A Note on Trigonometric Approximations of Bessel Functions of the First Kind, and Trigonometric Power Sums

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Article Info

Abstract

I reconsider the approximation of Bessel functions with finite sums of trigonometric functions, in the light of recent evaluations of Neumann-Bessel series with trigonometric coefficients. A proper choice of angle allows for an efficient choice of the trigonometric sum. Based on these series, I also obtain straightforward non-standard evaluations of new parametric sums with powers of cosine and sine functions.

1. Introduction

The aim of this paper is twofold: to investigate trigonometric approximations of Bessel functions via Bessel-Neumann series whose sums are finite trigonometric sums, and use the same series to provide new sums of powers of sines with cosines.
Bessel functions are among the most useful and studied special functions. Analytic expansions exist for different regimes [1], and numerical algorithms for their precise evaluation [2]-[5]. Their simplest approximations are polynomials [6]-[8] and finite trigonometric sums, that can be advantageous in applications [9].
Let's consider J_0. Several trigonometric sums appeared in the decades, sometimes being rediscovered. These very simple ones

$$J_0(x) \simeq \tfrac{1}{4}[1 + \cos x + 2\cos(\tfrac{\sqrt{2}}{2}x)] \tag{1.1}$$

$$J_0(x) \simeq \tfrac{1}{6}\left[1 + \cos x + 2\cos(\tfrac{1}{2}x) + 2\cos(\tfrac{\sqrt{3}}{2}x)\right] \tag{1.2}$$

have errors $\varepsilon = J_0 - J_0^{\text{approx}}$ with power series (the marvel of Mathematica)

$$\varepsilon(x) = -\frac{x^8}{2^8 \cdot 20160}(1 - \frac{x^2}{36} + ...), \qquad \varepsilon(x) = -\frac{x^{12}}{2^{12} \cdot 239500800}(1 - \frac{x^2}{52} + ...).$$

In practice, an error less than 0.001 is achieved for $x \leq 3$ or $x \leq 5.9$. These approximations were obtained by Fettis with the Poisson formula [10]. Rehwald [11] and later Waldron [12], Blachman and Mousavinezhad [13] and [14] used the strategy of truncating to the first term Neumann-Bessel series like this one

$$J_0(x) + 2J_8(x) + 2J_{16}(x) + ... = \tfrac{1}{4}\cos[1 + \cos x + 2\cos(\tfrac{\sqrt{2}}{2}x)],$$

that can be obtained from the Bessel generating function. The examples (1.1), (1.2) correspond to $n = 4, 6$ of eq.19 in [15]:

$$J_0(x) + 2\sum_{k=1}^{\infty}(-1)^{kn}J_{2kn}(x) = \frac{1}{n}\sum_{\ell=0}^{n-1}\cos(x\cos\tfrac{\pi}{n}\ell) \tag{1.3}$$

where only J_0 is kept, and the errors reflect the behaviour $J_{2n}(x) \approx \frac{(x/2)^{2n}}{(2n)!}$ of the first neglected term. The truncation yields J_0 as a sum of cosines that corresponds to the evaluation of the Bessel integral $J_0(x) = \int_0^\pi \frac{d\theta}{\pi} \cos(x\cos\theta)$ with the trapezoidal rule with n nodes [16]-[18]. Increasing n increases accuracy: $n = 15$ is a formula by Fettis [10] with 8 cosines (instead of 15, symmetries of the roots of unity reduce the number of terms):

$$J_0(x) \simeq \tfrac{1}{15}\cos x + \tfrac{2}{15}\sum_{k=1}^{7}\cos(x\cos\tfrac{k\pi}{15}). \tag{1.4}$$

The error now is order $x^{30} \times 10^{-42}$ and less than 10^{-6} for $x < 15$.

I reconsider the approximations for J_0 in the light of new Neumann-Bessel trigonometric series in [19]. They extend the series (1.3) by including an angular parameter, that is chosen to kill the term with J_{2n}, so that the truncation involves the next-to-next term J_{4n} of the series. This is presented in Section 2. The same strategy is then used in Section 3 to approximate Bessel functions J_n of low order by appropriate series. The quality of the approximations is the same as earlier ones with same number of terms, but the terms are different and the source of error is more clear.

In Section 4, I show that the same Neumann series give in very simple way some parametric sums of powers of sines and cosines. Some are found in the recent literature [20], while the following ones, to my knowledge, are new:

$$\sum_{\ell=0}^{n}\sin^p(\tfrac{\theta+2\pi\ell}{n})\{\begin{smallmatrix}\sin\\\cos\end{smallmatrix}\}(q\tfrac{\theta+2\pi\ell}{n}) \qquad (p,q = 0,1,...).$$

Many other trigonometric sums are available in the literature. For example, sums like $\sum_{k=1}^{m-1}\sin\frac{2kq\pi}{m}\cot^n\frac{k\pi}{m}$ and variants are studied in [21, 22]. Ratios of powers of sines and cosines are evaluated in [23], and many results are given in the remarkable paper [24].

2. The Bessel function J_0

Consider the Neumann trigonometric series eq.11 in [19]:

$$J_0(x) + 2\sum_{k=1}^{\infty}(-1)^{kn}J_{2kn}(x)\cos(2kn\theta) = \frac{1}{n}\sum_{\ell=0}^{n-1}\cos[x\cos(\theta+\tfrac{\pi}{n}\ell)]. \tag{2.1}$$

The approximations (1.1), (1.2) and (1.4) are obtained with $\theta = 0$, $n = 4, 6, 15$, and neglecting functions J_8, J_{12}, J_{30} and higher orders. However they are not optimal. The advantage of eq.(2.1) is the possibility to choose the angle $\theta = \pi/4n$ to kill all terms J_{2n}, J_{6n}, etc. Then:

$$J_0(x) - 2J_{4n}(x) + 2J_{8n}(x) - ... = \frac{1}{n}\sum_{\ell=0}^{n-1}\cos(z\cos\tfrac{1+4\ell}{4n}\pi).$$

An expansion for J_0 results, again, by neglecting the other terms.

Some examples:

● $n = 2$. It is $J_0(x) = \frac{1}{2}[\cos(x\cos\frac{\pi}{8}) + \cos(x\sin\frac{\pi}{8})] + \varepsilon_2(x)$. If we neglect the error, the first zero occurs at $\pi\sqrt{2-\sqrt{2}} = 2.4045$ ($j_{0,1} = 2.4048$).

● $n = 3$. The approximation has three cosines:

$$J_0(x) = \tfrac{1}{3}[\cos(x\tfrac{1}{\sqrt{2}}) + \cos(x\tfrac{\sqrt{3}-1}{2\sqrt{2}}) + \cos(x\tfrac{\sqrt{3}+1}{2\sqrt{2}})] + \varepsilon_3(x) \tag{2.2}$$

$$\varepsilon_3(x) = \tfrac{x^{12}}{2^{12}\cdot239500800}\left[1 - \tfrac{x^2}{52} + \tfrac{x^4}{52\cdot112} - \tfrac{x^6}{52\cdot112\cdot180} + ...\right].$$

Remarkably, the first powers of the error are opposite of those for the expansion eq.(1.2), that would involve 6 terms if not for the degeneracy of the roots of unity. The half-sum of (1.2) and (2.2),

$$J_0(x) \simeq \tfrac{1}{12}\left[1 + \cos x + 2\cos(\tfrac{1}{2}x) + 2\cos(\tfrac{\sqrt{3}}{2}x) + 2\cos(x\tfrac{1}{\sqrt{2}}) + 2\cos(x\tfrac{\sqrt{3}-1}{2\sqrt{2}}) + 2\cos(x\tfrac{\sqrt{3}+1}{2\sqrt{2}})\right] \tag{2.3}$$

has error $\varepsilon(x) = -\tfrac{x^{24}}{5.2047}\times 10^{-30}[1 - \tfrac{x^2}{100} + \tfrac{x^4}{20800} - ...]$.

● $n = 6$ gives a precision similar to the sum (2.3):

$$J_0(z) = \tfrac{1}{6}\left[\cos(x\cos\tfrac{\pi}{24}) + \cos(x\cos\tfrac{3\pi}{24}) + \cos(x\cos\tfrac{5\pi}{24}) + \cos(x\sin\tfrac{\pi}{24}) + \cos(x\sin\tfrac{3\pi}{24}) + \cos(x\sin\tfrac{5\pi}{24})\right] + \varepsilon_6(x). \tag{2.4}$$

The error has power expansion $\varepsilon_6(x) = \tfrac{x^{24}}{5.2047}\times 10^{-30}[1 - \tfrac{x^2}{100} + ...]$.

● $n = 8$ is a sum of 8 cosines and compares with the formula (1.4) by Fettis. The two approximations are different but with the same number of terms (because $\theta = 0$ produces degenerate terms) and similar precision.
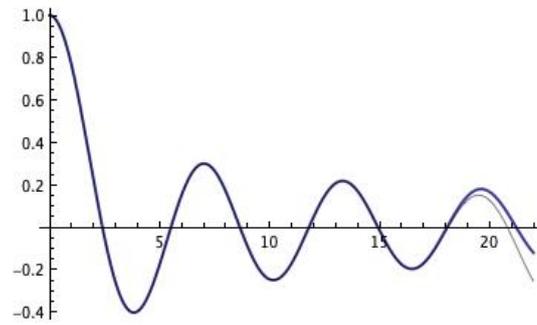
Figure 2.1: The Bessel function J_0 (thick) and the trigonometric expansion (2.4). The difference increases with x; it is less than 10^{-9} for $x < 8$ and 10^{-3} for $x < 15$.

3. Bessel functions J_n.

- **J_1** is evaluated via $J_1 = -J_0'$. Eq. (2.3) gives:

$$J_1(x) \simeq \tfrac{1}{12}\left[\sin x + \sin(\tfrac{1}{2}x) + \sqrt{3}\sin(\tfrac{\sqrt{3}}{2}x) + \sqrt{2}\sin(x\tfrac{1}{\sqrt{2}}) + (\tfrac{\sqrt{3}-1}{\sqrt{2}})\sin(x\tfrac{\sqrt{3}-1}{2\sqrt{2}}) + (\tfrac{\sqrt{3}+1}{\sqrt{2}})\sin(x\tfrac{\sqrt{3}+1}{2\sqrt{2}})\right]$$

with error $\varepsilon(x) \simeq (x/20)^{23} \times 3.87 \times [1 - \tfrac{13}{1200}x^2 + \ldots]$.

- **J_2, J_4** can be evaluated with the following identity (the real part of eq.(5) in [19]):

$$J_p(x) + \sum_{k=1}^{\infty}[J_{kn+p}(x) + (-1)^{kn+p}J_{kn-p}(x)]\cos(kn\theta) = \tfrac{1}{n}\sum_{\ell=0}^{n-1}\cos[x\sin(\theta + \tfrac{2\pi\ell}{n}) + p(\theta + \tfrac{2\pi\ell}{n})]. \qquad (3.1)$$

Because of the term J_{n-p}, we take $2p < n$. With $y = \tfrac{\pi}{2n}$:

$$J_p(x) - (-1)^p J_{2n-p}(x) + \ldots = \tfrac{1}{n}\sum_{\ell=0}^{n-1}\cos[x\sin(\tfrac{1+4\ell}{2n}\pi) + p\tfrac{1+4\ell}{2n}\pi].$$

If only J_p is kept, the approximation depends on the parity of p:

$$J_p(x) \simeq \begin{cases} \cos(p\tfrac{1+4\ell}{2n}\pi) \times \tfrac{1}{n}\sum_{\ell=0}^{n-1}\cos[x\sin(\tfrac{1+4\ell}{2n}\pi)] & p \text{ even} \\ -\sin(p\tfrac{1+4\ell}{2n}\pi) \times \tfrac{1}{n}\sum_{\ell=0}^{n-1}\sin[x\sin(\tfrac{1+4\ell}{2n}\pi)] & p \text{ odd} \end{cases}$$

$p = 2$, $n = 6$, give the short formula

$$J_2(x) \simeq \tfrac{1}{2\sqrt{3}}\left[\cos(x\sin\tfrac{\pi}{12}) - \cos(x\cos\tfrac{\pi}{12})\right]$$

with error $\varepsilon(x) = 2.69114 \times (x/10)^{-10}[1 - \tfrac{x^2}{44} + \ldots]$. The first zero is evaluated $\tfrac{2}{3}\pi\sqrt{6} \simeq 5.1302$ ($j_{2,1} = 5.13562$). A better approximation is $n = 8$, $y = \tfrac{\pi}{16}$:

$$J_2(x) \simeq \tfrac{1}{4}\cos(\tfrac{\pi}{8})[\cos(x\sin\tfrac{\pi}{16}) - \cos(x\cos\tfrac{\pi}{16})] + \tfrac{1}{4}\sin(\tfrac{\pi}{8})[\cos(x\cos\tfrac{5\pi}{16}) - \cos(x\sin\tfrac{5\pi}{16})]$$

with error $\varepsilon(x) = 7.00119 \times 10^{-16}x^{14}[1 - \tfrac{x^2}{60} + \ldots]$; $\varepsilon(5) = 3 \times 10^{-6}$, $\varepsilon(8) = 0.0010$.
For J_4 we select $p = 4$, $n = 8$, $\theta = \tfrac{\pi}{16}$. Now the lowest neglected term is J_{12}:

$$J_4(x) \simeq \tfrac{\sqrt{2}}{8}\left[\cos(x\sin\tfrac{\pi}{16}) + \cos(x\cos\tfrac{\pi}{16}) - \cos(x\sin\tfrac{5\pi}{16}) - \cos(x\cos\tfrac{5\pi}{16})\right].$$

The error is less that 10^{-3} at $x < 6.3$.

- **J_3, J_5**. A useful sum for odd-order Bessel functions is eq.(17) in [19]:

$$\sum_{k=0}^{\infty}(-1)^{n+k}J_{(2n+1)(2k+1)}(x)\cos[(2k+1)\theta] = \sum_{\ell=0}^{2n}\frac{\sin[x\cos(\tfrac{\theta+2\pi\ell}{2n+1})]}{2(2n+1)}. \qquad (3.2)$$

The angle $\theta = \tfrac{\pi}{6}$ cancels J_{6n+3}, J_{14n+7} etc. and gives the approximation

$$J_{2n+1}(x) \simeq \frac{(-1)^n}{\sqrt{3}}\sum_{\ell=0}^{2n}\frac{\sin[x\cos\tfrac{1+12\ell}{12n+6}\pi]}{2n+1}$$

that neglects J_{10n+5} etc. With $n = 1$ and $n = 2$ we obtain:

$$J_3(x) \simeq -\tfrac{1}{3\sqrt{3}}[\sin(x\cos\tfrac{\pi}{18}) - \sin(x\sin\tfrac{2\pi}{9}) - \sin(x\sin\tfrac{\pi}{9})]$$

$$J_5(x) \simeq \tfrac{1}{5\sqrt{3}}\left[\sin(x\cos\tfrac{\pi}{30}) + \sin(x\sin\tfrac{\pi}{15}) - \sin(\tfrac{\sqrt{3}}{2}x) - \sin(x\sin\tfrac{4\pi}{15}) + \sin(x\cos\tfrac{2\pi}{15})\right].$$

The expansion for J_3 has error $\varepsilon = 2.33373 \times 10^{-17}x^{15}[1 - \tfrac{x^2}{64} + \ldots]$. The second one has error $\varepsilon = 1.92134x^{25} \times 10^{-33} \times [1 - \tfrac{x^2}{104} + \ldots]$.
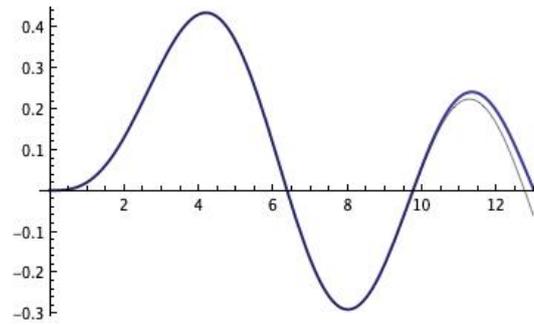
Figure 3.1: The Bessel function J_3 (thick) and the approximation (3.3). The difference is $\varepsilon(6) = 6 \times 10^{-6}$, $\varepsilon(8) = .0003$, $\varepsilon(10) = .0045$.

4. Trigonometric identities

The Neumann-Bessel series here used provide sums of powers of sines and cosines. They arise by expanding in powers of x the Bessel functions in the series,

$$J_n(x) = \sum_{k=0}^{\infty} (-1)^k \frac{(x/2)^{n+2k}}{k!(k+n)!}$$

and the trigonometric functions in the sum of the series.

• Consider the series eq.(2.1). At threshold powers x^{2n}, x^{4n} etc. new Bessel functions $(-1)^n 2J_{2n}\cos(2n\theta)$, $2J_{4n}\cos(4n\theta)$ etc. enter a term in the sum of cosines.

$$\frac{1}{n}\sum_{\ell=0}^{n-1} \left[\cos\frac{\theta+\ell\pi}{n}\right]^{2k} = \begin{cases} \frac{1}{4^k}\binom{2k}{k} & 0 \le k < n \\ \frac{1}{4^k}\left[\binom{2k}{k} + 2\binom{2k}{k-n}\cos(2\theta)\right] & n \le k < 2n \\ \frac{1}{4^k}\left[\binom{2k}{k} + 2\binom{2k}{k-n}\cos(2\theta) + 2\binom{2k}{k-2n}\cos(4\theta)\right] & 2n \le k < 3n \\ \dots & \dots \end{cases}.$$

By replacing θ with $\theta + n\pi$ we obtain:

$$\frac{1}{n}\sum_{\ell=0}^{n-1} \left[\sin\frac{\theta+\ell\pi}{n}\right]^{2k} = \begin{cases} \frac{1}{4^k}\binom{2k}{k} & 0 \le k < n \\ \frac{1}{4^k}\left[\binom{2k}{k} + (-1)^n 2\binom{2k}{k-n}\cos(2\theta)\right] & n \le k < 2n \\ \frac{1}{4^k}\left[\binom{2k}{k} + (-1)^n 2\binom{2k}{k-n}\cos(2\theta) + 2\binom{2k}{k-2n}\cos(4\theta)\right] & 2n \le k < 3n \\ \dots & \dots \end{cases}.$$

Examples:

$$\frac{1}{9}\sum_{\ell=0}^{8} \left[\sin\frac{\theta+\ell\pi}{9}\right]^{20} = \frac{1}{4^{10}}\left[\binom{20}{10} - 2\binom{20}{1}\cos(2\theta)\right].$$

$$\frac{1}{n}\sum_{\ell=0}^{n-1} \left[\cos(\tfrac{1+6\ell}{6n}\pi)\right]^{2n} = \frac{1}{4^n}\left[\binom{2n}{n} + 1\right],$$

$$\frac{1}{n}\sum_{\ell=0}^{n-1} \left[\cos(\tfrac{1+4\ell}{4n}\pi)\right]^{4n} = \frac{1}{16^n}\left[\binom{4n}{2n} - 2\right].$$

For $\theta = 0$ and $\theta = \frac{\pi}{2}$ these identities are eqs. 4.4.2 in [25], 2.1 and 2.2 (together with several other non-parametric sums) in [26]. The series had also been studied in [27]. Parametric averages on the full circle were recently evaluated by Jelitto [20], with a different method.

• With the Neumann series (3.2) we obtain:

$$\frac{1}{2n+1}\sum_{\ell=0}^{2n} \left[\cos\frac{\theta+2\pi\ell}{2n+1}\right]^{2k+1} = \begin{cases} 0 & 1 \le 2k+1 < 2n+1 \\ \frac{1}{4^k}\binom{2k+1}{k-n}\cos\theta & 2n+1 \le 2k+1 < 3(2n+1) \\ \frac{1}{4^k}\left[\binom{2k+1}{k-n}\cos\theta + \binom{2k+1}{k-3n-1}\cos(3\theta)\right] & 3(2n+1) \le 2k+1 < 5(2n+1) \\ \dots & \dots \end{cases}.$$

The sums of even powers of cosines are obtained from the series eq.16 in [19]:

$$J_0(x) + 2\sum_{k=1}^{\infty} (-1)^k J_{(4n+2)k}(x)\cos(2k\theta) = \sum_{\ell=0}^{2n} \frac{\cos[x\cos\frac{\theta+2\pi\ell}{2n+1}]}{2n+1}$$

$$\frac{1}{2n+1}\sum_{\ell=0}^{2n}\left[\cos\frac{\theta+2\pi\ell}{2n+1}\right]^{2k} = \begin{cases} \frac{1}{4^k}\binom{2k}{k} & 0 \le k < 2n+1 \\ \frac{1}{4^k}\left[\binom{2k}{k} + 2\binom{2k}{k-2n-1}\cos(2\theta)\right] & 2n+1 \le k < 4n+2 \\ \frac{1}{4^k}\left[\binom{2k}{k} + 2\binom{2k}{k-2n-1}\cos(2\theta) + 2\binom{2k}{k-4n-2}\cos(4\theta)\right] & 4n+2 \le k < 6n+3 \\ \dots & \dots \end{cases}.$$

Example: $(\cos\frac{\theta}{3})^{12} + (\cos\frac{\theta+\pi}{3})^{12} + (\cos\frac{\theta+2\pi}{3})^{12} = \frac{3}{4^6}[\binom{12}{6} + 2\binom{12}{3}\cos(2\theta) + 2\cos(4\theta)]$.

The formulae with sines are obtained by shifting the parameter θ.

• Now let's consider the sum eq.(3.1) with $p < n - p$.
The equations are new and are easier to state if we distinguish the parity of n and of p.

Case n = 2m and p = 2q. Eq.(3.1) now is:

$$\frac{1}{2m}\sum_{\ell=0}^{2m-1}\cos[x\sin(\tfrac{\theta+\pi\ell}{m})]\cos[2q\tfrac{\theta+\pi\ell}{m}] = J_{2q}(x) + [J_{2m-2q}(x) + J_{2m+2q}(x)]\cos(2\theta)$$
$$+[J_{4m-2q}(x) + J_{4m+2q}(x)]\cos(4\theta) + \dots$$

Separation of even and odd parts in x, and expansion in x give:

$$\frac{1}{2m}\sum_{\ell=0}^{2m-1}[\sin\tfrac{\theta+\pi\ell}{m}]^{2k+1}\sin(2q\tfrac{\theta+\pi\ell}{m}) = 0, \qquad k = 0,1,2,\dots$$

This result is obvious as the sum from 0 to $m-1$ is opposite of the rest of the sum. The symmetry is used also in the other result:

$$\frac{1}{m}\sum_{\ell=0}^{m-1}[\sin\tfrac{\theta+\pi\ell}{m}]^{2k}\cos(2q\tfrac{\theta+\pi\ell}{m}) =$$

$$= \frac{(-1)^q}{4^k}\begin{cases} 0 & k < q \\ \binom{2k}{k-q} & q \le k < m-q \\ \binom{2k}{k-q} + (-1)^m\binom{2k}{k-m+q}\cos(2\theta) & m-q \le k < m+q \\ \binom{2k}{k-q} + (-1)^m\left[\binom{2k}{k-m+q} + \binom{2k}{k-m-q}\right]\cos(2\theta) & m+q \le k < 2m-q \\ \dots & \dots \end{cases}$$

Case n = 2m, p = 2q + 1. Eq.(3.1) becomes:

$$-\frac{1}{2m}\sum_{\ell=0}^{2m-1}\sin[x\sin(\tfrac{\theta+\pi\ell}{m})]\sin[(2q+1)\tfrac{\theta+\pi\ell}{m}] = J_{2q+1}(x) +$$
$$+[-J_{2m-2q-1}(x) + J_{2m+2q+1}(x)]\cos(2\theta) + [J_{4m-2q-1}(x) + J_{4m+2q+1}(x)]\cos(4\theta) + \dots$$

The non trivial result is:

$$\frac{1}{m}\sum_{\ell=0}^{m-1}[\sin\tfrac{\theta+\pi\ell}{m}]^{2k+1}\sin[(2q+1)\tfrac{\theta+\pi\ell}{m}] =$$

$$= \frac{(-1)^q}{2^{2k+1}}\begin{cases} 0 & k < q \\ \binom{2k+1}{k-q} & q \le k < m-q-1 \\ \binom{2k+1}{k-q} + (-1)^m\binom{2k+1}{k+m-q}\cos(2\theta) & m-q-1 \le k < m+q \\ \binom{2k+1}{k-q} + (-1)^m\left[\binom{2k+1}{k+m-q} + \binom{2k+1}{k-m-q}\right]\cos(2\theta) & m+q \le k < 2m-q-1 \\ \dots & \dots \end{cases}$$

Example: $\frac{1}{5}\sum_{\ell=0}^{4}\sin^{13}(\tfrac{\pi\ell}{5})\sin(\tfrac{3\pi\ell}{5}) = -\frac{1}{2^{13}}\left[\binom{13}{5} - \binom{13}{10} - \binom{13}{0}\right] = -\frac{125}{1024}$.

Case n = 2m + 1 and p = 2q:

$$\frac{1}{2m+1}\sum_{\ell=0}^{2m}\cos[x\sin(\tfrac{\theta+2\pi\ell}{2m+1})]\cos(2q\tfrac{\theta+2\pi\ell}{2m+1}) = J_{2q}(x) + [J_{4m+2-2q}(x) + J_{4m+2+2q}(x)]\cos(2\theta) + \dots$$
$$\frac{1}{2m+1}\sum_{\ell=0}^{2m}\sin[x\sin(\tfrac{\theta+2\pi\ell}{2m+1})]\sin(2q\tfrac{\theta+2\pi\ell}{2m+1}) = [J_{2m+1-2q}(x) - J_{2m+1+2q}(x)]\cos\theta + \dots$$

$$\frac{1}{2m+1}\sum_{\ell=0}^{2m}[\sin\tfrac{\theta+2\pi\ell}{2m+1}]^{2k}\cos(2q\tfrac{\theta+2\pi\ell}{2m+1}) =$$

$$= \frac{(-1)^q}{4^k}\begin{cases} 0 & k < q \\ \binom{2k}{k-q} & q \le k < 2m+1-q \\ \binom{2k}{k-q} - \binom{2k}{k+q-1-2m}\cos(2\theta) & 2m+1-q \le k < 2m+1+q \\ \binom{2k}{k-q} - \left[\binom{2k}{k+q-1-2m} + \binom{2k}{k-q-1-2m}\right]\cos(2\theta) & 2m+1+q \le k < 4m+2-q \\ \dots & \dots \end{cases}$$

$$\frac{1}{2m+1}\sum_{\ell=0}^{2m}[\sin\tfrac{\theta+2\pi\ell}{2m+1}]^{2k+1}\sin(2q\tfrac{\theta+2\pi\ell}{2m+1}) = \frac{(-1)^{q+m+1}}{2^{2k+1}}\begin{cases}0 & k < m-q \\ \binom{2k+1}{k-m+q}\cos\theta & m-q \le k < m+q \\ \left[\binom{2k+1}{k-m+q} - \binom{2k+1}{k-m-q}\right]\cos\theta & m+q \le k < 3m+1-q \\ \dots & \dots\end{cases}$$

Case n $= 2m+1$ and p $= 2q+1$:

$$\frac{1}{2m+1}\sum_{\ell=0}^{2m}\cos[x\sin\tfrac{\theta+2\pi\ell}{2m+1}]\cos[(2q+1)\tfrac{\theta+2\pi\ell}{2m+1}] = [J_{2m-2q}(x)+J_{2m+2q+2}(x)]\cos\theta + \dots$$

$$-\frac{1}{2m+1}\sum_{\ell=0}^{2m}\sin[x\sin\tfrac{\theta+2\pi\ell}{2m+1}]\sin[(2q+1)\tfrac{\theta+2\pi\ell}{2m+1}] = J_{2q+1}(x)+[-J_{4m-2q+1}(x)+J_{4m+2q+3}(x)]\cos(2\theta) + \dots$$

$$\frac{1}{2m+1}\sum_{\ell=0}^{2m}\sin[\tfrac{\theta+2\pi\ell}{2m+1}]^{2k}\cos[(2q+1)\tfrac{\theta+2\pi\ell}{2m+1}] =$$

$$= \frac{(-1)^{m+q}}{4^k}\begin{cases}0 & k < m-q \\ \binom{2k}{k-m+q}\cos\theta & m-q \le k < m+q+1 \\ \left[\binom{2k}{k-m+q} - \binom{2k}{k-m-q-1}\right]\cos\theta & m+q+1 \le k < 3m+q+2 \\ \dots & \dots\end{cases}$$

$$\frac{1}{2m+1}\sum_{\ell=0}^{2m}\sin[\tfrac{\theta+2\pi\ell}{2m+1}]^{2k+1}\sin[(2q+1)\tfrac{\theta+2\pi\ell}{2m+1}] =$$

$$= \frac{(-1)^{q}}{2^{2k+1}}\begin{cases}0 & k < q \\ \binom{2k+1}{k-q} & q \le k < 2m-q \\ \binom{2k+1}{k-q} - \binom{2k+1}{k+q-2m}\cos(2\theta) & 2m-q \le k < 2m+q+1 \\ \binom{2k+1}{k-q} - \left[\binom{2k+1}{k+q-2m} + \binom{2k+1}{k-q-1-2m}\right]\cos(2\theta) & 2m+q+1 \le k < 4m-q+1 \\ \dots & \dots\end{cases}.$$

Examples:

$\frac{1}{13}\sum_{\ell=0}^{12}\sin^9(\tfrac{\theta+2\pi\ell}{13})\sin(5\tfrac{\theta+2\pi\ell}{13}) = \frac{1}{2^9}\binom{9}{2} = \frac{9}{128}$ for all θ,

$\frac{1}{13}\sum_{\ell=0}^{12}\sin^{31}(\tfrac{\theta+2\pi\ell}{13})\sin(5\tfrac{\theta+2\pi\ell}{13}) = \frac{1}{2^{31}}\left[\binom{31}{13} - \left(\binom{31}{5}+1\right)\cos(2\theta)\right]$.

Conclusion

Bessel functions of the first kind may be well approximated on an interval containing the origin, by the trapezoidal rule applied the Bessel integral. The result is a finite trigonometric sum. Here I show that comparable accuracy is obtained by exploiting some exact results for Neumann series of Bessel functions and cosines, as finite trigonometric sums. At appropriate angles, the second term of the Neumann series cancels and, by the rapid decay of third and next terms, the trigonometric sum well represents the first Bessel term in the Neumann series.

The same Neumann series allow for the evaluation of new trigonometric sums of powers of sine and cosine functions. They extend recent results by Jelitto [20], and are not included in the extensive paper by Al Jarrah et al. [15].

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