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A Note on the Composition of a Positive Integer Whose Parts are Odd Integers

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Abstract. In this study, we interested in the compostions of integers. Then, the combinations of an integer whose each part is odd were examined.

$$O_n = \{(2a_1 + 1, ..., 2a_t + 1) : \ 2a_1 + 1 + ... + 2a_t + 1 = n \text{ and } \ a_i \text{ positive integer}\}.$$

and we call the set as an odd combination set O_n set of an integer n . Then, an action on the set are defined. Then, the decomposition of the composition sets of a positive integer has been examined by using set theory. Then, we also focused on the combination of an integer n whose sum is less than a fixed integer m. We have obtained the composition set of an integer whose largest part is less than m. Using these sets, we obtained recurrence relations.

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1. Introduction

Partition Theory is an important area of additive number theory, a subject concerning the representation of integers as sum of other integers. Recently, many researcher have written many publication on the theory of partition of a number because the rich history of partition of a number has been gone back to not only to very famous mathematicians Leonard Euler, but also Jacobi and also Indian mathematicians S. Ramanujan and English mathematicians G. H. Hardy [7, 8, 10, 13, 15, 17, 20].

Partition of a positive integer is expressing that number as the sum of positive integers. The number of these parts will be denoted by $p(n)$. For a positive integer n, the partition function to be studied is the number of ways n can be written as a sum of positive integers n.

Partition are divided into compositions and partitions. The commutative of the sums in partition is not important, while the non-commutative of the sums in composition is important.

Example 1.1. The number 4 has 5 partitions and the number of compositions is 8.

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The set of partition of 5 is

$$\{5; (4, 1); (3, 2); (3, 1, 1); (2, 2, 1); (2, 1, 1, 1); (1, 1, 1, 1)\}$$

and 5 has 7 partitions.

The set of composition of 5 is

$$\{5; (4, 1); (1, 4); (3, 2); (2, 3); (3, 1, 1); (1, 3, 1); (1, 1, 3); (2, 2, 1); (2, 1, 2); \\ (1, 2, 2); (2, 1, 1, 1); (1, 2, 1, 1); (1, 1, 2, 1); (1, 1, 1, 2); (1, 1, 1, 1)\}$$

and 5 has 16 compositions.

2. Partitions

There is a lot of information about partitions in the literature. The expressions that will be used in our study from this information are briefly summarized.

Euler investigated the generating function of the number of partitions of an integer n, as follows

$$f(x) = \prod_{n=1}^{\infty} \frac{1}{(1 - x^n)} = \sum_{n=0}^{\infty} p(n)x^n,$$

where $0 < x < 1$ [11].

After Euler's iteration for the partition of an integer, many mathematicians worked to obtain a more efficient version of the recurrence relation. The main purpose of the studies is to aim to reach the result by reducing the processes. In studies conducted in this area, J.A.Ewell, M. Merca, B. Al, and M. Alkan ([14, Theorem 1.2], [22, Theorem 1], [2, Theorem 2.4 and 2.5]) have obtained more effective recurrence relations.

As the studies on the partition theory progressed, new information was obtained by restricted partition. In the literature, the restricted partitions are substantial as unrestricted partition of an integer [18, 19, 21].

From [11, page 309], we recall that the number of partitions of k into parts not exceeding m is denoted by $p_m(k)$ for integers m, k. Then, $p_m(k) = p(k)$ for $m \geq k$. It is clear that $p_m(k)$ is less than $p(k)$ and the computation of $p_m(k)$ is simpler for integers m, k. The generating function for the number of partitions of k into parts not exceeding m is defined as

$$F_m(x) = \prod_{i=1}^{m} \frac{1}{1 - x^i} = 1 + \sum_{i=1}^{\infty} p_m(i)x^i.$$

In the literature, the restricted partitions are substantial as unrestricted partition of an integer [9, 23–25]. In [11], the generating function of the number of partitions of an integer n into odd part is

$$\prod_{n=1}^{\infty} \frac{1}{1 - x^{2n-1}} = \sum_{n=0}^{\infty} Q(n)x^n.$$

The generating function of the number of partitions of an integer n into distinct odd parts is

$$\prod_{n=1}^{\infty}(1 + x^{2n-1}) = \sum_{n=0}^{\infty} Z(n)x^n.$$

The integer 6 has the partition $p(6) = 11$. Since the number of partitions, which are all odd numbers (the ones with odd partition are written in bold), is 4, $Q(6) = 4$ and the odd and irregular partition is $1 + 5$, that is, $Z(6) = 1$. In order to see these three values together, if the number 6 is analyzed by breaking it down, it becomes

$$\begin{array}{l} 6 \\ \mathbf{1}+\mathbf{5} \\ 4+2 \\ 4+1+1 \\ \mathbf{3}+\mathbf{3} \\ 3+2+1 \\ \mathbf{3}+\mathbf{1}+\mathbf{1}+\mathbf{1} \\ 2+2+2 \\ 2+2+1+1 \\ 2+1+1+1+1 \\ \mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{1}. \end{array}$$

3. Compositions

While there is a lot of information about partitions in the literature, information about compositions is limited. One of these information is the theorem that Gupta proved.

Theorem 3.1 ([16]). *, The number of partition of a composition of a positive integer n is* 2^{n-1}.

Proof. The partitions of n can be classified under two heads: Those in which the first part is 1; and those in which the first part is > 1. Removing 1 (the first part), from each partition of n of the first kind, we obtain all the different partitions of $(n-1)$, each once. Reducing by 1 the first part in the partitions of the second kind, we again get all the partitions of $(n-1)$ as before. Hence, the number of partitions of n, is twice the number of partitions of $(n-1)$ and the result follows readily by induction. □

We focus on decompositions of the composition sets. We recall some expressions from [1].

Let n be a positive integer and we define the set

$$P_n = \{(a_1, a_2, ..., a_t) : \ a_1 + a_2 + ... + a_t = n, \quad a_i, t \in \mathbb{Z}^+\}.$$

In fact, the element of P_n is a composition of integer n and so P_n is the set of a composition of an integer n.

Now, we want to construct P_{n+1} by using P_n. First, we define operations with the a composition $a = (a_1, a_2, ..., a_t)$ of integer n;

$$(1 \odot a) = (1, a_1, a_2, ..., a_t),$$
$$(1 \oplus a) = (a_1 + 1, a_2, ..., a_t).$$

Then, $1 \oplus a, 1 \odot a \in P_{n+1}$ and sowe also use the notaions $1 \oplus P_n$, $1 \odot P_n$ for the set of new type elements, i,e;

$$1 \oplus P_n = \{1 \oplus a : a \in P_n\}$$
$$1 \odot P_n = \{1 \odot a : a \in P_n\}.$$

Example 3.2. The number 3 has 4 compositions.The set of composition of 3 is $\{3; (1,2); (2,1); (1,1,1)\}$.Thus

$$1 \oplus P_3 = \{4; (2,2); (3,1); (2,1,1)\},$$
$$1 \odot P_3 = \{(1,3); (1,1,2); (1,2,1); (1,1,1,1)\}.$$

Theorem 3.3 ([1]). *, For a positive integer n, we have*

$$P_{n+1} = (1 \oplus P_n) \cup (1 \odot P_n).$$

Example 3.4.

$$1 \oplus P_3 = \{4; (2,2); (3,1); (2,1,1)\},$$
$$1 \odot P_3 = \{(1,3); (1,1,2); (1,2,1); (1,1,1,1)\}.$$

Thus,

$$\begin{aligned}(1 \oplus P_3) \cup (1 \odot P_3) &= \{4; (2,2); (3,1); (2,1,1)\} \cup \{(1,3); (1,1,2); (1,2,1); (1,1,1,1)\} \\ &= P_4.\end{aligned}$$

We can give an alternative proof with Theorem 3.3 for the theorem that Gupta proved.

Theorem 3.5. *The number of a composition of an integer n $(n > 1)$ is 2^{n-1}.*

Proof. Let n be a positive integer. It is clear that, the number of element of both $(1 \oplus P_n)$ and $(1 \odot P_n)$ are equal, i.e. $(|1 \oplus P_n| = |1 \odot P_n|$ and so $|P_{n+1}| = 2\,|1 \odot P_n|$. Since $|P_2| = 2$, we have that $|P_{n+1}| = 2^n$ by induction metod [3]. □

In [5], we generated the formula that gives the composition number of n patches consisting of m parts.

Theorem 3.6. *Number of k decompositions of n positive integers, where k and n are positive integers*

$$P_k(n) = \frac{(n-1)!}{(n-k)!(k-1)!}.$$

By using the non commutative partition set of an integer n, we define the notation $\bar{a} = a_1.a_2.a_3...a_t$ for multiplication of summand where $n = a_1 + ... + a_t$. The sum of multiplication of summand in the non commutative partition set P_n define the function from the non commutative partition sets of integers to to positive integers defined by

$$T_n := T(P_n) = \sum_{a \in P_n} \bar{a}.$$

We may assume that, $T_0 = 1$ and $T_n = T(P_n)$ is called the multipliction sum of the non commutative partition set P_n (or the multipliction sum of the integer n) [3]. It is well known that the number element of the non commutative partition set P_n is 2^n and now our aims are to investigate both the number element of the non commutative partition subset of P_n and the multipliction sum of the non commutative partition subset of P_n. For new notions, we state an easy numeric example;

Example 3.7. For $n = 3$, we have $P_3 = \{(3), (1, 1, 1), (1, 2), (2, 1)\}$ and $T_3 = T(P_3) = 8$. Moreover, it follows

$$1 \odot P_3 = \{(1, 3), (1, 1, 1, 1), (1, 1, 2), (1, 2, 1)\}$$

$$1 \oplus P_3 = \{(4), (2, 1, 1), (2, 2), (3, 1)\}$$

and so $P_4 = (1 \odot P_3) \cup (1 \oplus P_3)$. Then, $T_4 = T(P_4) = 13$.

By Theorem 3.3, we obtain a recurance for the multipliction sum of the non commutative partition sets;

Theorem 3.8 ([6])**.** *For a positive integer n, we have*

$$T_{n+1} = T_n + \sum_{i=0}^{n} T_{n-i}. \tag{3.1}$$

Proof. For an element $a \in P_{n+1}$, there is $b = (b_1, b_2, ..., b_l) \in P_n$ such that either $\overline{a} = \overline{1 \odot b} = \overline{b}$ or $\overline{a} = \overline{1 \oplus b}$ and so $\overline{a} = \overline{1 \odot b} = \overline{b}$ or $\overline{a} = \overline{1 \oplus b} = (b_2....b_l) + \overline{b}$. Hence, we have that

$$T(1 \odot P_n) = \sum_{1 \odot b \in 1 \odot P_n} \overline{b} = T_n.$$

Moreover, it follows that

$$\begin{aligned}
T(1 \oplus P_n) &= \sum_{a \in P_n} (1 + a_1).a_2.a_3...a_t \\
&= \sum_{a \in P_n} (a_1.a_2.a_3...a_t) + \sum_{i=1}^{n} \sum_{(a_2.a_3...a_t) \in P_{n-i}} (a_2 a_3...a_t) \\
&= T_n + \sum_{i=1}^{n} T_{n-i} = \sum_{i=0}^{n} T_{n-i}.
\end{aligned}$$

Therefore, we have that

$$T_{n+1} = T(P_{n+1}) = T(1 \odot P_n) + T(1 \oplus P_n) = T_n + \sum_{i=0}^{n} T_{n-i}.$$

Hence, we have completed the proof. □

By using Equations (4.1), (3.1) and the induction method, we obtain the generating function for T_n.

Theorem 3.9 ([6]). *The generating function of T_n for a positive integer n is*
$$\sum_{n=0} T_n x^n = \frac{x}{1-3x+x^2.}$$

Proof. Let $h(x) = \sum_{n=1} T_n x^n$. Then,
$$\begin{aligned}
h(x) &= \sum_{n=1} T_n x^n = T_1 x + \sum_{n=2} T_n x^n \\
&= x + x \sum_{n=1} \left(T_n + \sum_{i=0}^{n} T_{n-i} \right) x^n \\
&= x + xh(x) + x \left(T_1 x + T_0 x + \sum_{n=2} \sum_{i=0}^{n} T_{n-i} x^n \right) \\
&= x + xh(x) + x \left(2x + \sum_{n=1} \sum_{i=0}^{n+1} T_{n+1-i} x^{n+1} \right) \\
&= x + xh(x) + 2x^2 + x \sum_{n=1} T_{n+1} x^{n+1} + x \sum_{n=1} \sum_{i=0}^{n} T_{n-i} x^{n+1} \\
&= x + xh(x) + 2x^2 + x \sum_{n=1} T_{n+1} x^{n+1} + x \sum_{n=1} \left(T_{n+1} - T_n \right) x^{n+1} \\
&= x + xh(x) + 2x^2 - x^2 h(x) + 2x \sum_{n=2} T_n x^n \\
&= x + xh(x) - x^2 h(x) + 2xh(x).
\end{aligned}$$

Thus,
$$h(x) = \frac{x}{1-3x+x^2}.$$

Hence, we have completed the proof. □

It is well known that the generating function for even Fibonacci number is $\frac{x}{1-3x+x^2}$.

Theorem 3.10. *For any positive integer, we have that $T_n = f_{2n}$.*

4. The Odd Combinations Set of An Integer

Now, we focus on the combinations of an integer whose each part is odd. We will examine odd compositions with set theory. Let us use the notion
$$O_n = \{(2a_1 + 1, ..., 2a_t + 1) : \; 2a_1 + 1 + ... + 2a_t + 1 = n \text{ and } \; a_i \text{ positive integer}\}$$
and we call the set as an odd combination set O_n set of an integer n. It is clear that, the even combination set of an even integer $2n$ involved to the combination set of an integer n and so the number of element of the even combination set of $2n$ is 2^{n-1}. Now, we try to decompose the odd combination set as union of subset of odd combinations set of integers.

Theorem 4.1 ([6]). *For a positive integer n, we get the decomposition of an odd combination of an integer n as a disjoint union of subset of odd combinations set of integers;*
$$O_{2n+1} = \{(2n+1)\} \cup \bigcup_{i=0}^{n-1} ((2i+1) \odot O_{2(n-i)})$$
$$O_{2n} = \bigcup_{i=0}^{n-1} ((2i+1) \odot O_{2(n-i)-1}) .$$

Proof. It is enough to show that, one inclusion for the odd number $n = 2k + 1$, where k an integer. Let $x = (2a_1 + 1, ..., 2a_t + 1)$ and assume that t is different from 1. Then, $n - 2a_1 - 1 = 2m$ for an integer even and so the element $b = (2a_2 + 1, ...2a_t + 1)$ is O_{2m}. Therefore, $x = (2a_1 + 1) \odot O_{2n-2a_2}$ and this complete the proof. □

Corollary 4.2. *The number k_n of element of the odd combination set of an integer is the nth Fibonacci number.*

Proof. By Theorem 4.1, it is easy to prove that $k_{n+1} = k_n + k_{n-1}$ and $k_0 = 0, k_1 = 1$. □

Using Theorem 4.1, it is easy to reprove the well known identities

$$f_{2n+1} = 1 + \sum_{i=1}^{n} f_{2i} \tag{4.1}$$

and

$$f_{2n} = \sum_{i=0}^{n-1} f_{2i+1}$$

for both the even and odd Fibonacci number.

In [6], we investigated the product sum of an odd combination of an integer n, i.e.,

$$o_n := \sum_{a \in O_n} \bar{a}.$$

One may compute the sequence as

$$\begin{aligned} &o_1 = 1, o_2 = 1, o_3 = 4, o_4 = 7,\ o_5 = 15, o_6 = 32, \\ &o_7 = 65, o_8 = 137, o_9 = 284, o_{10} = 591, o_{11} = 1231... \end{aligned}$$

Theorem 4.3 ([6]). *The generating function for the product sum of an odd composition sets is*

$$o(x) = 1 + x^2 (x+1) \frac{-2x + x^2 - 1}{x + 2x^2 + x^3 - x^4 - 1},$$

where $|x| < 1$.

Proof. [6], we have the recurrence relations for either an even or an odd term of the product sum of an odd composition of an integer, for an integer n,

$$o_{2n+3} = 3o_{2n} + 3o_{2n+1} - o_{2n-2} \tag{4.2}$$

$$o_{2n+2} = o_{2n+1} + 2o_{2n} + o_{2n-1} - o_{2n-2}. \tag{4.3}$$

Let $o(x) = \sum_{n=1}^{\infty} o_n x^n = 1 + \sum_{n=1}^{\infty} o_{2n} x^{2n} + \sum_{n=1}^{\infty} o_{2n+1} x^{2n+1}$ be the generating function for the product sum of an odd composition of integers and so it is enough to investigate

$$A(x) = \sum_{n=1} o_{2n} x^{2n}$$

$$B(x) = \sum_{n=1}^{\infty} o_{2n+1} x^{2n+1}.$$

By using the recurrence identity (4.2), it is easy to compute that

$$(1 - 3x^2)B(x) = x^3(3 - x^2)A(x) + 4x^3. \tag{4.4}$$

Similarly, it is also easy to compute

$$A(x) = \frac{x\left(x^2+1\right)}{(x^2-1)^2} B + \frac{x^2\left(x^2+1\right)}{(x^2-1)^2}, \tag{4.5}$$

due to the recurrence identity (4.3). Then, combining the equations (4.4) and (4.5), we figure out both A and B and so it follows that

$$B(x) = -x^3 \frac{5x^2 - 6x^4 + x^6 - 4}{(x + 2x^2 + x^3 - x^4 - 1)(x - 2x^2 + x^3 + x^4 + 1)},$$

$$A(x) = x^2 \frac{\left(x^2+1\right)^2}{(x - 2x^2 + x^3 + x^4 + 1)(-x - 2x^2 - x^3 + x^4 + 1)}.$$

Therefore, we investigate the generating function

$$o(x) = 1 + x^2 (x+1) \frac{-2x + x^2 - 1}{x + 2x^2 + x^3 - x^4 - 1}.$$

□

Now, we focus on the composition of an integer n whose summands less than the fix integer $m = 2t + 1$. We recall some expressions from [4]. For positive integer n,t; we use the notation the subset $O_{n,2t+1}$ of O_n for the biggest summand $2t + 1$ of odd composition of n, i.e.,

$$O_{n,m} = \left\{(2a_1 + 1, 2a_2 + 1, ..., 2a_k + 1) : 2(a_1 + ... + a_k) + k = n,\ a_i \leq m \text{ for all } i, \text{ and } k \in \mathbb{Z}^+\right\}.$$

When $n \leq m$, it is clear that $O_{n,m} = O_n$. And again, if $m = 2l$ by definition, $O_{n,m} = O_{n,2t}$.

We will now define a new notation to derive the recurrence relation, which gives parts of the odd compositions of the positive integer n, the largest summand of which is m.

First, we define notation with the a composition $b = (2b_1 + 1, 2b_2 + 1, ..., 2b_k + 1)$ of integer n, i;

$$(i \odot b) = (i, 2b_1 + 1, 2b_2 + 1, ..., 2b_k + 1),$$

Then, $i \odot b \in O_{n+i}$ and sowe also use the notaion $i \odot O_n$ for the set of new type elements, i,e;

$$i \odot O_n = \{i \odot b : b \in O_n\}.$$

Example 4.4. The set of composition of 5 is

$$\{5; (4,1); (1,4); (2,3); (3,2); (3,1,1); (1,3,1); (1,1,3); (2,2,1); (2,1,2);$$
$$(1,2,2); (2,1,1,1); (1,1,1,2); (1,2,1,1); (1,1,2,1); (1,1,1,1,1)\}$$

and 5 has 16 compositions. The set of odd composition of 5 is

$$\{5; (3,1,1); (1,3,1); (1,1,3); (1,1,1,1,1)\}.$$

The set of odd compositions of the

$$O_{5,3} = \{5; (3,1,1); (1,3,1); (1,1,3); (1,1,1,1,1)\}.$$

Lemma 4.5 ([6]). *For positive integers t, n,*

$$O_{n,m} = \bigcup_{i=0}^{t} ((2i+1) \odot O_{n-2i-1,m}).$$

Proof. Because of the definition of $O_{n,m}$, summands of the compositions in the set must be odd positive integers. When obtaining $O_{n,m}$, the largest summand must be m. The sum of the remaining summands of the composition whose first summand is m must be $n-m$. And again, since these summands will be at largest m, the expression becomes $m \odot O_{n-m,m}$. The proof is completed. □

Example 4.6. For $n = 7$ and $t = 2$,

$$O_{7,5} = \bigcup_{i=0}^{2} ((2i+1) \odot O_{6-2i,5}) = (1 \odot O_{6,5}) \cup (3 \odot O_{4,5}) \cup (5 \odot O_{2,5})$$

considering the

$$O_{6,5} = \{(5,1); (3,3); (3,1,1,1); (1,5); (1,3,1,1); (1,1,3,1); (1,1,1,3), (1,1,1,1,1,1)\},$$

$$O_{4,5} = O_4 = \{(3,1); (1,3); (1,1,1,1)\}$$

and

$$O_{2,5} = O_2 = \{(1,1)\}$$

sets results in

$$O_{7,5} = \left\{ \begin{array}{c} (5,1,1); (3,3,1); (3,1,3); (3,1,1,1,1); (1,5,1); (1,1,5); (1,3,3); \\ (1,3,1,1,1), (1,1,3,1,1); (1,1,1,3,1); (1,1,1,1,3); (1,1,1,1,1,1,1) \end{array} \right\}.$$

Now, we investigate the product sum of an odd compositions of the positive integer n, the largest summand of which is m, i.e.,

$$o_{n,m} := \sum_{\substack{a_1+a_2+\ldots+a_k=n \\ a_i\leq m}} a_1.a_2.....a_k,$$

where $i, m, k \in \mathbb{Z}^+$. We may assume that, $o_{n,m} = 0$ for non-positive integers n. It is clear that, $o_{n,m} = o_n$ when $n \leq m$.

Example 4.7. For $n = 4$ and $m = 5$,

$$o_{4,5} = 3.1 + 1.3 + 1.1.1 = 7.$$

Theorem 4.8. *For an intergers n, t with $1 < n$, we have that*

$$o_{n,m} = \sum_{i=0}^{t}(2i+1)o_{n-2i-1,m}. \tag{4.6}$$

$$o_{n,m} = o_{n-2,m} + o_{n-1,m} - m.o_{n-2t-3,m} + 2\sum_{i=1}^{t} o_{n-2i-1,m}. \tag{4.7}$$

Proof. For the equation (4.6) is proof,

$$O_{n,m} = \bigcup_{i=0}^{t}\left((2i+1)\odot O_{n-2i-1,m}\right).$$

Then, we obtained from the sum of products function

$$\begin{aligned}
o_{n,2t+1} &= \sum_{i=0}^{t}\sum_{b\in O_{n-2i-1,2t+1}}(2i+1).\overbrace{a_1.a_2.....a_k}^{\bar{b}} \\
&= 2\sum_{i=0}^{t} i \sum_{b\in O_{n-2i-1,2t+1}} \bar{b} + \sum_{i=0}^{t}\sum_{b\in O_{n-2i-1,2t+1}} \bar{b} \\
&= 2\sum_{i=0}^{t} i.o_{n-2i-1,2t+1} + \sum_{i=0}^{t} o_{n-2i-1,2t+1} \\
&= \sum_{i=0}^{t}(2i+1).o_{n-2i-1,2t+1}.
\end{aligned}$$

Thus, the proof of equation (4.6) is completed. For equation (4.7) is proof;

$$\begin{aligned}
o_{n-2,2t+1} &= \sum_{i=0}^{t}\left((2i+1).o_{n-2i-3,2t+1}\right) \\
&= \sum_{i=1}^{t+1}\left((2i-1).o_{n-2i-1,2t+1}\right) \\
&= \sum_{i=1}^{t+1} 2i.o_{n-2i-1,2t+1} - \sum_{i=1}^{t+1} o_{n-2i-1,2t+1}.
\end{aligned}$$

After some elementary calculations,

$$\begin{aligned}
o_{n-2,2t+1} &= \sum_{i=1}^{t+1} 2i.o_{n-2i-1,2t+1} + \sum_{i=1}^{t+1} o_{n-2i-1,2t+1} - \sum_{i=1}^{t+1} o_{n-2i-1,2t+1} - \sum_{i=1}^{t+1} o_{n-2i-1,2t+1} \\
&= \sum_{i=0}^{t}(2i+1).o_{n-2i-1,2t+1} + (2t+3).o_{n-2t-3,2t+1} - o_{n-1,2t+1} - 2\sum_{i=1}^{t+1} o_{n-2i-1,2t+1} \\
&= o_{n,2t+1} + 2t.o_{n-2t-3,2t+1} + 3o_{n-2t-3,2t+1} - o_{n-1,2t+1} - 2o_{n-2t-3,2t+1} - 2\sum_{i=1}^{t} o_{n-2i-1,2t+1} \\
&= o_{n,2t+1} - o_{n-1,2t+1} + (2t+1).o_{n-2t-3,2t+1} - 2\sum_{i=1}^{t} o_{n-2i-1,2t+1}.
\end{aligned}$$

Then,

$$o_{n,2t+1} = o_{n-2,2t+1} + o_{n-1,2t+1} - (2t+1).o_{n-2t-3,2t+1} + 2\sum_{i=0}^{t} o_{n-2i-1,2t+1}.$$

□

Example 4.9. Because of the equation (4.5) for $n = 7$ and $t = 2$, the following was obtained:

$$\begin{aligned}
O_{7,5} &= \bigcup_{i=0}^{2}((2i+1)\odot O_{6-2i,5}) = (1\odot O_{6,5}) \cup (3\odot O_{4,5}) \cup (5\odot O_{2,5}), \\
O_{6,3} &= \bigcup_{i=0}^{1}((2i+1)\odot O_{5-2i,3}) = (1\odot O_{5,3}) \cup (3\odot O_{3,3}), \\
O_{7,3} &= \bigcup_{i=0}^{1}((2i+1)\odot O_{6-2i,3}) = (1\odot O_{6,3}) \cup (3\odot O_{4,3}).
\end{aligned}$$

From equation (4.6), $o_{7,5} = 58$, $o_{6,3} = 22$ and $o_{7,3} = 43$ are obtained.
To compute $o_{n,m}$, we use equation (4.7), in the following. For $n = 7$ and $t = 2$,

$$\begin{aligned}
o_{7,5} &= o_{6,5} + o_{5,5} - 5.o_{2,5} + 2(o_{4,5} + o_{2,5}) \\
&= 32 + 15 - 5.1 + 2.8 \\
&= 58.
\end{aligned}$$

For $n = 6$ and $t = 1$,

$$\begin{aligned}
o_{6,3} &= o_{4,3} + o_{5,3} - 3.o_{1,3} + 2o_{3,3} \\
&= 7 + 10 - 3.1 + 2.4 \\
&= 22.
\end{aligned}$$

For $n = 7$ and $t = 1$,

$$\begin{aligned}
o_{7,3} &= o_{6,3} + o_{5,3} - 3.o_{2,3} + 2o_{4,3} \\
&= 22 + 10 - 3.1 + 2.7 \\
&= 43.
\end{aligned}$$

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All authors have contributed sufficiently in the planning, execution, or analysis of this study to be included as authors. All authors have read and agreed to the published version of the manuscript.

Conflicts of Interest

All authors jointly worked on the results and they have read and agreed to the published version of the manuscript.

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