Turk. J. Math. Comput. Sci.
15(1)(2023) 157–163

DOI : 10.47000/tjmcs.1167007

Some New Improvements of Huygen's Inequality

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Received: 25-08-2022 • Accepted: 09-05-2023

Abstract. First, some improvements of Young's inequality are given in this article. Then, using these improvements, stronger results are obtained from the Huygens inequality.

2010 AMS Classification: 26D05

Keywords: Young inequality, hyperbolic functions, Huygen inequality.

1. Introduction

The Huygen inequality is given as

$$2\frac{sinx}{x}+\frac{tanx}{x}>3, x\in\left(0,\frac{\pi}{2}\right). \tag{1.1}$$

Information about the history of this inequality is found in reference [1, 3]. Then, many mathematicians worked on the generalization and improvement of this inequality, and proved the analogues of this inequality for other special functions [4–6, 9–11].These inequalities allow us to compare trigonometric functions with linear functions over a given range.

Now we give some facts essential to prove our results.

Mitrinovic and Adamovic gave the following inequality in [8]:

$$(cosx)^{\frac{1}{3}}<\frac{sinx}{x}<\frac{2+cosx}{3}, x\in\left(0,\frac{\pi}{2}\right). \tag{1.2}$$

Lazarevic gave an analogue of the inequality (1.2) for hyperbolic functions in [7]:

$$(coshx)^{\frac{1}{3}}<\frac{sinhx}{x}<\frac{2+coshx}{3}, x\neq 0. \tag{1.3}$$

Neuman and Sandor gave an analogue of the inequality (1.1) for hyperbolic functions in [11] as

$$2\frac{sinhx}{x}+\frac{tanhx}{x}>3, x\neq 0.$$

Also, Neuman and Sandor gave the following refinement of Huygens inequality in [11]:

$$2\frac{sinx}{x}+\frac{tanx}{x}>2\frac{x}{sinx}+\frac{x}{tanx}>3, x\in\left(0,\frac{\pi}{2}\right),$$

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$$2\frac{sinhx}{x} + \frac{tanhx}{x} > 2\frac{x}{sinhx} + \frac{x}{tanhx} > 3, x \neq 0.$$

In this article, we will prove some improvements of Huygens inequality by means of some modifications of Young's inequality.

2. Preliminaries

Lemma 2.1 ([2] Young's inequality). *If $x, y > 0$, and $\mu \in [0,1]$, then*

$$\mu x + (1-\mu)y \geq x^{\mu}y^{1-\mu}.$$

Lemma 2.2 (Cauchy-Schwarz inequality). *If $x_i, y_i > 0$, then*

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2.$$

Lemma 2.3. *If $x_i, y_i > 0$, $i = 1, 2, ..., n$, then*

$$\left(\sum_{i=1}^{n}(x_i + y_i)\right)^2 \geq 4\left(\sum_{i=1}^{n}\sqrt{x_i y_i}\right)\left(\sum_{i=1}^{n}\sqrt{\frac{x_i^2 + y_i^2}{2}}\right).$$

Proof. We know that, $4xy \leq (x+y)^2, \forall x, y > 0$

$$\begin{aligned} 4\left(\sum_{i=1}^{n}\sqrt{x_i y_i}\right)\left(\sum_{i=1}^{n}\sqrt{\frac{x_i^2+y_i^2}{2}}\right) &\leq \left[\left(\sum_{i=1}^{n}\sqrt{x_i y_i}\right) + \left(\sum_{i=1}^{n}\sqrt{\frac{x_i^2+y_i^2}{2}}\right)\right]^2 \\ &= \left[\sum_{i=1}^{n}\left(\sqrt{x_i y_i} + \sqrt{\frac{x_i^2+y_i^2}{2}}\right)\right]^2. \end{aligned}$$

And by using Lemma 2.2, we get

$$\left[\sum_{i=1}^{n}\left(\sqrt{x_i y_i} + \sqrt{\frac{x_i^2+y_i^2}{2}}\right)\right]^2 \leq \left[\sum_{i=1}^{n}\sqrt{(1+1)\left(x_i y_i + \frac{x_i^2+y_i^2}{2}\right)}\right]^2 = \left(\sum_{i=1}^{n}(x_i+y_i)\right)^2$$

The Lemma is proved. □

3. Results

Theorem 3.1. *If $x, y > 0$, $x \geq y$ and $\mu \in [\frac{1}{2}, 1]$, then the below inequality is satisfied*

$$\mu x + (1-\mu)y \geq x^{1-\mu}y^{\mu} + (2\mu - 1)(x - y) \geq x^{\mu}y^{1-\mu}.$$

Proof. We obtain the first part of the inequality directly from Lemma 2.1

$$\begin{aligned} \mu x + (1-\mu)y &= (2\mu-1)(x-y) + (1-\mu)x + \mu y \\ &\geq x^{1-\mu}y^{\mu} + (2\mu-1)(x-y). \end{aligned}$$

Now, we show that

$$x^{1-\mu}y^{\mu} + (2\mu-1)(x-y) \geq x^{\mu}y^{1-\mu}.$$

For this, let's define a function $f : [1, \infty) \to R$ such that

$$\begin{aligned} f(t) &= t^{1-\mu} + (2\mu-1)(t-1) - t^{\mu}, \\ f'(t) &= (1-\mu)t^{-\mu} + (2\mu-1) - \mu t^{\mu-1}, \\ f''(t) &= (1-\mu)(-\mu)t^{-\mu-1} - \mu(\mu-1)t^{\mu-2} = \mu(\mu-1)\left[\frac{1}{t^{\mu+1}} - \frac{1}{t^{2-\mu}}\right] \geq 0, \forall t \geq 1. \end{aligned}$$

Then, we obtain

$$\forall t \geq 1, f'(t) \geq f'(1) = 0.$$

Thus, $f(t)$ is an increasing and positive function for all $t \geq 1$. If we take $t = \dfrac{x}{y}$ and multiply both sides of the inequality by y, then we obtain

$$\left(\frac{x}{y}\right)^{1-\mu} y + (2\mu - 1)(x - y) \geq \left(\frac{x}{y}\right)^{\mu} y$$

or

$$x^{1-\mu}y^{\mu} + (2\mu - 1)(x - y) \geq x^{\mu}y^{1-\mu}.$$

The Theorem is proved. □

Theorem 3.2. *If $x, y > 0$, $x \geq y$ and $\mu \in [\frac{1}{2}, \frac{3}{4}]$, then the following inequality is satisfied*

$$\mu x + (1-\mu)y \geq x^{\mu-\frac{1}{2}}y^{\frac{3}{2}-\mu} + \frac{x-y}{2} \geq x^{\mu}y^{1-\mu}.$$

Proof. By using Lemma 2.1, we obtain

$$\mu x + (1-\mu)y = (\mu - \frac{1}{2})x + (\frac{3}{2} - \mu)y + \frac{x-y}{2} \geq x^{\mu-\frac{1}{2}}y^{\frac{3}{2}-\mu} + \frac{x-y}{2}.$$

Now, we show

$$x^{\mu-\frac{1}{2}}y^{\frac{3}{2}-\mu} + \frac{x-y}{2} \geq x^{\mu}y^{1-\mu}. \tag{3.1}$$

By using Lemma 2.1, we acquire

$$\frac{1}{2}\left[\left(\frac{x}{y}\right)^{\frac{1}{2}} + 1\right] \geq \left(\frac{x}{y}\right)^{\frac{1}{4}}.$$

Also, we know for all $\mu \in [\frac{1}{2}, \frac{3}{4}]$

$$\left(\frac{x}{y}\right)^{\frac{3}{4}-\mu} \geq 1$$

or

$$\left(\frac{x}{y}\right)^{\frac{1}{4}} \geq \left(\frac{x}{y}\right)^{\mu-\frac{1}{2}}$$

is true. Then, we acquire

$$\frac{1}{2}\left[\left(\frac{x}{y}\right)^{\frac{1}{2}} + 1\right] \geq \left(\frac{x}{y}\right)^{\frac{1}{4}} \geq \left(\frac{x}{y}\right)^{\mu-\frac{1}{2}}. \tag{3.2}$$

It is clear that the inequality (3.1) is equivalent to the following inequality:

$$\frac{x-y}{2} \geq x^{\mu-\frac{1}{2}}y^{1-\mu}\left[\sqrt{x} - \sqrt{y}\right].$$

If $x = y$, the inequality is trivial. So, let's assume $x > y$ and divide both side of the inequality by $\sqrt{y}(\sqrt{x} - \sqrt{y})$, we get

$$\frac{1}{2}\left[\left(\frac{x}{y}\right)^{\frac{1}{2}} + 1\right] > \left(\frac{x}{y}\right)^{\mu-\frac{1}{2}}.$$

this inequality is true according to the inequality (3.2).
The Theorem is proved. □

Theorem 3.3. *If $x \neq 0$, then the below inequality holds*

$$\frac{2}{3}\frac{sinh(x)}{x} + \frac{1}{3}\frac{tanh(x)}{x} > \left(\frac{sinh(x)}{x}\right)^{\frac{1}{6}}\left(\frac{tanh(x)}{x}\right)^{\frac{5}{6}} + \frac{sinh(x) - tanh(x)}{2x} > 1.$$

Proof. It is sufficient to prove the theorem for $x > 0$ according to the properties of hyperbolic functions by using Theorem 3.1 we get

$$\begin{aligned} \frac{2}{3}\frac{sinh(x)}{x} + \frac{1}{3}\frac{tanh(x)}{x} &> \left(\frac{sinh(x)}{x}\right)^{\frac{1}{6}}\left(\frac{tanh(x)}{x}\right)^{\frac{5}{6}} + \frac{sinh(x) - tanh(x)}{2x} \\ &> \left(\frac{sinh(x)}{x}\right)^{\frac{2}{3}}\left(\frac{tanh(x)}{x}\right)^{\frac{1}{3}}. \end{aligned}$$

Because $\frac{2}{3} \in [\frac{1}{2}, \frac{3}{4}]$ and $\frac{sinh(x)}{x} > \frac{tanh(x)}{x}, \forall x \neq 0$. Also, we acquire from inequality (1.3)

$$\left(\frac{sinh(x)}{x}\right)^{\frac{2}{3}}\left(\frac{tanh(x)}{x}\right)^{\frac{1}{3}} = \frac{sinh(x)}{x}.\frac{1}{\sqrt[3]{cosh(x)}} > 1.$$

The Theorem is proved. □

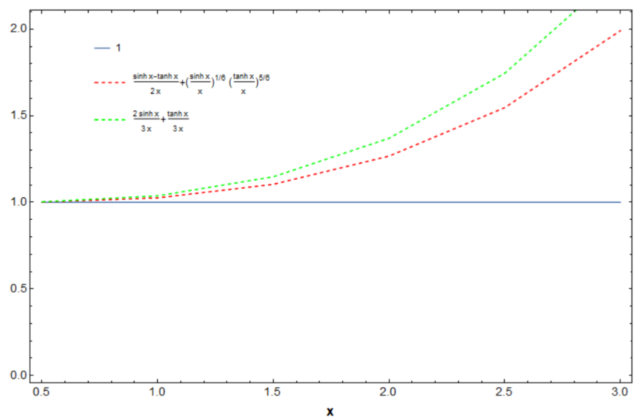

FIGURE 1. A refinement of Huygens inequality with $\left(\frac{sinh(x)}{x}\right)^{\frac{1}{6}}\left(\frac{tanh(x)}{x}\right)^{\frac{5}{6}} + \frac{sinh(x) - tanh(x)}{2x}$.

Theorem 3.4. *If $x \neq 0$, then then below inequality is satisfied*

$$\frac{2}{3}\frac{sinh(x)}{x} + \frac{1}{3}\frac{tanh(x)}{x} > \left(\frac{sinh(x)}{x}\right)^{\frac{1}{3}}\left(\frac{tanh(x)}{x}\right)^{\frac{2}{3}} + \frac{sinh(x) - tanh(x)}{3x} > 1.$$

Proof. It is sufficent to prove the theorem for $x > 0$ according to the properties of hyperbolic functions by using Theorem 3.1 we acquire,

$$\begin{aligned} \frac{2}{3}\frac{sinh(x)}{x} + \frac{1}{3}\frac{tanh(x)}{x} &> \left(\frac{sinh(x)}{x}\right)^{\frac{1}{3}}\left(\frac{tanh(x)}{x}\right)^{\frac{2}{3}} + \frac{sinh(x) - tanh(x)}{3x} \\ &> \left(\frac{sinh(x)}{x}\right)^{\frac{2}{3}}\left(\frac{tanh(x)}{x}\right)^{\frac{1}{3}}. \end{aligned}$$

Because $\frac{2}{3} \in [\frac{1}{2}, 1]$ and $\frac{sinh(x)}{x} > \frac{tanh(x)}{x}, \forall x \neq 0$. Also, we obtain from (1.3)

$$\left(\frac{sinh(x)}{x}\right)^{\frac{2}{3}}\left(\frac{tanh(x)}{x}\right)^{\frac{1}{3}} = \frac{sinh(x)}{x}.\frac{1}{\sqrt[3]{cosh(x)}} > 1.$$

The Theorem is proved. □

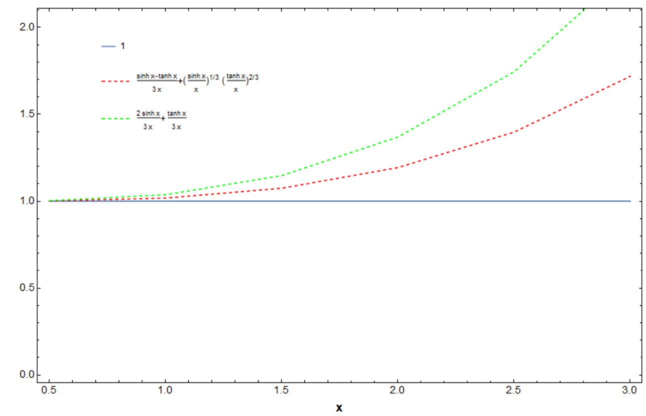

FIGURE 2. A refinement of Huygens inequality with $\left(\frac{sinh(x)}{x}\right)^{\frac{1}{3}}\left(\frac{tanh(x)}{x}\right)^{\frac{2}{3}}+\frac{sinh(x)-tanh(x)}{3x}$.

Theorem 3.5. *If $x \in (0, \frac{\pi}{2})$, then the following inequality holds*

$$\frac{2}{3}\frac{x}{sinx}+\frac{1}{3}\frac{x}{tanx}>\left(\frac{x}{sinx}\right)^{\frac{1}{3}}\left(\frac{x}{tanx}\right)^{\frac{2}{3}}+\frac{x}{3}\left(\frac{1}{sinx}-\frac{1}{tanx}\right)>1.$$

Proof. By using Theorem 3.1 we obtain

$$\frac{2}{3}\frac{x}{sinx}+\frac{1}{3}\frac{x}{tanx}>\left(\frac{x}{sinx}\right)^{\frac{1}{3}}\left(\frac{x}{tanx}\right)^{\frac{2}{3}}+\frac{x}{3}\left(\frac{1}{sinx}-\frac{1}{tanx}\right)$$
$$>\left(\frac{x}{sinx}\right)^{\frac{2}{3}}\left(\frac{x}{tanx}\right)^{\frac{1}{3}}.$$

Also, by using (1.2), we acquire

$$\left(\frac{x}{sinx}\right)^{\frac{2}{3}}\left(\frac{x}{tanx}\right)^{\frac{1}{3}}=\frac{x}{sinx}.\sqrt[3]{cosx}>\frac{sinx}{x}.\sqrt[3]{cosx}>1.$$

The Theorem is proved. □

Theorem 3.6. *If $x \in (0, \frac{\pi}{2})$, then the below inequality holds*

$$\frac{2}{3}\frac{x}{sinx}+\frac{1}{3}\frac{x}{tanx}>\left(\frac{x}{sinx}\right)^{\frac{1}{6}}\left(\frac{x}{tanx}\right)^{\frac{5}{6}}+\frac{x}{2}\left(\frac{1}{sinx}-\frac{1}{tanx}\right)>1.$$

Proof. by using Theorem 3.2 we get

$$\frac{2}{3}\frac{x}{sinx}+\frac{1}{3}\frac{x}{tanx}>\left(\frac{x}{sinx}\right)^{\frac{1}{6}}\left(\frac{x}{tanx}\right)^{\frac{5}{6}}+\frac{x}{3}\left(\frac{1}{sinx}-\frac{1}{tanx}\right)$$
$$>\left(\frac{x}{sinx}\right)^{\frac{2}{3}}\left(\frac{x}{tanx}\right)^{\frac{1}{3}}.$$

Also by using (1.2) we obtain

$$\left(\frac{x}{sinx}\right)^{\frac{2}{3}}\left(\frac{x}{tanx}\right)^{\frac{1}{3}}=\frac{x}{sinx}.\sqrt[3]{cosx}>\frac{sinx}{x}.\sqrt[3]{cosx}>1.$$

The Theorem is proved. □

Now, we will give a new inequality.

Theorem 3.7. *If $x \in (0, \frac{\pi}{2})$, then the following inequality holds*

$$\left(\frac{\sin x}{x} + \frac{x}{tanx}\right)\frac{1}{1 + cosx} < 1.$$

Proof. To prove the above inequality, we need to prove this inequality.

$$sinx \cdot tanx + x^2 < x \cdot tanx(1 + cosx).$$

Above inequality equivalent to below inequality

$$(x - sinx)(tanx - x) > 0.$$

This inequality holds.
The Theorem is proved. □

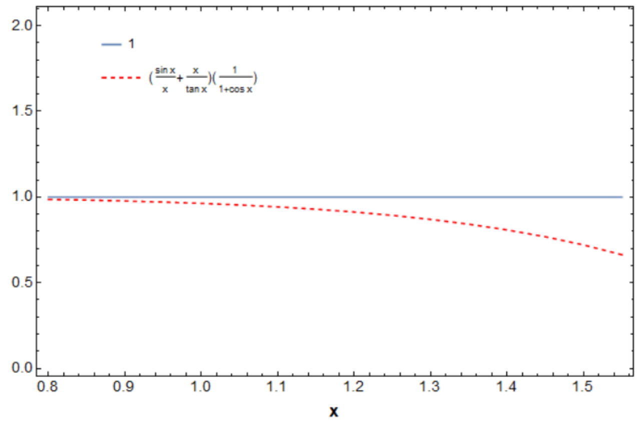

Figure 3. The upper bounds of $\left(\frac{\sin x}{x} + \frac{x}{tanx}\right)\frac{1}{1 + cosx}$.

Corollary 3.8. *If $x \neq 0$, then the following inequality is satisfied*

$$2\frac{sinh(x)}{x} + \frac{tanh(x)}{x} > \frac{sinh(x)}{x}\left(1 + 2\sqrt[4]{\frac{1 + cosh^2(x)}{2cosh^3(x)}}\right) > \frac{sinh(x)}{x}\left(1 + \frac{2}{\sqrt{cosh(x)}}\right) > 3.$$

Proof. From Lemma 2.3, we acquire

$$\begin{aligned} 2\frac{sinh(x)}{x} + \frac{tanh(x)}{x} &= \frac{sinh(x)}{x} + \frac{sinh(x)}{x} + \frac{tanh(x)}{x} \\ &> \frac{sinh(x)}{x}\left(1 + 2\sqrt[4]{\frac{1 + cosh^2(x)}{2cosh^3(x)}}\right). \end{aligned}$$

By using Lemma 2.1, we get

$$1 + cosh^2(x) > 2cosh(x)$$

and

$$\left(1 + \frac{2}{\sqrt{cosh(x)}}\right) > \frac{3}{\sqrt[3]{cosh(x)}}, \forall x > 0.$$

These inequalities show that

$$\frac{sinh(x)}{x}\left(1 + 2\sqrt[4]{\frac{1 + cosh^2(x)}{2cosh^3(x)}}\right) > \frac{sinh(x)}{x}\left(1 + \frac{2}{\sqrt{cosh(x)}}\right) > \frac{3sinh(x)}{x}\frac{1}{\sqrt[3]{cosh(x)}} > 3.$$

□

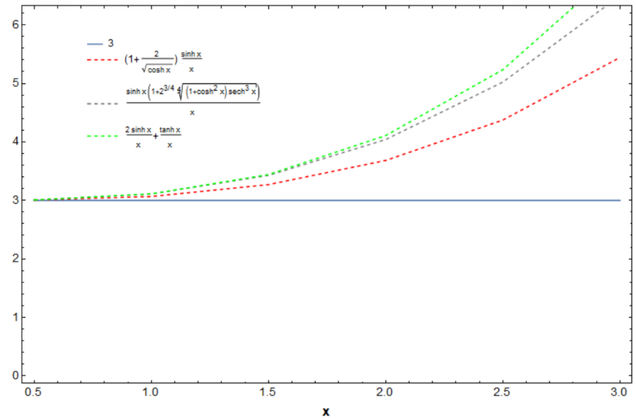

Figure 4. A refinement of Huygens inequality.

Conflicts of Interest

All the authors declare no conflict of interest.

Authors Contribution Statement

All authors jointly worked on the results and they have read and agreed to the published version of the manuscript.

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