Turk. J. Math. Comput. Sci.
15(2)(2023) 237–246
© MatDer
DOI : 10.47000/tjmcs.1166651

New Results on a Partial Differential Equation with General Piecewise Constant Argument

Marat Akhmet[1], Duygu Aruğaslan Çinçin[2], Zekeriya Özkan[3,*]

[1]Department of Mathematics, Faculty of Arts and Sciences, Middle East Technical University, 06531, Ankara, Turkey.
[2]Department of Mathematics, Faculty of Arts and Sciences, Süleyman Demirel University, 32260, Isparta, Turkey.
[3]Department of Ortaköy Vocational School, Aksaray University, 68400, Aksaray, Turkey.

Received: 25-08-2022 • Accepted: 14-08-2023

Abstract. There have been very few analyses on partial differential equations with piecewise constant arguments and as far as we know, there is no study conducted on heat equation with piecewise constant argument of generalized type. Motivated by this fact, this study aims to solve and analyse heat equation with piecewise constant argument of generalized type. We obtain formal solution of heat equation with piecewise constant argument of generalized type by separation of variables. We apply the Laplace transform method using unit step function and method of steps on each consecutive intervals. We investigate stability, oscillation, boundedness properties of solutions.

2020 AMS Classification: 35K05, 35K20, 35B05

Keywords: Partial differential equation, piecewise constant argument, Laplace transformation, heat equation, zeros of solutions, oscillation.

1. Introduction and Preliminaries

In the literature, there are many mathematical models to examine real life problems using differential equations. Most of these models include only the current states of the processes, but, in some cases real problems cannot be expressed by these models with a realistic approach since current and future states can significantly be influenced by the past states. A systematical study on mathematical models with piecewise constant arguments is established in [16]. Differential equations with piecewise constant arguments can be stated as hybrid dynamical systems [19], since they are very closely related to difference and differential equations. Since the early 1980s, differential equations with piecewise constant arguments have attracted great deal of attention of researchers in science such as mathematics, physics, biology, engineering, economics, health and other fields [8, 11, 25]. Oscillation, periodicity and convergence of solutions of ordinary differential equations with piecewise constant arguments have been studied in [1–7, 9, 10, 12–14, 21–24, 26, 28, 40].
However, there are very few studies for partial differential equations with piecewise constant arguments [15, 17, 18, 27, 29–39]. The first basic work [33] was published in 1991. It has been shown that partial differential equations with piecewise constant time naturally occur in the approximating process of partial differential equations by using

*Corresponding Author
Email addresses: marat@metu.edu.tr (M. Akhmet), duyguarugaslan@sdu.edu.tr (D. Aruğaslan Çinçin), zekeriyaozkan@gmail.com (Z. Özkan)

piecewise constant arguments. Heat flow in a rod with both diffusion $a^2u_{xx}(x,t)$ along the rod and heat loss/gain across the lateral sides of the rod is described by the equation

$$u_t = a^2u_{xx} - b(u - u_0).$$

Heat loss (for $b < 0$) or heat gain (for $b > 0$) is proportional to the difference between the temperature $u(x,t)$ of the rod and u_0 of the surrounding medium. For example, in the chemistry u may stand for concentration, the above equation says that the rate of change u_t of the substance is due both to the diffusion a^2u_{xx} (in the x-direction) and to the fact that the substance is being created ($b < 0$) or destroyed ($b > 0$) by a chemical reaction proportional to the difference between two concentrations u and u_0 [20]. One can change $u - u_0$ by u and consider the equation

$$u_t(x,t) = a^2u_{xx}(x,t) - bu(x,t).$$

Thus, lateral heat change (or substance change due to a chemical reaction) measurement at discrete moments of time leads to a partial differential equation with piecewise constant argument given by

$$u_t(x,t) = a^2u_{xx}(x,t) - bu(x,[t/h]h),$$

for $t \in [nh,(n+1)h]$, $n = 0,1,2,...,$ where h is a positive constant and $[\cdot]$ stands for the greatest integer function [33].

The existence of solutions of partial differential equations with piecewise constant arguments and qualitative properties of solutions of the problems such as stability, instability, oscillation, convergence, boundedness, unboundness and periodicity were investigated in [15, 17, 18, 27, 29–39]. One of the useful sources considering both ordinary and partial differential equations with piecewise constant arguments is Wiener's book [34].

It is well known that if, in an insulated rod of length L, the temperature flows from $x = 0$ to $x = L$ provided that the heat energy is neither created nor destroyed in the interior of the rod, then the temperature satisfies the heat equation. Such an equation becomes more meaningful but more complicated when the diffusion term depends on the present time and also on previous times. Moreover, there is not much work conducted on partial differential equations with piecewise constant arguments.

As far as we know, there is no study concerning a heat equation with piecewise constant arguments of generalized type through the Laplace transform. With this motivation, in this study, we consider the following initial boundary value problem

$$\frac{\partial u(x,t)}{\partial t} = a^2\frac{\partial^2 u(x,t)}{\partial x^2} - bu(x,\beta(t)), \qquad 0 \le x \le 1, \quad \theta_0 \le t < \infty, \tag{1.2}$$

with boundary conditions

$$u(0,t) = 0, \quad u(1,t) = 0, \quad \theta_0 \le t < \infty, \tag{1.3}$$

and initial condition

$$u(x,\theta_0) = u_0(x), \quad 0 \le x \le 1. \tag{1.4}$$

Here, a and b are nonzero real parameters, $u : G = [0,1] \times [\theta_0,\infty) \to (-\infty,\infty)$, $\beta(t)$ represents the generalized type piecewise constant function such that for $\theta_i \le t < \theta_{i+1}$, $i = 0,1,2,\cdots$, $\beta(t) = \theta_i$, $|\theta_i| \to \infty$ as $i \to \infty$ and $u_0(x)$ is a continuous function on $[0,1]$.

From now on without loss of generality we will assume that $\theta_0 = 0$ and there are two positive numbers $\underline{\theta}$ and $\overline{\theta}$ such that $\underline{\theta} \le \theta_{i+1} - \theta_i \le \overline{\theta}$, $i = 0,1,2,\cdots$.

This study is organized as follows. In Section 2, solution of the initial boundary value problem (1.2)-(1.4) is obtained, and also, the oscillatory and nonoscillatory conditions of the problem are investigated. In Section 3, we give 3 graphs of the solutions of the problem (1.2)-(1.4) with respect to various coefficients. In Section 4, concluding remarks and some open problems are given for further study.

2. Heat Equation with Piecewise Constant Argument of Generalized Type

Before starting to solve the problem (1.2)-(1.4), let us define the properties of the solution $u(x,t)$ in G [37].

Definition 2.1. A function $u(x,t)$ is said to be a solution of the initial boundary value problem (1.2)-(1.4) in G if it satisfies the following three conditions:

(i) $u(x,t)$ is continuous in G,

(ii) u_t and u_{xx} exist and are continuous in G, there may be exceptional points (x,θ_i), $i = 0,1,2,\cdots$, where one-sided derivatives exist with respect to second argument,

(iii) $u(x,t)$ satisfies Eq. (1.2) in G, with the possible exception of points (x,θ_i), $i = 0, 1, 2, \cdots$, and conditions (1.3), (1.4).

We can write $x(\beta(t))$ as a piecewise-defined function as follows

$$x(\beta(t)) = \begin{cases} x(0) = x_0 & \text{if} \quad \theta_0 = 0 \leq t < \theta_1, \\ x(\theta_1) & \text{if} \quad \theta_1 \leq t < \theta_2, \\ \quad\vdots & \\ x(\theta_n) & \text{if} \quad \theta_n \leq t < \theta_{n+1}, \\ \quad\vdots & \end{cases}.$$

Using this piecewise-defined function, we can write the following equality

$$x(\beta(t)) = x_0u_0(t) + \Big(x(\theta_1) - x_0\Big)u_{\theta_1}(t) + \Big(x(\theta_2) - x(\theta_1)\Big)u_{\theta_2}(t) + \cdots + \Big(x(\theta_{n+1}) - x(\theta_n)\Big)u_{\theta_{n+1}}(t) + \cdots,$$

where the unit step function $u_n(t)$ is defined in the following way

$$u_n(t) = \begin{cases} 0 & \text{if} \quad t < n, \\ 1 & \text{if} \quad t \geq n. \end{cases}$$

Now, we can rewrite $x(\beta(t))$, using series, as follows

$$x(\beta(t)) = x(0) + \sum_{n=0}^{\infty} \Big(x(\theta_{n+1}) - x(\theta_n)\Big)u_{\theta_{n+1}}(t). \tag{2.1}$$

Let us start to seek the solution of problem (1.2)-(1.4) in the form

$$u(x,t) = X(x)T(t). \tag{2.2}$$

Now taking partial derivatives of (2.2), we have

$$\frac{\partial u(x,t)}{\partial t} = X(x)T'(t)$$

and

$$\frac{\partial^2 u(x,t)}{\partial x^2} = X''(x)T(t).$$

Substituting these into Eq. (1.2), we have

$$X(x)T'(t) = a^2X''(x)T(t) - bX(x)T(\beta(t)).$$

Rearranging the last equation, we get

$$\frac{T'(t) + bT(\beta(t))}{a^2T(t)} = \frac{X''(x)}{X(x)} = -\lambda^2,$$

where λ is a real constant. Then, evaluating (2.2) at the boundary conditions, we have

$$X(0) = 0, \; X(1) = 0.$$

Then, separation of variables gives a boundary value problem

$$\begin{cases} X''(x) + \lambda^2X(x) = 0, \\ X(0) = X(1) = 0, \end{cases}$$

whose orthonormal set of solutions is given by

$$X_j(x) = \sqrt{2}\sin(\pi jx), \; j = 1, 2, \cdots,$$

on $[0, 1]$, and moreover, it gives the following differential equation with piecewise constant argument of generalized type

$$T_j'(t) + a^2\pi^2j^2T_j(t) = -bT_j(\beta(t)). \tag{2.3}$$

Now, using the series definition given in (2.1) for $T_j(\beta(t))$, we can rewrite Eq. (2.3) as

$$T_j'(t) + a^2\pi^2j^2T_j(t) = -bT_j(0) - b\sum_{n=0}^{\infty}\Big(T_j(\theta_{n+1}) - T_j(\theta_n)\Big)u_{\theta_{n+1}}(t). \tag{2.4}$$

Then, after applying Laplace transform to Eq. (2.4), we get for $s > 0$,

$$s\mathcal{L}\{T_j(t)\} - T_j(0) + a^2\pi^2 j^2 \mathcal{L}\{T_j(t)\} = \frac{-bT_j(0)}{s} - b\sum_{n=0}^{\infty}\Big(T_j(\theta_{n+1}) - T_j(\theta_n)\Big)\frac{e^{-(\theta_{n+1})s}}{s}.$$

If we solve this equation for $\mathcal{L}\{T_j(t)\}$, we get

$$\mathcal{L}\{T_j(t)\} = \frac{1 - b/s}{s + a^2\pi^2 j^2}T_j(0) - b\sum_{n=0}^{\infty}\Big(T_j(\theta_{n+1}) - T_j(\theta_n)\Big)\frac{e^{-(\theta_{n+1})s}}{s(s + a^2\pi^2 j^2)}.$$

Applying the inverse Laplace transform to the last equation, we obtain the solution of Eq. (2.3) in the following way

$$T_j(t) = \left(e^{-a^2\pi^2 j^2 t} - \frac{b}{a^2\pi^2 j^2}\left(1 - e^{-a^2\pi^2 j^2 t}\right)\right)T_j(0) - \frac{b}{a^2\pi^2 j^2}\sum_{n=0}^{\infty}\Big(T_j(\theta_{n+1}) - T_j(\theta_n)\Big)\left(1 - e^{-a^2\pi^2 j^2(t-\theta_{n+1})}\right)u_{\theta_{n+1}}(t). \tag{2.5}$$

Next, we aim to obtain a non-recursive relation for $T_j(\theta_{n+1}) - T_j(\theta_n)$. For this reason, we give the following proposition.

Proposition 2.2. *The solution of Eq. (2.3) on the interval* $[0, \infty)$ *is given by*

$$T_j(t) = \left(e^{-a^2\pi^2 j^2(t-\beta(t))} - \frac{b}{a^2\pi^2 j^2}\left(1 - e^{-a^2\pi^2 j^2(t-\beta(t))}\right)\right)\prod_{i=1}^{\delta(t)}\left(e^{-a^2\pi^2 j^2(\theta_i-\theta_{i-1})} - \frac{b}{a^2\pi^2 j^2}\left(1 - e^{-a^2\pi^2 j^2(\theta_i-\theta_{i-1})}\right)\right)T_j(0).$$

Here, $\delta(t)$ *denotes the number of discontinuity moments* θ_i *on the interval* $(0, t]$ *and it is assumed that* $\prod_{i=1}^{0}(\cdot) = 1$.

Proof. To prove the proposition, let us start on an arbitrary interval $[\theta_n, \theta_{n+1})$. Then,

$$T'_{nj}(t) + a^2\pi^2 j^2 T_{nj}(t) = -bT_{nj}(\theta_n), \tag{2.6}$$

where $T_{nj}(t)$ denotes the solution of Eq. (2.3) on this interval. If we solve this equation, we obtain the general solution of (2.6) as follows

$$T_{nj}(t) = \left(e^{-a^2\pi^2 j^2(t-\theta_n)} - \frac{b}{a^2\pi^2 j^2}\left(1 - e^{-a^2\pi^2 j^2(t-\theta_n)}\right)\right)T_{nj}(\theta_n).$$

Since solution $T_j(t)$ is continuous on the interval $[0, \infty)$, we have

$$T_{nj}(\theta_{n+1}) = T_{n+1,j}(\theta_{n+1}).$$

In an explicit way, it is true that

$$T_{n+1,j}(\theta_{n+1}) = \left(e^{-a^2\pi^2 j^2(\theta_{n+1}-\theta_n)} - \frac{b}{a^2\pi^2 j^2}\left(1 - e^{-a^2\pi^2 j^2(\theta_{n+1}-\theta_n)}\right)\right)T_{nj}(\theta_n).$$

Then, using the last equality, we obtain

$$T_{nj}(\theta_n) = \prod_{k=1}^{n}\left(e^{-a^2\pi^2 j^2(\theta_k-\theta_{k-1})} - \frac{b}{a^2\pi^2 j^2}\left(1 - e^{-a^2\pi^2 j^2(\theta_k-\theta_{k-1})}\right)\right)T_{0j}(0).$$

Hence,

$$T_{nj}(t) = \left(e^{-a^2\pi^2 j^2(t-\theta_n)} - \frac{b}{a^2\pi^2 j^2}\left(1 - e^{-a^2\pi^2 j^2(t-\theta_n)}\right)\right)\prod_{k=1}^{n}\left(e^{-a^2\pi^2 j^2(\theta_k-\theta_{k-1})} - \frac{b}{a^2\pi^2 j^2}\left(1 - e^{-a^2\pi^2 j^2(\theta_k-\theta_{k-1})}\right)\right)T_{0j}(0).$$

Since $T_{nj}(t)$ represents the solution on an arbitrary interval $\theta_n \le t < \theta_{n+1}$ and $T_j(t)$ is continuous on the interval $[0, \infty)$, the solution of (2.3) on the interval $t \in [0, \infty)$ can be expressed in the following way

$$T_j(t) = \left(e^{-a^2\pi^2 j^2(t-\beta(t))} - \frac{b}{a^2\pi^2 j^2}\left(1 - e^{-a^2\pi^2 j^2(t-\beta(t))}\right)\right)\prod_{k=1}^{\delta(t)}\left(e^{-a^2\pi^2 j^2(\theta_k-\theta_{k-1})} - \frac{b}{a^2\pi^2 j^2}\left(1 - e^{-a^2\pi^2 j^2(\theta_k-\theta_{k-1})}\right)\right)T_j(0). \tag{2.7}$$

Hence, proposition is proved. □

Now, we are ready to obtain a non-recursive relation instead of recursive relation in Eq. (2.5). To do this, if we write $t = \theta_{n+1}$ and $t = \theta_n$ in Eq. (2.7), respectively, we obtain the following two equalities

$$\begin{aligned} T_j(\theta_{n+1}) = &\left(e^{-a^2\pi^2 j^2(\theta_{n+1}-\theta_n)} - \frac{b}{a^2\pi^2 j^2}\left(1 - e^{-a^2\pi^2 j^2(\theta_{n+1}-\theta_n)}\right)\right) \\ &\times \prod_{k=1}^{n}\left(e^{-a^2\pi^2 j^2(\theta_k-\theta_{k-1})} - \frac{b}{a^2\pi^2 j^2}\left(1 - e^{-a^2\pi^2 j^2(\theta_k-\theta_{k-1})}\right)\right)T_j(0) \end{aligned} \tag{2.8a}$$

and

$$T_j(\theta_n) = \prod_{k=1}^{n}\left(e^{-a^2\pi^2 j^2(\theta_k-\theta_{k-1})} - \frac{b}{a^2\pi^2 j^2}(1 - e^{-a^2\pi^2 j^2(\theta_k-\theta_{k-1})})\right)T_j(0). \tag{2.8b}$$

Then, after subtracting (2.8b) from (2.8a), we have

$$\begin{aligned} T_j(\theta_{n+1}) - T_j(\theta_n) = &\left(e^{-a^2\pi^2 j^2(\theta_{n+1}-\theta_n)} - \frac{b}{a^2\pi^2 j^2}\left(1 - e^{-a^2\pi^2 j^2(\theta_{n+1}-\theta_n)}\right) - 1\right) \\ &\times \prod_{k=1}^{n}\left(e^{-a^2\pi^2 j^2(\theta_k-\theta_{k-1})} - \frac{b}{a^2\pi^2 j^2}\left(1 - e^{-a^2\pi^2 j^2(\theta_k-\theta_{k-1})}\right)\right)T_j(0). \end{aligned} \tag{2.9}$$

Introducing (2.9) into (2.5), we can rewrite the solution of Eq. (2.3) with the non-recursive relation as follows

$$\begin{aligned} T_j(t) =&\Bigg\{e^{-a^2\pi^2 j^2 t} - \frac{b}{a^2\pi^2 j^2}\left(1 - e^{-a^2\pi^2 j^2 t}\right) - \frac{b}{a^2\pi^2 j^2}\sum_{n=0}^{\infty}\left(e^{-a^2\pi^2 j^2(\theta_{n+1}-\theta_n)} - \frac{b}{a^2\pi^2 j^2}\left(1 - e^{-a^2\pi^2 j^2(\theta_{n+1}-\theta_n)}\right) - 1\right) \\ &\times \prod_{k=1}^{n}\left(e^{-a^2\pi^2 j^2(\theta_k-\theta_{k-1})} - \frac{b}{a^2\pi^2 j^2}\left(1 - e^{-a^2\pi^2 j^2(\theta_k-\theta_{k-1})}\right)\right)\left(1 - e^{-a^2\pi^2 j^2(t-\theta_{n+1})}\right)u_{\theta_{n+1}}(t)\Bigg\}T_j(0). \end{aligned} \tag{2.10}$$

So, the solutions of the heat equation (1.2) satisfying the boundary conditions (1.3) are obtained as follows

$$u_j(x,t) = X_j(x)T_j(t),\ j = 1, 2, \cdots .$$

Since the equation (1.2) is linear, with the superposition principle the solution of boundary value problem (1.2)-(1.3) on the region $[0,1]\times[0,\infty)$ is given by

$$\begin{aligned} u(x,t) &= \sum_{j=1}^{\infty} T_j(t)X_j(x) \\ &= \sum_{j=1}^{\infty}\Bigg\{e^{-a^2\pi^2 j^2 t} - \frac{b}{a^2\pi^2 j^2}\left(1 - e^{-a^2\pi^2 j^2 t}\right) - \frac{b}{a^2\pi^2 j^2}\sum_{n=0}^{\infty}\left(e^{-a^2\pi^2 j^2(\theta_{n+1}-\theta_n)} - \frac{b}{a^2\pi^2 j^2}\left(1 - e^{-a^2\pi^2 j^2(\theta_{n+1}-\theta_n)}\right) - 1\right) \\ &\times \prod_{k=1}^{n}\left(e^{-a^2\pi^2 j^2(\theta_k-\theta_{k-1})} - \frac{b}{a^2\pi^2 j^2}\left(1 - e^{-a^2\pi^2 j^2(\theta_k-\theta_{k-1})}\right)\right)\left(1 - e^{-a^2\pi^2 j^2(t-\theta_{n+1})}\right)u_{\theta_{n+1}}(t)\Bigg\}C_jT_j(0)\sqrt{2}\sin(\pi jx). \end{aligned} \tag{2.11}$$

Now, it is time to check the initial condition. To do this, putting $t = 0$ in (2.11) gives

$$u(x,0) = u_0(x) = \sum_{j=1}^{\infty}\sqrt{2}T_j(0)\sin(\pi jx),$$

where

$$T_j(0) = \sqrt{2}\int_0^1 u_0(x)\sin(\pi jx)\mathrm{d}x. \tag{2.12}$$

Hence, equality (2.11) with equality (2.12) is the solution of problem (1.2), (1.3), (1.4) in $[0,1]\times[0,\infty)$.

Theorem 2.3. *If*

$$-a^2\pi^2 < b < a^2\pi^2\frac{e^{a^2\pi^2\overline{\theta}} + 1}{e^{a^2\pi^2\overline{\theta}} - 1}, \tag{2.13}$$

then the solution (2.11) of the initial value problem (1.2)-(1.4) tends to zero as $t \to \infty$ uniformly with respect to x.

Proof. From (2.13), we get

$$-a^2\pi^2 j^2 < b < a^2\pi^2 j^2 \frac{1+e^{-a^2\pi^2 j^2(\theta_i-\theta_{i-1})}}{1-e^{-a^2\pi^2 j^2(\theta_i-\theta_{i-1})}}, \quad j=1,2,\cdots.$$

After rearranging this inequality, we obtain

$$-1 < e^{-a^2\pi^2 j^2(\theta_i-\theta_{i-1})} - \frac{b}{a^2\pi^2 j^2}\left(1-e^{-a^2\pi^2 j^2(\theta_i-\theta_{i-1})}\right) < 1.$$

With the help of the last inequality, the solution $T_j(t)$ given by (2.10) tends to zero. Since in the solution (2.11), $X_j(x)$ is bounded and $T_j(t)$ tends to zero, the solution (2.11) tends to zero as $t\to\infty$ uniformly with respect to x. Hence, theorem is proved. □

Theorem 2.4. *Each solution given by (2.10) has a zero on the interval (θ_n,θ_{n+1}) if*

$$b > \frac{a^2\pi^2}{e^{a^2\pi^2\underline{\theta}}-1}. \tag{2.14}$$

Proof. Using the inequality (2.14), we can write the following inequality

$$b > \frac{a^2\pi^2 j^2}{e^{a^2\pi^2 j^2(\theta_i-\theta_{i-1})}-1}.$$

From this equality, we have

$$e^{-a^2\pi^2 j^2(\theta_i-\theta_{i-1})} - \frac{b}{a^2\pi^2 j^2}(1-e^{-a^2\pi^2 j^2(\theta_i-\theta_{i-1})}) < 0.$$

Then, using (2.7), we have

$$T_{nj}(\theta_{n+1}) = \left(e^{-a^2\pi^2 j^2(\theta_{n+1}-\theta_n)} - \frac{b}{a^2\pi^2 j^2}\left(1-e^{-a^2\pi^2 j^2(\theta_{n+1}-\theta_n)}\right)\right)\prod_{i=1}^{n}\left(e^{-a^2\pi^2 j^2(\theta_i-\theta_{i-1})} - \frac{b}{a^2\pi^2 j^2}\left(1-e^{-a^2\pi^2 j^2(\theta_i-\theta_{i-1})}\right)\right)T_j(0)$$

and

$$T_{nj}(\theta_n) = \prod_{i=1}^{n}\left(e^{-a^2\pi^2 j^2(\theta_i-\theta_{i-1})} - \frac{b}{a^2\pi^2 j^2}\left(1-e^{-a^2\pi^2 j^2(\theta_i-\theta_{i-1})}\right)\right)T_j(0).$$

Multiplying both of these two equalities, we get

$$T_{nj}(\theta_{n+1})T_{nj}(\theta_n) = \left(e^{-a^2\pi^2 j^2(\theta_{n+1}-\theta_n)} - \frac{b}{a^2\pi^2 j^2}(1-e^{-a^2\pi^2 j^2(\theta_{n+1}-\theta_n)})\right)\left(\prod_{i=1}^{n}\left(e^{-a^2\pi^2 j^2(\theta_i-\theta_{i-1})} - \frac{b}{a^2\pi^2 j^2}(1-e^{-a^2\pi^2 j^2(\theta_i-\theta_{i-1})})\right)T_j(0)\right)^2.$$

Hence,

$$T_{nj}(\theta_{n+1})T_{nj}(\theta_n) < 0.$$

This inequality shows that each solution given by (2.10) has a zero on the interval (θ_n,θ_{n+1}). □

A solution is said to be *oscillatory* if it has arbitrarily large zeros. It follows immediately from Theorem 2.4 that (2.14) is sufficient for oscillation of the solutions of Eq. (2.3) as stated by the following statement.

Corollary 2.5. *If the inequality (2.14) holds true, then each solution of Eq. (2.3) is oscillatory for all values of j.*

Theorem 2.6. *If $b<0$, then for each $j=1,2,\cdots$, any solution of Eq. (2.3) is nonoscillatory. If $b>0$ and j is large enough, then functions T_j are oscillatory.*

Proof. Let $b<0$. Hence, the inequality

$$e^{-a^2\pi^2 j^2(\theta_i-\theta_{i-1})} - \frac{b}{a^2\pi^2 j^2}\left(1-e^{-a^2\pi^2 j^2(\theta_i-\theta_{i-1})}\right) > 0$$

is valid, which implies in turn that any solution of Eq. (2.3) is nonoscillatory. Solving the last inequality, we obtain

$$b < \frac{a^2\pi^2 j^2}{e^{a^2\pi^2 j^2\underline{\theta}}-1},$$

for each $j = 1, 2, \cdots$. Since

$$\lim_{j\to\infty} \frac{a^2\pi^2 j^2}{e^{a^2\pi^2 j^2 \underline{\theta}} - 1} = 0,$$

the last inequality fails for $b > 0$ and large enough j. So, in this case, functions $T_j(t)$ oscillate. □

3. Examples

Example 3.1. In the problem (1.2)-(1.4), if we take $a = 1/2$, $b = -3/2$, and $\theta_i = i/4$, $i = 0, 1, 2, \cdots$, $u_0(x) = -2x^3 + 2x$, we obtain the following problem

$$\begin{aligned} &\frac{\partial u(x,t)}{\partial t} = \frac{1}{4}\frac{\partial^2 u(x,t)}{\partial x^2} + \frac{3}{2}u(x,\beta(t)), \qquad 0 \le x \le 1, \quad 0 \le t < \infty, \\ &u(0,t) = 0, \quad u(1,t) = 0, \quad 0 \le t < \infty, \\ &u_0(x) = -2x^3 + 2x, \quad 0 \le x \le 1. \end{aligned} \tag{3.1}$$

Since the condition (2.13) is satisfied for $a = 1/2$, $b = -3/2$, and $\overline{\theta} = 1/4$, the solution $u(x,t)$ of the problem (3.1) tends to zero as $t \to \infty$ based on Theorem 2.3 as shown in Figure 1 obtained by using Mathematica.

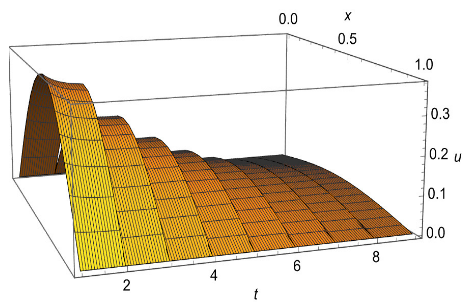

Figure 1. For $a = 0.5$, $b = -1.5$, $\theta_i = i/4$ and $u_0(x) = -2x^3 + 2x$, the graph of the solution of (3.1).

Example 3.2. In the problem (1.2)-(1.4), if we take $a = 1/4$, $b = -5.1$, and $\theta_i = i/4$, $i = 0, 1, 2, \cdots$, $u_0(x) = x^4 - x^2$, we obtain the following problem

$$\begin{aligned} &\frac{\partial u(x,t)}{\partial t} = \frac{1}{16}\frac{\partial^2 u(x,t)}{\partial x^2} + \frac{51}{10}u(x,\beta(t)), \qquad 0 \le x \le 1, \quad 0 \le t < \infty, \\ &u(0,t) = 0, \quad u(1,t) = 0, \quad 0 \le t < \infty, \\ &u_0(x) = x^4 - x^2, \quad 0 \le x \le 1. \end{aligned} \tag{3.2}$$

Since the condition of the Theorem (2.6) is satisfied for $a = 1/4$, $b = -5.1 < 0$, and $\overline{\theta} = 1/4$, the solution $u(x,t)$ of the problem (3.2) does not oscillate as shown in Figure 2.

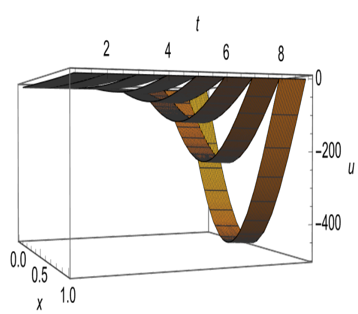

Figure 2. For $a = 0.25$, $b = -5.1$, $\theta_i = i/4$ and $u_0(x) = x^4 - x^2$, the graph of the solution of (3.2).

Example 3.3. In the problem (1.2)-(1.4), if we take $a = 1/2$, $b = 6.1$, and $\theta_i = i/4$, $i = 0, 1, 2, \cdots$, $u_0(x) = x^4 - x^2$, we obtain the following problem

$$\begin{aligned} &\frac{\partial u(x,t)}{\partial t} = \frac{1}{4}\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{61}{10}u(x,\beta(t)), \qquad 0 \le x \le 1, \quad 0 \le t < \infty, \\ &u(0,t) = 0, \quad u(1,t) = 0, \quad 0 \le t < \infty, \\ &u_0(x) = x^4 - x^2, \quad 0 \le x \le 1. \end{aligned} \tag{3.3}$$

Since the condition of the Theorem (2.6) is satisfied for $a = 1/2$, $b = 6.1 > 0$, and $\overline{\theta} = 1/4$, the solution $u(x,t)$ of the problem (3.3) is oscillatory as shown in Figure 3.

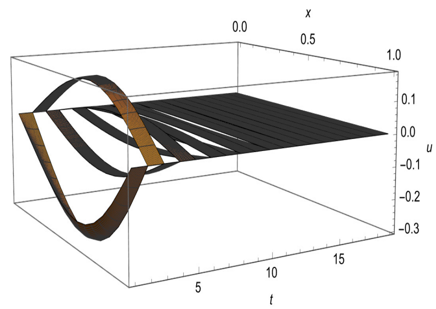

Figure 3. For $a = 0.5$, $b = 6.1$, $\theta_i = i/4$ and $u_0(x) = x^4 - x^2$, the graph of the solution of (3.3).

4. Conclusion

In this paper, we investigated solutions of the initial boundary value problem (1.2)-(1.4), where a and b are nonzero real parameters, $u_0(x)$ is a continuous function on the interval $[0, 1]$.

The oscillatory and nonoscillatory conditions, as well as an explicit formula for the solutions of (2.3), are obtained. Besides, the criteria for solutions to converge to zero are discovered for the issue problem. Moreover, conditions for the solutions of (2.3) to possess zeros on each interval $[\theta_i, \theta_{i+1}]$, $i = 0, 1, 2, \cdots$, are discussed.

In the future studies, similar analysis for the following initial boundary problem

$$\begin{aligned} &\frac{\partial^2 u(x,t)}{\partial t} = a^2\frac{\partial^2 u(x,t)}{\partial x^2} - bu(x,\beta(t)), \qquad 0 \le x \le 1, \quad \theta_0 \le t < \infty, \\ &u(0,t) = 0, \quad u(1,t) = 0, \quad \theta_0 \le t < \infty, \\ &u(x,\theta_0) = u_0(x), \quad u_t(x,\theta_0) = u_1(x), \quad 0 \le x \le 1. \end{aligned}$$

can be handled.

5. Acknowledgements

M. Akhmet has been supported by 2247-A National Leading Researchers Program of TÜBİTAK (The Scientific and Technological Research Council of Turkey), Turkey, N 120A138.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The authors have contributed equally. All authors have read and agreed to the published version of the manuscript.

References

[1] Aftabizadeh, A.R., Wiener, J., Ming Xu, J., *Oscillatory and periodic solutions of delay differential equations with piecewise constant argument*, Proc. Amer. Math. Soc., **99**(4)(1987), 673–679.

[2] Aftabizadeh, A.R., Wiener, J., *Oscillatory and periodic solutions for systems of two first order linear differential equations with piecewise constant argument*, Appl. Anal., **26**(4)(1988), 327–333.

[3] Akhmet, M.U., *Integral manifolds of differential equations with piecewise constant argument of generalized type*, Nonlinear Anal, **66**(2007), 367–383.

[4] Akhmet, M.U., *Stability of differential equations with piecewise constant arguments of generalized type*, Nonlinear Anal., **68**(2008), 794–803.

[5] Akhmet, M.U., *Almost periodic solutions of the linear differential equation with piecewise constant argument*, Discrete and Impulsive Systems, Series A, Mathematical Analysis, **16**(2009), 743–753.

[6] Akhmet, M.U., Nonlinear Hybrid Continuous Discrete-Time Models, Atlantis Press: Amsterdam-Paris, 2011.

[7] Akhmet, M.U., Functional Differential Equations with Piecewise Constant Argument, In: Regularity and Stochasticity of Nonlinear Dynamical Systems, Springer, 2018.

[8] Akhmet, M.U., Aruğaslan, D., Yılmaz, E., *Stability in cellular neural networks with a piecewise constant argument*, Journal of Computational and Applied Mathematics, **233**(2010), 2365–2373.

[9] Akhmet, M.U., Büyükadalı, C., *Differential equations with state-dependent piecewise constant argument*, Nonlinear Analysis: Theory, methods and applications, **72(11)**(2010), 4200–4211.

[10] Akhmet, M., Dauylbayev, M., Mirzakulova, A., *A singularly perturbed differential equation with piecewise constant argument of generalized type*, Turkish Journal of Mathematics, **42**(1)(2018), 1680–1685.

[11] Akhmet, M.U., Yılmaz, E., Neural Networks with Discontinuous/Impact Activations, Springer: New York, 2013.

[12] Aruğaslan, D., Cengiz, N., *Green's function and periodic solutions of a Spring-Mass system in which the forces are functionally dependent on piecewise constant argument*, Süleyman Demirel University Journal of Natural and Applied Sciences, **21**(1)(2017), 266–278.

[13] Aruğaslan, D., Cengiz, N., *Existence of periodic solutions for a mechanical system with piecewise constant forces*, Hacet. J. Math. Stat., **47**(3)(2018), 521–538.

[14] Bainov, D.D., Simeonov, P.S., Impulsive Differential Equations: Asymptotic Properties of the Solutions, World Scientific: Singapore, New Jersey, London, 1995.

[15] Bereketoğlu, H., Lafcı, M., *Behavior of the solutions of a partial differential equation with a piecewise constant argument*, Filomat, **31**(19)(2017), 5931–5943.

[16] Busenberg, S., Cooke, K.L., *Models of vertically transmitted diseases with sequential-continuous dynamics*, in Nonlinear Phenomena in Mathematical Sciences, Lakshmikantham, V. (editor), Academic Press, New York, (1982), 179–187.

[17] Büyükkahraman, M.L., Bereketoğlu, H., *On a partial differential equation with piecewise constant mixed arguments*, Iranian Journal of Science and Technology, Transactions A: Science **44**(6)(2020), 1791–1801.

[18] Chi, H., Poorkarimi, H., Wiener, J., Shah, S.M., *On the exponential growth of solutions to nonlinear hyperbolic equations*, Internat. J. Math. Math. Sci., **12**(3)(1989), 539–545.

[19] Cooke, K.L., Wiener, J., *Retarded differential equations with piecewise constant delays*, J. Math. Anal. and Appl., **99**(1)(1984), 265–297.

[20] Farlow, S.J., Partial Differential Equations for Scientists and Engineers, John Wiley & Sons, 1982.

[21] Győri, I., *On approximation of the solutions of delay differential equations by using piecewise constant arguments*, Internat. J. Math. Math. Sci., **14**(1)(1991), 111–126.

[22] Győri, I., Ladas, G., *Linearized oscillations for equations with piecewise constant arguments*, Differential and Integral Equations, **2**(2)(1989), 123–131.

[23] Huang, Y.K., *Oscillations and asymptotic stability of solutions of first order neutral differential equations with piecewise constant argument*, J. Math. Anal. Appl., **149**(1)(1990), 70–85.

[24] Liang, H., Wang, G., *Existence and uniqueness of periodic solutions for a delay differential equation with piecewise constant arguments*, Port. Math., **66**(1)(2009), 1–12.

[25] Muroya, Y., *New contractivity condition in a population model with piecewise constant arguments*, J. Math. Anal. Appl., **346**(1)(2008), 65–81.

[26] Pinto, M., *Asymptotic equivalence of nonlinear and quasi linear differential equations with piecewise constant arguments*, Math. Comput. Modelling, **49**(9-10)(2009), 1750–1758.

[27] Poorkarimi, H., Wiener, J., Bounded solutions of nonlinear parabolic equations with time delay, Proceedings of the 15th Annual Conference of Applied Mathematics (Edman, OK, 1999), 87-91 (electronic), Electron. J. Differ. Equ, Conf., 2, Southwest Texas State Univ., San Marcos, TX, 1999.

[28] Samoilenko, A.M., Perestyuk, N.A., Impulsive Differential Equations, World Scientific: Singapore, New Jersey, London, Hong Kong, 1995.

[29] Shah, S., Poorkarimi, M H., Wiener, J., *Bounded solutions of retarded nonlinear hyperbolic equations*, Bull. Allahabad. Math. Soc., **1**(1986), 1–14.

[30] Veloz, T., Pinto, M., *Existence, computability and stability for solutions of the diffusion equation with general piecewise constant argument*, J. Math. Anal. Appl., **426**(1)(2015), 330–339.

[31] Wang, Q., *Stability of numerical solution for partial differential equations with piecewise constant arguments*, Advances in Difference Equations, **2018**(1)(2018), 1–13.

[32] Wang, Q., Wen, J., *Analytical and numerical stability of partial differential equations with piecewise constant arguments*, Numer. Methods Partial Differential Equations, **30**(1)(2014), 1–16.

[33] Wiener, J., *Boundary value problems for partial differential equations with piecewise constant delay*, Internat. J. Math. Math. Sci., **14**(2)(1991), 363–379.

[34] Wiener, J., Generalized Solutions of Functional Differential Equations, World Scientific, Publishing Co., Inc, River Edge, NJ, 1993.

[35] Wiener, J., Debnath, L., *A wave equation with discontinuous time delay*, Internat. J. Math. Math. Sci. **15**(4)(1992), 781–788.

[36] Wiener, J., Debnath, L., *A survey of partial differential equations with piecewise continuous arguments*, Internat. J. Math. Math. Sci., **18**(2)(1995), 209–228.

[37] Wiener, J., Debnath, L., *Boundary value problems for the diffusion equation with piecewise continuous time delay*, Internat. J. Math. Math. Sci. **20**(1)(1997), 187–195.

[38] Wiener, J., Heller, W., *Oscillatory and periodic solutions to a diffusion equation of neutral type*, Internat. J. Math. Math. Sci. **22**(2)(1999), 313–348.

[39] Wiener, J., Lakshmikantham, V., *Complicated dynamics in a delay Klein-Gordon equation*, Nonlinear Anal. **38**(1), Ser. B:Real World Appl., (1999), 75–85.

[40] Yuan, R., *The existence of almost periodic solutions of retarded differential equations with piecewise constant argument*, Nonlinear Anal. Ser. A:Theory Methods, **48**(7)(2002), 1013–1032.