mathbf{Ric}_1 = {} & 2(\rho A_\rho - A)[\alpha_1^2]_{x^a}[ln\varphi]_{x^d}\bar{g}^{ad} - 2sA[ln\varphi]_{x^a x^d}\bar{g}^{ad} - 2sA[ln\varphi]_{x^d}[\bar{g}^{ad}]_{x^a} \\ & + [ln\varphi]_{x^d x^a}\bar{g}^{ad}\left(4\rho B_\rho - 2(\rho A_\rho - A) - 2B(n_1 - 1)\right) \\ & + 4s[ln\varphi]_{x^d}[ln\varphi]_{x^a}\bar{g}^{ad}\{\rho A_\rho(C - 1 - A - 2B) + A(A + 2\rho B_\rho) - AB(n_1 - 1)\} \\ & - \rho\Pi\,(\Pi - s^{-1}M)\{\rho A_{\rho\rho} - 2\rho B_{\rho\rho} + (n_1 - 3)B_\rho\} - s^{-1}\rho N\Pi(A_{\rho\rho} - 2B_{\rho\rho}) \\ & + 2(A - 2B + 2C)\Pi^2\{\rho^2(A_{\rho\rho} - 2B_{\rho\rho}) + (n_1 - 1)\rho B_\rho\} \\ & + 2B\Pi^2\{(\rho A_\rho - A) - 2\rho B_\rho + (n_1 + 1)B\} \\ & - \Pi^2\{(\rho A_\rho - A)^2 - 4(\rho B_\rho - B)^2 + 2(\rho A_\rho - A)(2\rho B_\rho - B) - (n_1 - 1)B^2 \\ & - 2\rho^2 C_\rho(A_\rho - 2B_\rho) - 2\rho B_\rho C\},\end{aligned} \tag{7}$$

where $\mathbf{Ric}_1$ is the Ricci curvature of the Riemannian metric $\alpha_1(x_1, y_1)$, and

$$\begin{aligned}\mathbf{Ric}_2 = {} & [ln\varphi]_{x^a x^d}\, y^a y^d y^\alpha\left(2\rho\, C_\rho + C\right) \\ & + [ln\varphi]_{x^a}[ln\varphi]_{x^d}\bar{g}^{ad}4sy^\alpha\left(\rho\, A_\rho C - A(C - 2\rho\, C_\rho)\right) \\ & + y^\alpha\Pi^2\left\{\begin{matrix} -(3C_\rho + 2\rho C_{\rho\rho})(\rho(1 - (s\Pi)^{-1}M) + (s\Pi)^{-1}N) + 4\rho^2 C_{\rho\rho}\,(A - 2B + 2C) \\ -2\rho^2 C_\rho(A_\rho - 2B_\rho + 2C_\rho) + 2\rho\, C_\rho(3A - 4B + 4C) - 2\rho\, B_\rho C + C(2B - C)\end{matrix}\right\}\end{aligned} \tag{8}$$

where $\mathbf{Ric}_2$ is the Ricci curvature of the Riemannian metric $\alpha_2(x_2, y_2)$, and

$$A(\rho) = \frac{\rho\phi_\rho}{2(\phi - \rho\phi_\rho)}, \qquad B(\rho) = \frac{\rho^2\phi\phi_\rho\phi_{\rho\rho}}{2\Delta(\phi - \rho\phi_{\rho\rho})}, \qquad C(\rho) = \frac{\phi\phi_\rho - \rho\phi_\rho^2 + \rho\phi\phi_{\rho\rho}}{2\Delta},$$

$$\Delta(\rho) = \phi\phi_\rho - \rho{\phi_\rho}^2 + 2\rho\phi\phi_{\rho\rho}, \quad \rho = \frac{\varphi^2\alpha_2^2}{\alpha_1^2}.$$

$$\mathrm{M} = [{\alpha_1}^2]_{x^a}y^a, \qquad N = \varphi^2\,[{\alpha_2}^2]_{x^a}y^\alpha, \qquad \Pi = 2[\ln\varphi]_{x^a}y^a,$$

Example 1.1. [4] When we choose $f(s,t)$ as given below

$$f(s,t) = \frac{1}{1+\epsilon}\left\{s + t + \epsilon(s^k + t^k)^{\frac{1}{k}}\right\},$$

where ϵ is a nonnegative number and k is a positive integer, we get an example for the warped f-product Finsler metric.

II. PRELIMINARIES

A nonnegative scalar function $F = F(x, y)$ defined on the tangent bundle TM^n is a Finsler metric on a manifold M^n where x, a point in M^n and y, a point in $T_x M^n$, namely a tangent vector at x. The characterization of geodesics for a Finsler metric $F = F(x, y)$ in local coordinates are given by

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0,$$

where $g^{il} = g^{il}(x,y)$, $g_{ij} = g_{ij}(x,y)$, $F^2 = F^2(x,y)$, $g_{ij}(x,y) = (\frac{1}{2}F^2)_{y^iy^j}$, and

$$G^i = \frac{1}{4}g^{il}\{ [F^2]_{x^ky^l}\, y^k - [F^2]_{x^l} \}. \qquad (9)$$

The spray G of F is a vector field defined by using local functions G^i on TM^n as follows

$$G = y^i\frac{\partial}{\partial x^i} - 2G^i\frac{\partial}{\partial y^i}.$$

These local functions $G^i = G^i(x,y)$ are called **spray coefficients of F for the spray G**. For any $x \in M^n$ and $y \in T_x M^n \backslash\{0\}$, the Riemann curvature $R_y : T_x M^n \to T_x M^n$ is defined by $R_y(u) = R^i{}_k(x,y)\, u^k \frac{\partial}{\partial x^i}\Big|_x$, where

$$R^i{}_k = 2\frac{\partial G^i}{\partial x^k} - y^j\frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j\frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

The Ricci curvature is given by

$$Ric = 2\frac{\partial G^m}{\partial x^m} - y^j\frac{\partial^2 G^m}{\partial x^j \partial y^m} + 2G^j\frac{\partial^2 G^m}{\partial y^j \partial y^m} - \frac{\partial G^m}{\partial y^j}\frac{\partial G^j}{\partial y^m}.$$

Definition: Let (M_1, F_1) and (M_2, F_2) be given Finsler manifolds. Then a Finsler metric F on $M = M_1 \times M_2$ is called a **product Finsler metric of F_1 and F_2** if at any point $x = (x_1, x_2) \in M$, we have

$$F(x,y) = \begin{cases} F_1(x_1,y_1), & if\ y = y_1 \oplus 0 \ \in\ T_xM \\ F_2(x_2,y_2), & if\ y = 0 \oplus y_2 \ \in\ T_xM' \end{cases}$$

where $T_xM \cong T_{x_1} \oplus T_{x_2}$. In this case, (M,F) is called **a product Finsler manifold** of (M_1,F_1) and (M_2,F_2).

Unfortunately, there is no canonical way to define product Finsler metrics on the product manifold. In the case that the Finsler metrics are Riemannian, we define the product Finsler metrics in the following way below.

Definition: Let α_1 and α_2 be Euclidean norms on vector spaces V_1 and V_2, respectively and $V = V_1 \oplus V_2$. Let $f : [0,\infty) \times [0,\infty) \longrightarrow [0,\infty)$ be a C^∞ function satisfying

i. $f(\lambda s, \lambda t) = \lambda f(s,t)$, $(\lambda > 0)$
ii. $f(s,t) \neq 0$, if $(s,t) \neq 0$.

We define a function $F : V \longrightarrow [0,\infty)$ by

$$F(y) = \sqrt{f([\alpha_1(y_1)]^2, [\alpha_2(y_2)]^2)},$$

where $y = y_1 \oplus y_2 \in V_1 \oplus V_2$. $F = F(y)$ has the following properties:

(a) $F(y) \geq 0$ for any $y \in V$, and

$F(y) = 0$ if and only if $y = 0$;

(b) $F(\lambda y) = \lambda F(y)$, for any $y \in V$ and $\lambda > 0$;

(c) F is C^∞ on $V\backslash\{0\}$.

Let $n_i = \dim V_i$, for $i = 1,2$, $\dim V = n$, with $n = n_1 + n_2$, and the ranges of indices are given below,

$$1 \leq a,b,c \leq n_1, \qquad n_1 + 1 \leq \alpha,\beta,\gamma \leq n, \qquad 1 \leq i,j,k \leq n.$$

We let $\{\boldsymbol{b}_a\}, \{\boldsymbol{b}_\alpha\}$ and $\{\boldsymbol{b}_i\}$ be the bases for V_1, V_2 and V, respectively. We express

$$\alpha_1(y_1) = \sqrt{\bar{g}_{ab}y^ay^b}, \text{ and } \alpha_2(y_2) = \sqrt{\bar{g}_{\alpha\beta}y^\alpha y^\beta},$$

where $y_1 = y^a\boldsymbol{b}_a$, and $y_2 = y^\alpha\boldsymbol{b}_\alpha$, then $g_{ij} = \frac{1}{2}[F^2]_{y^iy^j}$ are given by

$$(g_{ij}) = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \qquad (10)$$

where

$G_{11} = 2f_{ss}\bar{y}_a\bar{y}_b + f_s\bar{g}_{ab}$,	$G_{21} = 2f_{st}\bar{y}_\alpha\bar{y}_\beta$,
$G_{12} = 2f_{st}\bar{y}_a\bar{y}_\beta$,	$G_{22} = 2f_{tt}\bar{y}_\alpha\bar{y}_\beta + f_t\bar{g}_{\alpha\beta}$,

and $\bar{y}_a = \bar{g}_{ab}y^b$, $\bar{y}_\alpha = \bar{g}_{\alpha\beta}y^\beta$. By the elementary argument, one can see that (g_{ij}) is positive definite if and only if $f(s,t)$ satisfies the following conditions i) and ii) given below,

i) $f_s, f_t > 0, f_s + 2sf_{ss} > 0, f_t + 2sf_{tt} > 0$ (11)

ii) $f_sf_t - 2\,f\,f_{st} > 0.$ (12)

We let $D_{ab} = det(\bar{g}_{ab}), D_{\alpha\beta} = det(\bar{g}_{\alpha\beta})$, then

$$det(g_{ij}) = h([\alpha_1]^2[\alpha_2]^2)D_{ab}\,D_{\alpha\beta}, \quad (13)$$

where $h = (f_s)^{n_1-1}(f_t)^{n_2-1}\{f_sf_t - 2\,f\,f_{st}\}$.

By using the above construction, for any given pair of Riemannian manifolds (M_1,α_1) and (M_2,α_2) one can construct uncountably many product Finsler metrics on $M = M_1 \times M_2$. We just take a function f as in (12) and we define

$$F(x,y) = \sqrt{f([\alpha_1(x_1,y_1)]^2,[\alpha_2(x_2,y_2)]^2)},$$

where $(x_1,x_2) \in M$ and $y_1 \oplus y_2 \in T_xM$. F is called a **reversible warped f- product Finsler metric** on M.

III. PROOF OF THEOREM 1.1

Let F be a given warped f- product Finsler metric on M as given below,

$$F = \alpha_1(x_1,y_1)\sqrt{\phi\left(\varphi(x_1)^2\left[\frac{\alpha_2(x_2,y_2)}{\alpha_1(x_1,y_1)}\right]^2\right)}$$

where each (M_i,α_i) is an arbitrary Riemannian manifold for $i = 1,2,$ and $M = M_1 \times M_2$, the product manifold, with an arbitrary C^∞ warping function f

$$f : [0,\infty) \times [0,\infty) \longrightarrow [0,\infty)$$

satisfying the conditions in (1), (2) and (3). Next, we find additional conditions on $f = f(s,t)$ under which the matrix given below,

$$g_{ij} = \frac{1}{2}[F^2]_{y^iy^j}$$

is positive-definite. We take standard local coordinate systems (x^a,y^a) in TM_1 and (x^α,y^β) in TM_2. We express

$$\alpha_1(x_1,y_1) = \sqrt{\bar{g}_{ab}(x_1)y^ay^b}, \quad \text{and} \quad \alpha_2(x_2,y_2) = \sqrt{\bar{g}_{\alpha\beta}(x_2)y^\alpha y^\beta},$$

where $y_1 = y^a\frac{\partial}{\partial x^a}$, $y_2 = y^\alpha\frac{\partial}{\partial x^\alpha}$ and $\rho = \frac{\varphi^2\alpha_2^2}{\alpha_1^2}$ for a standard local coordinate system

$(x^i,y^i) = (x^a,x^\alpha,y^a,y^\beta)$ in TM.

Lemma 3.1. Let (M_i,α_i) be arbitrary Riemannian manifolds for $i = 1,2,$ and $M = M_1 \times M_2$, a product manifold. For a warped $f-$ product Finsler metric F given below

$$F = \alpha_1(x_1,y_1)\sqrt{\phi\left(\varphi(x_1)^2\left[\frac{\alpha_2(x_2,y_2)}{\alpha_1(x_1,y_1)}\right]^2\right)}$$

where $x \in M, y \in T_xM$, and $x = (x_1,x_2), y = y_1 \oplus y_2$, we have

$$(g_{ij}) = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \quad (13)$$

$G_{11} = 2s^{-1}\rho^2\phi_{\rho\rho}\bar{y}_a\bar{y}_b + (\phi - \rho\phi_\rho)\bar{g}_{ab}$,	$G_{21} = -2s^{-1}\varphi^2\rho\phi_{\rho\rho}\bar{y}_\alpha\bar{y}_b$,
$G_{12} = -2s^{-1}\varphi^2\rho\phi_{\rho\rho}\bar{y}_a\bar{y}_\beta$,	$G_{22} = 2s^{-1}\varphi^4\phi_{\rho\rho}\bar{y}_\alpha\bar{y}_\beta + \varphi^2\phi_\rho\bar{g}_{\alpha\beta}$,

and

$$(g^{ij}) = \begin{bmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{bmatrix} \quad (14)$$

$G^{11} = \Delta^{-1}(A\bar{g}^{ab} + s^{-1}By^a y^b)$,	$G^{21} = \Delta^{-1}\left(s^{-1}Cy^\alpha y^\beta\right)$,
$G^{12} = \Delta^{-1}\left(s^{-1}Cy^a y^\beta\right)$,	$G^{22} = \Delta^{-1}\left(s^{-1}Ey^\alpha y^\beta + \varphi^{-2}D\bar{g}^{\alpha\beta}\right)$,

where $A = \frac{\phi\phi_\rho - \rho{\phi_\rho}^2 + 2\rho\phi\phi_{\rho\rho}}{\phi - \rho\phi_\rho}$, $B = -\frac{2\rho^2\phi_\rho\phi_{\rho\rho}}{\phi - \rho\phi_\rho}$,

$C = 2\rho\phi_{\rho\rho}$, $D = \frac{\phi\phi_\rho - \rho{\phi_\rho}^2 + 2\rho\phi\phi_{\rho\rho}}{\phi_\rho}$,

$E = -\frac{2\phi_{\rho\rho}(\phi - \rho\phi_\rho)}{\phi_\rho}$, $\Delta = \phi\phi_\rho - \rho{\phi_\rho}^2 + 2\rho\phi\phi_{\rho\rho}$.

Proof. Using $(g_{ij}) = \left(\frac{1}{2}F^2\right)_{y^i y^j}$ and the conditions in (1), (2) and (3), we easily obtain (13). Moreover, by using basic application of linear algebra and using the following, the inverse matrix form is given by

$$(g_{ij})^{-1} = (g^{ij}) = \begin{bmatrix} G^{11} & G^{12} \\ G^{21} & G^{22} \end{bmatrix}$$

$G^{11} = \Delta^{-1}(A\bar{g}^{ab} + s^{-1}By^a y^b)$	$G^{21} = \Delta^{-1}\left(s^{-1}Cy^\alpha y^\beta\right)$
$G^{12} = \Delta^{-1}(s^{-1}Cy^a y^\beta$	$G^{22} = \Delta^{-1}(s^{-1}Ey^\alpha y^\beta + \varphi^{-2}D\bar{g}^{\alpha\beta})$

and we also have the following equations we got while computing the inverse matrix,

$[a.]$ $A(\phi - \rho\phi_\rho) = \Delta$,

$[b.]$ $2A\rho^2\phi_{\rho\rho} + 2B\rho^2\phi_{\rho\rho} + B(\phi - \rho\phi_\rho) - 2C\rho^2\phi_{\rho\rho} = 0$,

$[c.]$ $-2A\rho\phi_{\rho\rho} - 2B\rho\phi_{\rho\rho} + 2C\rho\phi_{\rho\rho} + C\phi_\rho = 0$,

$[d.]$ $2C\rho^2\phi_{\rho\rho} + C(\phi - \rho\phi_\rho) - 2D\rho\phi_{\rho\rho} - 2E\rho^2\phi_{\rho\rho} = 0$,

$[e.]$ $-2C\rho\phi_{\rho\rho} + 2D\phi_{\rho\rho} + 2E\rho\phi_{\rho\rho} + E\phi_\rho = 0$,

$[f.]$ $D\phi_\rho = \Delta$.

The coefficients A, B, C D, E and Δ in $(g_{ij})^{-1}$ can be found in terms of ϕ by using the above equations.

Lemma 3.2. Let (M_i, α_i) be arbitrary Riemannian manifolds for $i = 1, 2,$ and $M = M_1 \times M_2$, a product manifold. For a warped $f-$ product Finsler metric F given below

$$F = \alpha_1(x_1, y_1)\sqrt{\phi\left(\varphi(x_1)^2\left[\frac{\alpha_2(x_2, y_2)}{\alpha_1(x_1, y_1)}\right]^2\right)}$$

where $x \in M, y \in T_x M$, and $x = (x_1, x_2), y = y_1 \oplus y_2$, the spray coefficients of $(M, F_1 \oplus F_2)$ are given as follows,

a) For (M_1, F_1), $1 \le a, b \le n_1$, $G^a = \bar{G}^a - sA[ln\varphi]_{x^b}\bar{g}^{ab} + B\Pi y^a$

$$A(\rho) = \frac{\rho\phi_\rho}{2(\phi - \rho\phi_\rho)}, \quad B(\rho) = \frac{\rho^2\phi\phi_\rho\phi_{\rho\rho}}{2\Delta(\phi - \rho\phi_{\rho\rho})}, \quad (15)$$

b) For (M_2, F_2), $n_1 + 1 \le \alpha, \beta \le n$, $G^\alpha = \bar{G}^\alpha + C\Pi y^\alpha$

$$C(\rho) = \frac{\phi\phi_\rho - \rho\phi_\rho^2 + \rho\phi\phi_{\rho\rho}}{2\Delta}. \quad (16)$$

Proof. By using (9), we have

$$a)\ G^a = \frac{1}{4}g^{aj}[F^2]_{x^m y^j}\, y^m - [F^2]_{x^j}, \quad j = b, \beta,\ 1 \le a, b \le n_1,\ n_1 + 1 \le \alpha, \beta \le \mathrm{n} \quad (17)$$

$$[F^2]_{x^m y^b}\, y^m \;-\; [F^2]_{x^b} = 2\{s^{-1}\rho^2\phi_{\rho\rho}M - s^{-1}\rho\phi_{\rho\rho}N - \rho^2\phi_{\rho\rho}\Pi\}\bar{y}_b + 4(\phi - \rho\phi_\rho)\bar{G}_b - 2s\rho\phi_\rho\,[\ln\varphi]_{x^b}$$

where $\bar{y}_b = \bar{g}_{ab}y^a$. By using (17) and Lemma 3.1, we obtain (15). By using (9), we have

$$b) \qquad G^\alpha = \tfrac{1}{4}g^{\alpha j}[F^2]_{x^m y^j}\, y^m - [F^2]_{x^j}, \quad j = b,\beta, \quad 1 \le a,b \le n_1, \; n_1 + 1 \le \alpha,\beta \le n. \tag{18}$$

$$[F^2]_{x^m y^\beta}\, y^m \;-\; [F^2]_{x^\beta} = 2\varphi^2\begin{Bmatrix} -s^{-1}\rho\phi_{\rho\rho}M - s^{-1}\phi_{\rho\rho}N \\ +\;(\rho\phi_{\rho\rho} + \phi_\rho)\Pi \end{Bmatrix}\bar{y}_\beta + 4\varphi^2\phi_\rho\bar{G}_\beta, \quad \text{where} \quad \bar{y}_\beta = \bar{g}_{\alpha\beta}y^\alpha.$$

By using (18) and Lemma 3.1, we obtain (16) where $M = [\alpha_1{}^2]_{x^a}y^a$, $N = \varphi^2\,[\alpha_2{}^2]_{x^\alpha}y^\alpha$, and $\Pi = 2[\ln\varphi]_{x^a}y^a$.

For any x ∈ M and y ∈ $T_x M \setminus\{0\}$, the Riemannian curvature $R_y : T_x M, \to T_x M$ is defined by

$R_y(u) = R^j_k(x,y)u^k \frac{\partial}{\partial x^i}|_x$, where

$$R^i_k = \bar{R}^i_k + 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^m y^k}y^m + 2G^m\frac{\partial^2 G^i}{\partial y^m \partial y^k} - \frac{\partial G^i}{\partial y^m}\frac{\partial G^m}{\partial y^k}$$

We construct the warped f-product Finsler space that is Ricci-flat, but not flat:

$$0 = \boldsymbol{Ric_1} + 2Q^a_{;a} - Q^a_{;m.a}y^m + 2Q^m Q^a_{.m.a} - Q^a_{.m}Q^m_{.a} \tag{19}$$

$$Q^a_{;a} = [ln\varphi]_{x^d}\bar{g}^{ad}(1 - 2s\rho A_\rho) - sA[ln\varphi]_{x^a x^d}\bar{g}^{ad} - sA[ln\varphi]_{x^d}[\bar{g}^{ad}]_{x^a} + \rho B_\rho \Pi(\Pi - s^{-1}M) + 2B[ln\varphi]_{x^d x^a}y^d y^a, \tag{20}$$

$$Q^a_{;m.a}y^m = 2[ln\varphi]_{x^c x^d}y^c y^d\{(\rho A_\rho - A) - 2(\rho B_\rho - B) + (n_1 - 1)B\} + \rho\Pi(\Pi - s^{-1}M)\{\rho(A_{\rho\rho} - 2B_{\rho\rho})(1 + s^{-1}\Pi N) + B_\rho(n_1 - 1)\} + s^{-1}B_\rho\Pi N(n_1 - 1) \tag{21}$$

$$Q^m Q^a_{.m.a} = -2sA[ln\varphi]_{x^a}[ln\varphi]_{x^d}\bar{g}^{ad}\{(\rho A_\rho - A) - 2(\rho B_\rho - B) + B(n_1 - 1)\} + \Pi^2(A - 2B + 2C)\{\rho^2(A_{\rho\rho} - 2B_{\rho\rho}) + (n_1 - 1)\rho B_\rho\} + B\Pi^2\{(\rho A_\rho - A) - 2(\rho B_\rho - B) + (n_1 - 1)B\}, \tag{22}$$

$$Q^a_{.m}Q^m_{.a} = 4s[ln\varphi]_{x^a}[ln\varphi]_{x^d}\bar{g}^{ad}\{2(\rho A_\rho - A)B - \rho A_\rho C\} + \Pi^2\{2(\rho A_\rho - A)(B - 2\rho B_\rho) + (\rho A_\rho - A)^2 + (n_1 - 1)B^2 + 2\rho^2 C_\rho(A_\rho - 2B_\rho) + 2\rho B_\rho C + 4(\rho B_\rho - B)^2\} \tag{23}$$

where $m = c,\beta$, $1 \le a,c,d \le n_1$, $n_1 + 1 \le \beta \le n$.

We plug (20)-(23) into (19), to obtain (7)

$$0 = \boldsymbol{Ric_2} + 2Q^\alpha_{;\alpha} - Q^\alpha_{;m.\alpha}y^m + 2Q^m Q^\alpha_{.m.\alpha} - Q^\alpha_{.m}Q^m_{.\alpha}, \tag{24}$$

$$Q^\alpha_{;\alpha} = s^{-1}C_\rho N\Pi, \tag{25}$$

$$Q^\alpha_{;m.\alpha}y^m = s^{-1}N\Pi\{(n_2 + 2)C_\rho + 2\rho C_{\rho\rho}\} + (4\rho C_\rho + 2n_2 C)[ln\varphi]_{x^d x^a}y^d y^a$$

$$+ \quad \rho\Pi(\Pi - s^{-1}M)\{(n_2+2)C_\rho + \quad 2\rho C_{\rho\rho}\}$$

$$+ \quad (n_2+2)s^{-1}\rho C_\rho M\Pi \tag{26}$$

$$Q^m Q^\alpha_{.m.\alpha} \;=\; (n_2+2)\rho C_\rho \Pi^2(A+2C+2B) \;+2\rho^2 C_{\rho\rho}\Pi^2(A+2C-2B)$$

$$+ \; B\big(2\rho C_\rho + \; n_2 C\big)\Pi^2$$

$$- \quad 2sA\big(2\rho C_\rho + n_2 C\big)[ln\varphi]_{x^d}[ln\varphi]_{x^a}\bar{g}^{ad}, \tag{27}$$

$$Q^\alpha_{.m}Q^m_{.\alpha} \;=\; 2\rho^2 C_\rho \Pi^2\big(A_\rho - 2B_\rho\big) \;+\; \big(2\rho C_\rho +\; C\big)^2 \,\Pi^2 \;+\; (n_2-1)C^2\Pi^2$$

$$- \; 4s\rho A_\rho C[ln\varphi]_{x^d}[ln\varphi]_{x^a}\bar{g}^{ad} + \quad 2\rho B_\rho C\Pi^2\,, \tag{28}$$

We plug (25)-(28) into (24), and we obtain (8).

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