Turk. J. Math. Comput. Sci.
16(1)(2024) 240–254
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DOI : 10.47000/tjmcs.1167568

Boubaker Collocation Method for Approximate Solutions of the Model of Pollution for a System of Lakes

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Received: 27-08-2022 • Accepted: 16-05-2023

Abstract. This paper focuses on a numerical approach for the solution of the pollution problem for a system of lakes. The pollution problem consists of three lakes with interconnecting channels and this model corresponds to a system of linear differential equations. The main purpose of this study is to present a collocation method based on the Boubaker polynomials to obtain approximate solutions of this pollution model. Firstly, the approximation solutions are assumed in the forms of the truncated series of the Boubaker polynomials. The solution forms and their derivatives are written in the matrix forms. By means of these matrix forms, the matrix operations and the collocation points, the pollution model is reduced to a system of the algebraic linear equations. In addition, the error estimation method is presented by using the residual function. The parameters in the pollution model are selected according to the datas in the literature. For the selected parameters, the applications of the presented method are made by using a code written in MATLAB. The application results are compared with the results of other methods in the literature. The effectiveness and reliability of the presented method are observed from the obtained results.

2020 AMS Classification: 33C47, 65H10, 65L05, 65L60, 65L80

Keywords: Approximate solutions, Boubaker polynomials, collocation method, error estimation, the pollution model of lakes.

1. Introduction

Pollution has become a very serious threat to our environment. Monitoring pollution is the first step in planning to save the environment. The use of differential equations makes it possible to monitoring pollution. In this study, a system of three lakes, interconnected by channels flowing between them, is introduced using a system of differential equations.

Recently, many methods related to system of differential equations such as Chebyshev collocation method [8, 24], high-order collocation methods [26], hybrid collocation method [2], Taylor collocation method [19], hybrid block method [27], Legendre–Gauss collocation method [12,13], two-step almost collocation method [7], spectral collocation method [11, 34], rational Chebyshev collocation method [18], Legendre-Gauss-Radau collocation method [23], an exponential matrix method [30], block hybrid collocation method [29], Taylor collocation and Adomian decomposition method [6], RKN-type Fourier collocation method [22], Laguerre collocation method [32, 33], exponential Fourier collocation method [25], Haar wavelet collocation method [1, 21], Jacobi collocation method [9] have been studied.

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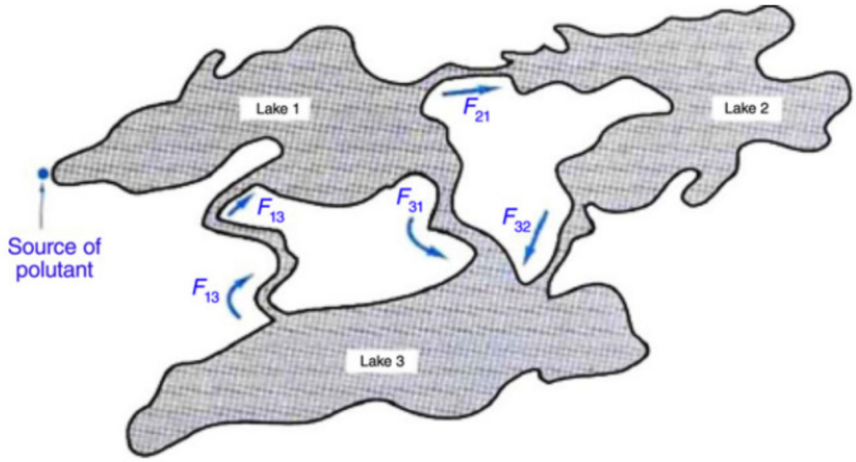

Figure 1. System of three lakes with interconnecting channels [4, 10]

In addition, many methods related to the problem of pollution for a system of lakes such as Adomian method [4], homotopy perturbation method [16], modified differential transformation method [15], variational iteration method [5], Bessel collocation method [31], modified differential transform method [3], Bernoulli Ritz-collocation method [20], Laplace transform method [17] have been studied by many researchers.

The model is created with the following some assumptions [4]:

- Each lake is considered a large compartment.
- Channels connected as pipes between compartments are considered.
- The direction of flow in channels or pipes is demonstrated with arrows in the Figure 1.
- A pollutant is introduced into the first lake and $p(t)$ shows the rate at which the pollutant enters the lake per unit time.
- The function $p(t)$ may be constant or change over time.
- We are interested in knowing the pollution level in each lake at any given time.
- Let $y_k(t)$ demonstrate the amount of pollution in lake k at any time $t \geq 0$, where $k = 1, 2, 3$.
- We assume that the pollutant in each lake is evenly distributed throughout the lake by some mixing process.
- The water volume V_k in lake k rests constant for each lake.
- We also assume that the type of pollution is permanent and not reduced to other forms.
- Then, the pollutant concentration in lake k at any given time is shown by

$$n_k(t) = \frac{y_k(t)}{V_k}.$$

- Initially each lake is assumed to be free of any pollutants, so $y_k(0) = \mu_k$ for each $k = 1, 2, 3$. Here, μ_k are appropriate constants.
- Let F_{mk} be the flow rate from lake k to lake m. Here, F_{mk} are appropriate constants.
- The pollutant flow from lake k to lake m at any time t is demonstrated by $x_{mk}(t)$, defined by

$$x_{mk}(t) = F_{mk} n_k(t) = F_{mk} \frac{y_k(t)}{V_k}.$$

Thus, $x_{mk}(t)$ measures the rate at which the pollutant concentration in lake k flows into lake m at time t. We will observe that

The pollutant change rate = Input rate - Output rate.

This principle is applied to each lake. Thus, the problem of pollution for a system of lakes is modeled with the system of first-order equations given by [4]

$$\begin{cases} y_1'(t) = \frac{F_{13}}{V_3}y_3(t) + p(t) - \frac{F_{31}}{V_1}y_1(t) - \frac{F_{21}}{V_1}y_1(t) & \\ y_2'(t) = \frac{F_{21}}{V_1}y_1(t) - \frac{F_{32}}{V_2}y_2(t), & 0 \le t \le b, \\ y_3'(t) = \frac{F_{31}}{V_1}y_1(t) + \frac{F_{32}}{V_2}y_2(t) - \frac{F_{13}}{V_3}y_3(t) & \end{cases} \tag{1.1}$$

and initial conditions

$$y_1(0) = \mu_1, \quad y_2(0) = \mu_2, \quad \text{and} \quad y_3(0) = \mu_3. \tag{1.2}$$

Since the volume of each lake to remain constant, the flow rate into each lake must balance the flow out of the lake. Hence, we write

$$\begin{aligned} &\text{Lake 1:} F_{13} = F_{21} + F_{31}, \\ &\text{Lake 2:} F_{21} = F_{32}, \\ &\text{Lake 3:} F_{13} = F_{31} + F_{33}. \end{aligned}$$

Here, $y_k(t)(k = 1, 2, 3)$ are the function representing the amount of pollution in k-th lake. $F_{13}, F_{21}, F_{31}, F_{32}, V_1, V_2, V_3$ are the appropriate constants. $p(t)$ is a function defined in interval $0 \le t \le b$, which represents the rate at which the pollutant enters the lake per unit time. In this study, the approximate solutions of (1.1) are investigated in form of the truncated Boubaker polynomial series given by

$$y_k(t) = \sum_{i=0}^{N} a_{k,i}B_i(t), (k = 1, 2, 3), \quad 0 \le t \le b. \tag{1.3}$$

Here, $N \ge 1$ is chosen to be any positive integer. $a_{k,i}$ are the unknown Boubaker coefficients. $B_i(t)$ are the Boubaker polynomials defined by [14]

$$B_i(t) = \sum_{j=0}^{[\![i/2]\!]} (-1)^j \frac{(i-4j)}{i-j}\binom{i-j}{j} t^{i-2j}. \tag{1.4}$$

The recurrence relation of the Boubaker polynomials is [14]

$$B_i(t) = tB_{i-1}(t) - B_{i-2}(t), \quad i \ge 3,$$

where $B_0(t) = 1$, $B_1(t) = t$, $B_2(t) = t^2 + 2$.

Let's summarize rest of this paper as follows: The collocation method is presented in Section 2. For this, the collocation points are defined and the assumed solutions are written in matrix forms. The error estimation method is given in Section 3. In Section 4, the application of the method is made and the results are interpreted. The results of the paper are summarized in Section 5.

2. Boubaker Collocation Method

Let's start this section by defining the collocation points by

$$t_i = \frac{b}{N}i, \quad i = 0, 1, ..., N. \tag{2.1}$$

Secondly, let's write the Boubaker polynomial solutions $y_k(t), (k = 1, 2, 3)$ in (1.3) in matrix forms as

$$y_k(t) = \mathbf{B}(t)\mathbf{A}_k; \quad \mathbf{A}_k = \begin{bmatrix} a_{k,0} & a_{k,1} & \cdots & a_{k,N} \end{bmatrix}^T, \quad \mathbf{B}(t) = \begin{bmatrix} B_0(t) & B_1(t) & \cdots & B_N(t) \end{bmatrix}. \tag{2.2}$$

On the other hand, the Boubaker polynomials in (1.4) can be expressed in matrix form

$$\mathbf{B}(t) = \mathbf{Y}(t)\mathbf{M}. \tag{2.3}$$

Here, $\mathbf{Y}(t) = \begin{bmatrix} 1 & t & \cdots & t^N \end{bmatrix}$ and if N is odd [28],

$$\mathbf{M}^T = \begin{bmatrix} \phi_{0,0} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \phi_{1,0} & 0 & \cdots & 0 & 0 \\ \phi_{2,1} & 0 & \phi_{2,0} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi_{N-1,\frac{N-1}{2}} & 0 & \phi_{N-1,\frac{N-3}{2}} & \cdots & \phi_{N-1,0} & 0 \\ 0 & \phi_{N,\frac{N-1}{2}} & 0 & \cdots & 0 & \phi_{N,0} \end{bmatrix},$$

and if N is even [28],

$$\mathbf{M}^T = \begin{bmatrix} \phi_{0,0} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \phi_{1,0} & 0 & \cdots & 0 & 0 \\ \phi_{2,1} & 0 & \phi_{2,0} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \phi_{N-1,\frac{N-1}{2}} & 0 & \cdots & \phi_{N-1,0} & 0 \\ \phi_{N,\frac{N}{2}} & 0 & \phi_{N,\frac{N-2}{2}} & \cdots & 0 & \phi_{N,0} \end{bmatrix},$$

$$\phi(m,n) = \left[\frac{m-4n}{m-n}\binom{m-n}{n}\right](-1)^n, \quad m,n = 0,1,\ldots,N.$$

Using (2.3), the Boubaker polynomial solutions in matrix form (2.2) can be expressed as

$$y_k(t) = \mathbf{Y}(t)\mathbf{M}\mathbf{A}_k, \quad (k = 1,2,3). \tag{2.4}$$

As a next step, let's write the terms $y_k^{'}(t)$, $(k = 1,2,3)$ in model (1.1) in matrix forms. In that case, the derivative of (2.4) is taken and the first derivatives of the Boubaker polynomial solutions become in the next matrix forms

$$y_k^{'}(t) = \mathbf{Y}(t)\mathbf{D}\mathbf{M}\mathbf{A}_k, \quad (k = 1,2,3), \tag{2.5}$$

where [31]

$$\mathbf{D} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Substituting matrix relations (2.4)-(2.5) in (1.1), we get

$$\begin{aligned} \mathbf{Y}(t)\mathbf{D}\mathbf{M}\mathbf{A}_1 &= \tfrac{F_{13}}{V_3}\mathbf{Y}(t)\mathbf{M}\mathbf{A}_3 + p(t) - \tfrac{F_{31}}{V_1}\mathbf{Y}(t)\mathbf{M}\mathbf{A}_1 - \tfrac{F_{21}}{V_1}\mathbf{Y}(t)\mathbf{M}\mathbf{A}_1 \\ \mathbf{Y}(t)\mathbf{D}\mathbf{M}\mathbf{A}_2 &= \tfrac{F_{21}}{V_1}\mathbf{Y}(t)\mathbf{M}\mathbf{A}_1 - \tfrac{F_{32}}{V_2}\mathbf{Y}(t)\mathbf{M}\mathbf{A}_2 \\ \mathbf{Y}(t)\mathbf{D}\mathbf{M}\mathbf{A}_3 &= \tfrac{F_{31}}{V_1}\mathbf{Y}(t)\mathbf{M}\mathbf{A}_1 + \tfrac{F_{32}}{V_2}\mathbf{Y}(t)\mathbf{M}\mathbf{A}_2 - \tfrac{F_{13}}{V_3}\mathbf{Y}(t)\mathbf{M}\mathbf{A}_3. \end{aligned} \tag{2.6}$$

By using the collocation points (2.1) in (2.6), the system (2.6) is expressed as

$$
\begin{aligned}
\mathbf{Y}(t_0)\mathbf{DMA}_1 &= \tfrac{F_{13}}{V_3}\mathbf{Y}(t_0)\mathbf{MA}_3 + p(t_0) - \tfrac{F_{31}}{V_1}\mathbf{Y}(t_0)\mathbf{MA}_1 - \tfrac{F_{21}}{V_1}\mathbf{Y}(t_0)\mathbf{MA}_1\\
\mathbf{Y}(t_0)\mathbf{DMA}_2 &= \tfrac{F_{21}}{V_1}\mathbf{Y}(t_0)\mathbf{MA}_1 - \tfrac{F_{32}}{V_2}\mathbf{Y}(t_0)\mathbf{MA}_2\\
\mathbf{Y}(t_0)\mathbf{DMA}_3 &= \tfrac{F_{31}}{V_1}\mathbf{Y}(t_0)\mathbf{MA}_1 + \tfrac{F_{32}}{V_2}\mathbf{Y}(t_0)\mathbf{MA}_2 - \tfrac{F_{13}}{V_3}\mathbf{Y}(t_0)\mathbf{MA}_3\\
\mathbf{Y}(t_1)\mathbf{DMA}_1 &= \tfrac{F_{13}}{V_3}\mathbf{Y}(t_1)\mathbf{MA}_3 + p(t_1) - \tfrac{F_{31}}{V_1}\mathbf{Y}(t_1)\mathbf{MA}_1 - \tfrac{F_{21}}{V_1}\mathbf{Y}(t_1)\mathbf{MA}_1\\
\mathbf{Y}(t_1)\mathbf{DMA}_2 &= \tfrac{F_{21}}{V_1}\mathbf{Y}(t_1)\mathbf{MA}_1 - \tfrac{F_{32}}{V_2}\mathbf{Y}(t_1)\mathbf{MA}_2\\
\mathbf{Y}(t_1)\mathbf{DMA}_3 &= \tfrac{F_{31}}{V_1}\mathbf{Y}(t_1)\mathbf{MA}_1 + \tfrac{F_{32}}{V_2}\mathbf{Y}(t_1)\mathbf{MA}_2 - \tfrac{F_{13}}{V_3}\mathbf{Y}(t_1)\mathbf{MA}_3\\
&\ \,\vdots\\
\mathbf{Y}(t_N)\mathbf{DMA}_1 &= \tfrac{F_{13}}{V_3}\mathbf{Y}(t_N)\mathbf{MA}_3 + p(t_N) - \tfrac{F_{31}}{V_1}\mathbf{Y}(t_N)\mathbf{MA}_1 - \tfrac{F_{21}}{V_1}\mathbf{Y}(t_N)\mathbf{MA}_1\\
\mathbf{Y}(t_N)\mathbf{DMA}_2 &= \tfrac{F_{21}}{V_1}\mathbf{Y}(t_N)\mathbf{MA}_1 - \tfrac{F_{32}}{V_2}\mathbf{Y}(t_N)\mathbf{MA}_2\\
\mathbf{Y}(t_N)\mathbf{DMA}_3 &= \tfrac{F_{31}}{V_1}\mathbf{Y}(t_N)\mathbf{MA}_1 + \tfrac{F_{32}}{V_2}\mathbf{Y}(t_N)\mathbf{MA}_2 - \tfrac{F_{13}}{V_3}\mathbf{Y}(t_N)\mathbf{MA}_3.
\end{aligned}
\tag{2.7}
$$

On the other hand, the matrix relations for the initial conditions (1.2) are written as

$$
\begin{aligned}
\mathbf{Y}(0)\mathbf{MA}_1 &= \mu_1\\
\mathbf{Y}(0)\mathbf{MA}_2 &= \mu_2\\
\mathbf{Y}(0)\mathbf{MA}_3 &= \mu_3.
\end{aligned}
\tag{2.8}
$$

The system (2.7) and the system (2.8) are combined as a single system. The obtained new system is solved by using a program created in MATLAB. The solution of this system gives the coefficients matrices A_1, A_2, A_3. By substituting these Boubaker coefficient matrices A_1, A_2, A_3 in (2.4), the Boubaker polynomial solutions $y_1(t), y_2(t), y_3(t)$ are obtained.

3. Error Estimation Method

In this section, an error problem based on the approximate solution is presented. This error problem is solved according to the method in Section 2 and thus the estimated errors are obtained.

Let $y_k(t)$ $(k = 1,2,3)$ and $y_{k,N}(t)$ $(k = 1,2,3)$ be, respectively, the exact solutions and the Boubaker polynomial solutions of the problem (1.1).

Since the approximate solutions $y_{1,N}(t)$, $y_{2,N}(t)$, $y_{3,N}(t)$ provide the problem (1.1), the residual functions $R_{k,N}(t)(k = 1,2,3)$ are represented by

$$
\begin{aligned}
R_{1,N}(t) &= y'_{1,N}(t) - \tfrac{F_{13}}{V_3}y_{3,N}(t) - p(t) + \tfrac{F_{31}}{V_1}y_{1,N}(t) + \tfrac{F_{21}}{V_1}y_{1,N}(t)\\
R_{2,N}(t) &= y'_{2,N}(t) - \tfrac{F_{21}}{V_1}y_{1,N}(t) + \tfrac{F_{32}}{V_2}y_{2,N}(t)\\
R_{3,N}(t) &= y'_{3,N}(t) - \tfrac{F_{31}}{V_1}y_{1,N}(t) - \tfrac{F_{32}}{V_2}y_{2,N}(t) + \tfrac{F_{13}}{V_3}y_{3,N}(t).
\end{aligned}
\tag{3.1}
$$

Similarly, since the approximate solutions $y_{1,N}(t)$, $y_{2,N}(t)$, $y_{3,N}(t)$ provide the conditions (1.2), we have

$$
y_{1,N}(0) = \mu_1, \quad y_{2,N}(0) = \mu_2, \quad y_{3,N}(0) = \mu_3. \tag{3.2}
$$

Hence, it can be easily checked the accuracy of the Boubaker polynomial solutions by means of Eq. (3.1).

On the other hand, by subtracting the problem (3.1)-(3.2) from the problem (1.1)-(1.2), the error problem is obtained as

$$
\begin{aligned}
&e'_{1,N}(t) - \tfrac{F_{13}}{V_3}e_{3,N}(t) + \tfrac{F_{31}}{V_1}e_{1,N}(t) + \tfrac{F_{21}}{V_1}e_{1,N}(t) = -R_{1,N}(t)\\
&e'_{2,N}(t) - \tfrac{F_{21}}{V_1}e_{1,N}(t) + \tfrac{F_{32}}{V_2}e_{2,N}(t) = -R_{2,N}(t)\\
&e'_{3,N}(t) - \tfrac{F_{31}}{V_1}e_{1,N}(t) - \tfrac{F_{32}}{V_2}e_{2,N}(t) + \tfrac{F_{13}}{V_3}e_{3,N}(t) = -R_{3,N}(t)\\
&e_{1,N}(0) = 0, \quad e_{2,N}(0) = 0, \quad \text{and} \quad e_{3,N}(0) = 0,
\end{aligned}
\tag{3.3}
$$

where $e_{1,N}(t) = y_1(t) - y_{1,N}(t)$, $e_{2,N}(t) = y_2(t) - y_{2,N}(t)$ and $e_{3,N}(t) = y_3(t) - y_{3,N}(t)$.

Finally, the error problem (3.3) is solved according to the method in Section 2. Here, the estimated error functions are calculated for the M cut-off limit for $M > N$. In other words, the estimated error functions $e_{1,N,M}(t)$, $e_{2,N,M}(t)$ and $e_{3,N,M}(t)$ are expresssed in the form

$$e_{k,N,M}(t) = \sum_{i=0}^{M} a^*_{k,i} B_i(t), \quad (k = 1, 2, 3).$$

Here, $a^*_{k,i}$ are the new Boubaker coefficients for $M = 0, 1, ..., N$.

4. Application

In this section, the numerical methods in Sections 2 and 3 are applied to some numerical values of the parameters in problem (1.1)-(1.2) and also the obtained results are discussed in tables and graphs. All the computations are made using a program created for the method in MATLAB. In problem (1.1)-(1.2), $p(t) = 1+sin(t)$, $F_{13} = 38mi^3$/year, $F_{21} = 18mi^3$/year, $F_{31} = 20mi^3/year$, $F_{32} = 18mi^3/year$, $V_1 = 2900mi^3$, $V_2 = 850mi^3$, $V_3 = 1180mi^3$, $y_1(0) = 0$, $y_2(0) = 0$, $y_3(0) = 0$ are chosen. According to the selected parameters, the model becomes

$$\begin{cases} y_1'(t) = \frac{38}{1180} y_3(t) + (1 + sin(t)) - \frac{20}{2900} y_1(t) - \frac{18}{2900} y_1(t) & \\ y_2'(t) = \frac{18}{2900} y_1(t) - \frac{18}{850} y_2(t), & 0 \le t \le 1, \\ y_3'(t) = \frac{20}{2900} y_1(t) + \frac{18}{850} y_2(t) - \frac{38}{1180} y_3(t) & \\ y_1(0) = 0, \quad y_2(0) = 0, \quad y_3(0) = 0. & \end{cases} \tag{4.1}$$

Using (2.4) for $N = 3$, the approximate solutions $y_{k,3}(t)$ $(k = 1, 2, 3)$ of the problem (4.1) are sought in the form

$$y_{k,3}(t) = \mathbf{Y}(t)\mathbf{M}\mathbf{A}_k, \quad (k = 1, 2, 3), \tag{4.2}$$

where

$$\mathbf{Y}(t) = \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix}, \quad \mathbf{A}_k = \begin{bmatrix} a_{k,0} & a_{k,1} & a_{k,2} & a_{k,3} \end{bmatrix}^T, \quad \mathbf{M} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

From (2.1) for $N = 3$, the collocation points are computed as

$$t_0 = 0, \quad t_1 = \frac{1}{3}, \quad t_2 = \frac{2}{3}, \quad t_3 = 1. \tag{4.3}$$

By utilizing the collocation points (4.3) in (2.7) and (2.8) for the selected parameters, we have

$$\begin{aligned}
\mathbf{Y}(0)\mathbf{DMA}_1 &= \tfrac{F_{13}}{V_3}\mathbf{Y}(0)\mathbf{MA}_3 + p(0) - \tfrac{F_{31}}{V_1}\mathbf{Y}(0)\mathbf{MA}_1 - \tfrac{F_{21}}{V_1}\mathbf{Y}(0)\mathbf{MA}_1 \\
\mathbf{Y}(0)\mathbf{DMA}_2 &= \tfrac{F_{21}}{V_1}\mathbf{Y}(0)\mathbf{MA}_1 - \tfrac{F_{32}}{V_2}\mathbf{Y}(0)\mathbf{MA}_2 \\
\mathbf{Y}(0)\mathbf{DMA}_3 &= \tfrac{F_{31}}{V_1}\mathbf{Y}(0)\mathbf{MA}_1 + \tfrac{F_{32}}{V_2}\mathbf{Y}(0)\mathbf{MA}_2 - \tfrac{F_{13}}{V_3}\mathbf{Y}(0)\mathbf{MA}_3 \\
\mathbf{Y}(\tfrac{1}{3})\mathbf{DMA}_1 &= \tfrac{F_{13}}{V_3}\mathbf{Y}(\tfrac{1}{3})\mathbf{MA}_3 + p(\tfrac{1}{3}) - \tfrac{F_{31}}{V_1}\mathbf{Y}(\tfrac{1}{3})\mathbf{MA}_1 - \tfrac{F_{21}}{V_1}\mathbf{Y}(\tfrac{1}{3})\mathbf{MA}_1 \\
\mathbf{Y}(\tfrac{1}{3})\mathbf{DMA}_2 &= \tfrac{F_{21}}{V_1}\mathbf{Y}(\tfrac{1}{3})\mathbf{MA}_1 - \tfrac{F_{32}}{V_2}\mathbf{Y}(\tfrac{1}{3})\mathbf{MA}_2 \\
\mathbf{Y}(\tfrac{1}{3})\mathbf{DMA}_3 &= \tfrac{F_{31}}{V_1}\mathbf{Y}(\tfrac{1}{3})\mathbf{MA}_1 + \tfrac{F_{32}}{V_2}\mathbf{Y}(\tfrac{1}{3})\mathbf{MA}_2 - \tfrac{F_{13}}{V_3}\mathbf{Y}(\tfrac{1}{3})\mathbf{MA}_3 \\
\mathbf{Y}(\tfrac{2}{3})\mathbf{DMA}_1 &= \tfrac{F_{13}}{V_3}\mathbf{Y}(\tfrac{2}{3})\mathbf{MA}_3 + p(\tfrac{2}{3}) - \tfrac{F_{31}}{V_1}\mathbf{Y}(\tfrac{2}{3})\mathbf{MA}_1 - \tfrac{F_{21}}{V_1}\mathbf{Y}(\tfrac{2}{3})\mathbf{MA}_1 \\
\mathbf{Y}(\tfrac{2}{3})\mathbf{DMA}_2 &= \tfrac{F_{21}}{V_1}\mathbf{Y}(\tfrac{2}{3})\mathbf{MA}_1 - \tfrac{F_{32}}{V_2}\mathbf{Y}(\tfrac{2}{3})\mathbf{MA}_2 \\
\mathbf{Y}(\tfrac{2}{3})\mathbf{DMA}_3 &= \tfrac{F_{31}}{V_1}\mathbf{Y}(\tfrac{2}{3})\mathbf{MA}_1 + \tfrac{F_{32}}{V_2}\mathbf{Y}(\tfrac{2}{3})\mathbf{MA}_2 - \tfrac{F_{13}}{V_3}\mathbf{Y}(\tfrac{2}{3})\mathbf{MA}_3 \\
\mathbf{Y}(1)\mathbf{DMA}_1 &= \tfrac{F_{13}}{V_3}\mathbf{Y}(1)\mathbf{MA}_3 + p(1) - \tfrac{F_{31}}{V_1}\mathbf{Y}(1)\mathbf{MA}_1 - \tfrac{F_{21}}{V_1}\mathbf{Y}(1)\mathbf{MA}_1 \\
\mathbf{Y}(1)\mathbf{DMA}_2 &= \tfrac{F_{21}}{V_1}\mathbf{Y}(1)\mathbf{MA}_1 - \tfrac{F_{32}}{V_2}\mathbf{Y}(1)\mathbf{MA}_2 \\
\mathbf{Y}(1)\mathbf{DMA}_3 &= \tfrac{F_{31}}{V_1}\mathbf{Y}(1)\mathbf{MA}_1 + \tfrac{F_{32}}{V_2}\mathbf{Y}(1)\mathbf{MA}_2 - \tfrac{F_{13}}{V_3}\mathbf{Y}(1)\mathbf{MA}_3
\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}
\mathbf{Y}(0)\mathbf{MA}_1 &= 0 \\
\mathbf{Y}(0)\mathbf{MA}_2 &= 0 \\
\mathbf{Y}(0)\mathbf{MA}_3 &= 0,
\end{aligned} \tag{4.5}$$

where

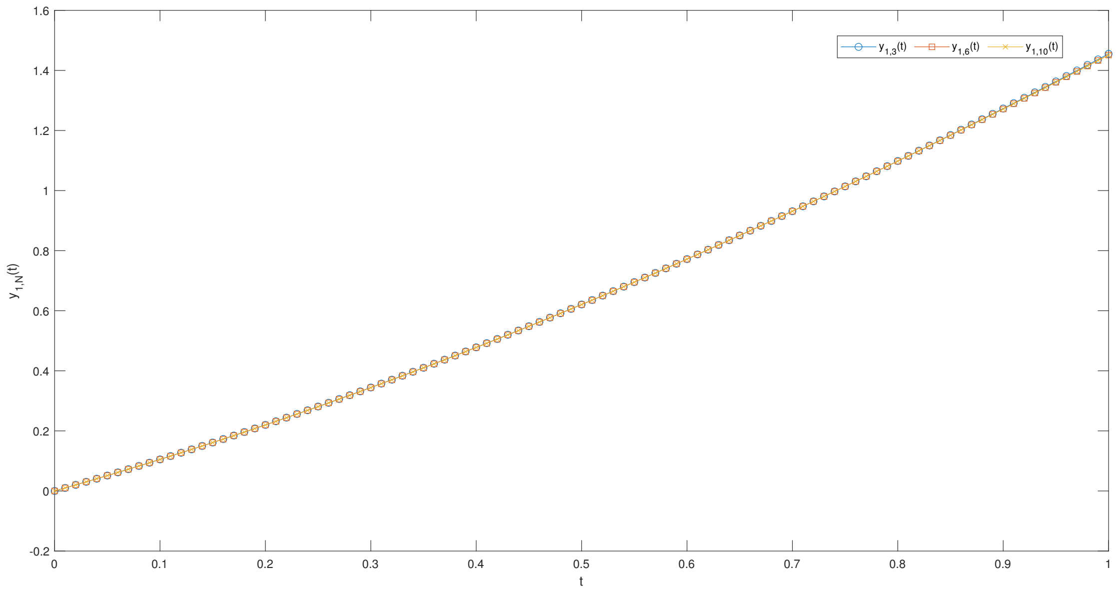

FIGURE 2. Graphics of the Boubaker polynomial solutions of model (4.1) obtained using $N = 3, 6, 10$ corresponding to $y_{1,N}(t)$

$$\mathbf{Y}(0) = [\ 1 \quad 0 \quad 0 \quad 0\ ], \quad \mathbf{Y}\left(\frac{1}{3}\right) = \left[\ 1 \quad \tfrac{1}{3} \quad \tfrac{1}{9} \quad \tfrac{1}{27}\ \right], \quad \mathbf{Y}\left(\frac{2}{3}\right) = \left[\ 1 \quad \tfrac{2}{3} \quad \tfrac{4}{9} \quad \tfrac{8}{27}\ \right], \quad \mathbf{Y}(1) = [\ 1 \quad 1 \quad 1 \quad 1\ ],$$

$$\mathbf{A}_1 = [\ a_{1,0} \quad a_{1,1} \quad a_{1,2} \quad a_{1,3}\ ]^T, \quad \mathbf{A}_2 = [\ a_{2,0} \quad a_{2,1} \quad a_{2,2} \quad a_{2,3}\ ]^T, \quad \mathbf{A}_3 = [\ a_{3,0} \quad a_{3,1} \quad a_{3,2} \quad a_{3,3}\ ]^T,$$

$$\mathbf{D}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The system (4.4) and the system (4.5) are combined as a single system. The obtained new system is solved and the coefficients matrices are found as below

$$\begin{aligned} \mathbf{A}_1 &= [\ -1.0223 \quad 1.056 \quad 0.51117 \quad -0.055969\ ]^T, \\ \mathbf{A}_2 &= [\ -0.0062884 \quad -0.00091315 \quad 0.0031442 \quad 0.00091315\ ]^T, \\ \mathbf{A}_3 &= [\ -0.0069854 \quad -0.0010266 \quad 0.0034927 \quad 0.0010266\ ]^T. \end{aligned}$$

By substituting these Boubaker coefficient matrices A_1, A_2, A_3 in (4.2), the Boubaker polynomial solutions are calculated as

$$\begin{aligned} y_{1,3}(t) &= -0.0559691071449t^3 + 0.511169853145t^2 + t \\ y_{2,3}(t) &= 0.000913154523724t^3 + 0.00314419625898t^2 + 2.64697796017e - 23 \\ y_{3,3}(t) &= 0.00102656683731t^3 + 0.00349268868257t^2 - 2.64697796017e - 23. \end{aligned}$$

In Figure 2, the Boubaker polynomial solutions $y_{1,N}(t)$ of model (4.1) are shown for $N = 3, N = 6, N = 10$. In Figure 3, the Boubaker polynomial solutions $y_{2,N}(t)$ of model (4.1) are given for $N = 3, N = 6, N = 10$. In Figure 4, the Boubaker polynomial solutions $y_{3,N}(t)$ of model (4.1) are presented for $N = 3, N = 6, N = 10$.

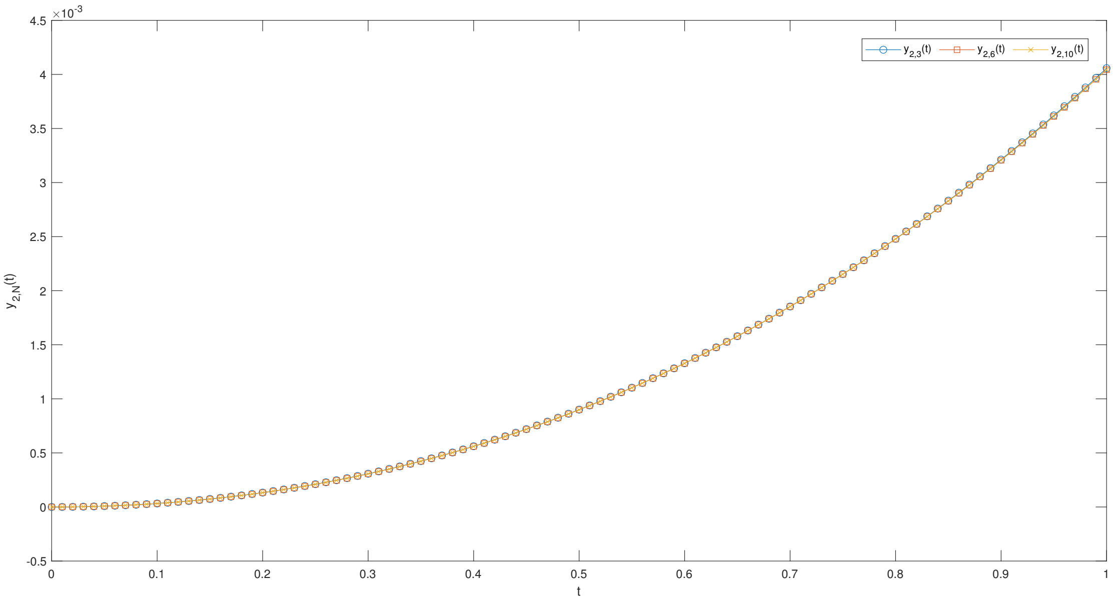

FIGURE 3. Graphics of the Boubaker polynomial solutions of model (4.1) obtained using $N = 3, 6, 10$ corresponding to $y_{2,N}(t)$

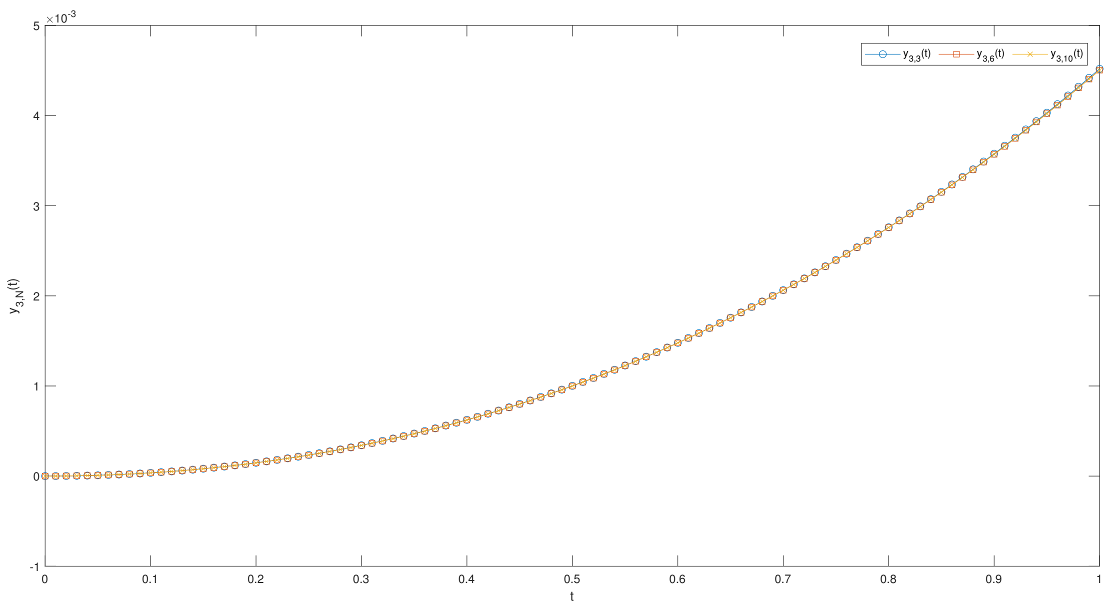

FIGURE 4. Graphics of the Boubaker polynomial solutions of model (4.1) obtained using $N = 3, 6, 10$ corresponding to $y_{3,N}(t)$

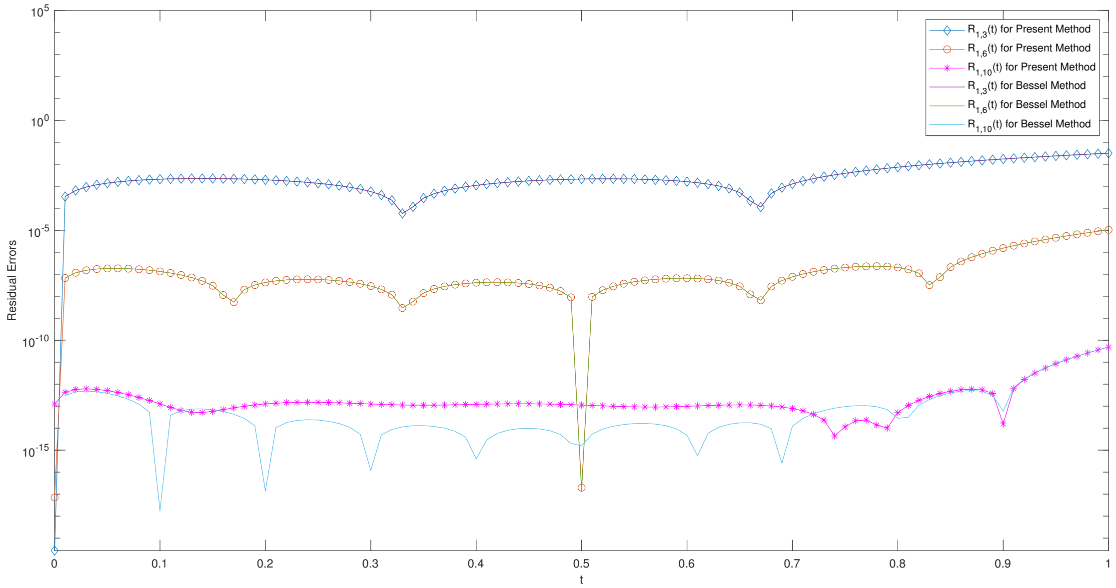

FIGURE 5. Comparison of the residual error functions $R_{1,N}(t)$ with Bessel collocation method [31] for $N = 3$, $N = 6$, $N = 10$

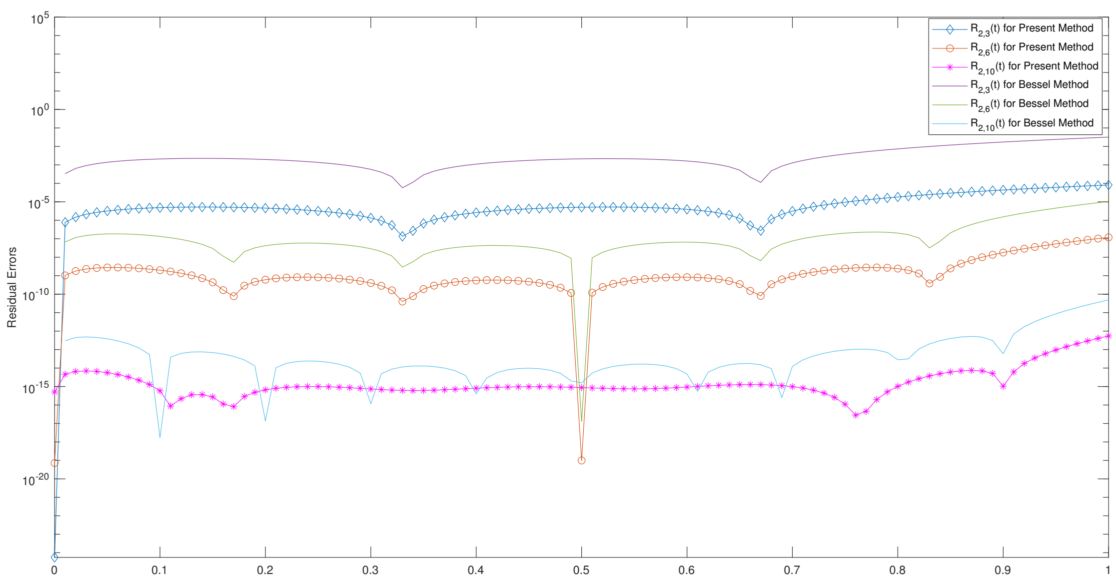

FIGURE 6. Comparison of the residual error functions $R_{2,N}(t)$ with Bessel collocation method [31] for $N = 3$, $N = 6$, $N = 10$

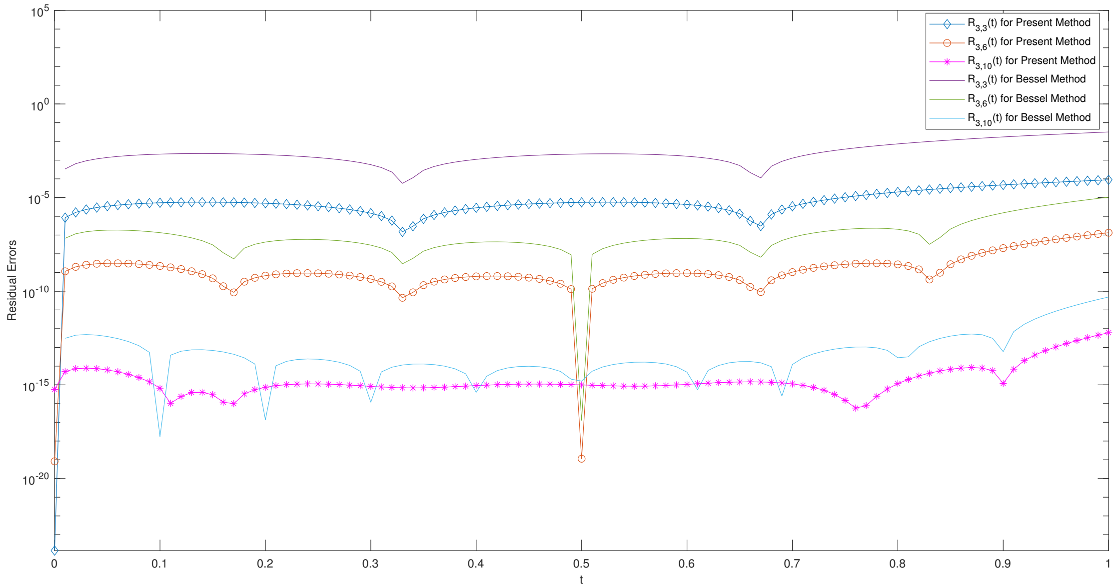

FIGURE 7. Comparison of the residual error functions $R_{3,N}(t)$ with Bessel collocation method [31] for $N = 3$, $N = 6$, $N = 10$

TABLE 1. Comparison of the residual absolute errors $|R_{1,N}(t)|$, $|R_{2,N}(t)|$, $|R_{3,N}(t)|$ corresponding to $N = 3$, $N = 6$, $N = 10$ in model (4.1)

	$\|R_{1,N}(t)\|$			$\|R_{2,N}(t)\|$			$\|R_{3,N}(t)\|$		
t_i	$N=3$	$N=6$	$N=10$	$N=3$	$N=6$	$N=10$	$N=3$	$N=6$	$N=10$
0.2	1.9603e-03	4.2273e-08	1.2676e-13	4.5638e-06	6.0769e-010	6.6115e-16	4.9742e-06	6.7800e-010	7.4546e-16
0.4	1.1016e-03	4.1197e-008	1.2107e-13	2.6079e-06	5.5439e-010	8.0919e-16	2.8424e-006	6.1853e-010	9.1416e-16
0.6	1.6179e-03	6.5658e-008	9.8588e-14	3.9118e-06	8.3158e-010	9.3706e-16	4.2636e-006	9.2779e-010	1.0639e-15
0.8	7.3601e-03	2.0293e-007	5.0737e-14	1.8255e-05	2.4308e-009	1.0197e-15	1.9897e-005	2.7120e-009	1.1667e-15
1	3.1884e-02	1.0456e-005	4.8928e-11	8.1496e-05	1.1899e-007	5.4596e-13	8.8826e-005	1.3275e-007	6.0929e-13

TABLE 2. Comparison of the estimated absolute errors $|e_{1,N,M}(t)|$, $|e_{2,N,M}(t)|$, $|e_{3,N,M}(t)|$ corresponding to $(N, M) = (3, 4)$, $(N, M) = (6, 7)$, $(N, M) = (10, 11)$

	$\|e_{1,N,M}(t)\|$			$\|e_{2,N,M}(t)\|$			$\|e_{3,N,M}(t)\|$		
t_i	$(N,M)=(3,4)$	$(N,M)=(6,7)$	$(N,M)=(10,11)$	$(N,M)=(3,4)$	$(N,M)=(6,7)$	$(N,M)=(10,11)$	$(N,M)=(3,4)$	$(N,M)=(6,7)$	$(N,M)=(10,11)$
0.2	3.3675e-04	1.9604e-08	4.9923e-14	1.1043e-06	2.4969e-10	3.7808e-16	1.2115e-06	2.7906e-10	4.2427e-16
0.4	4.4368e-04	1.5631e-08	7.5458e-14	1.8459e-06	1.7380e-10	4.5382e-16	2.0331e-06	1.9481e-10	5.1173e-16
0.6	6.7597e-05	1.4350e-08	9.7390e-14	1.2837e-06	1.4038e-10	5.1669e-16	1.4304e-06	1.5793e-10	5.8611e-16
0.8	4.2440e-04	3.0324e-08	1.0911e-13	2.3650e-06	3.1902e-10	5.3819e-16	2.6157e-06	3.5757e-10	6.1630e-16
1	4.1991e-03	4.6473e-07	1.1463e-12	1.1925e-05	5.9402e-09	1.4433e-14	1.3047e-05	6.6249e-09	1.6083e-14

Figure 5 compares the residual error functions $R_{1,N}(t)$ of the model (4.1) with Bessel collocation method [31] for $N = 3, N = 6, N = 10$. Figure 6 shows the comparison of the residual error functions $R_{2,N}(t)$ of model (4.1) with the Bessel collocation method [31] for $N = 3, N = 6, N = 10$. Figure 7 compares the residual error functions $R_{3,N}(t)$ of model (4.1) with the Bessel collocation method [31] for $N = 3, N = 6, N = 10$.

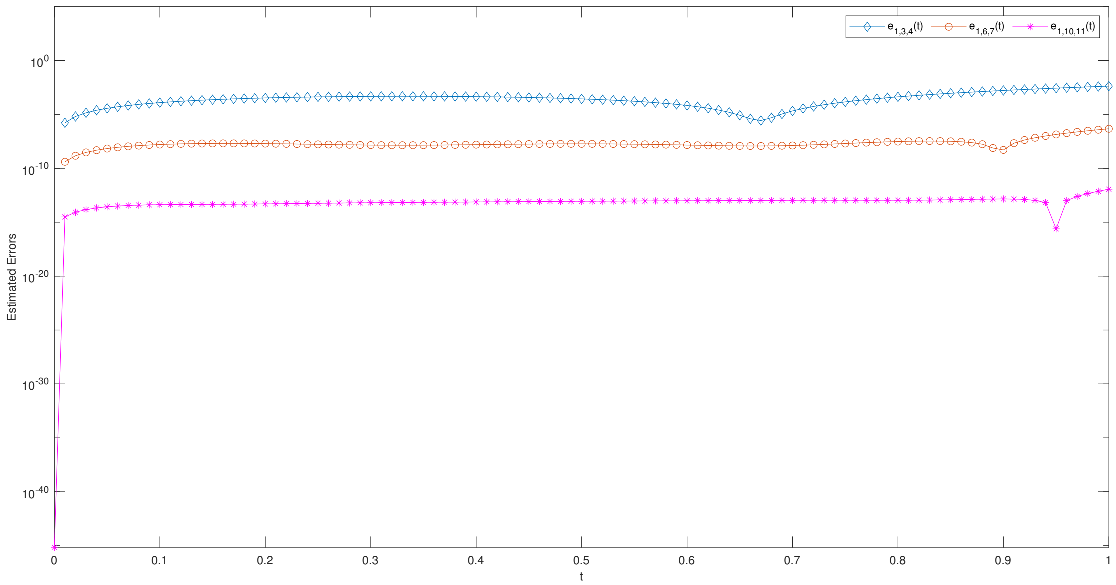

FIGURE 8. Comparison of the estimated error functions $e_{1,N,M}(t)$ for $(N,M) = (3,4)$, $(N,M) = (6,7)$, $(N,M) = (10,11)$

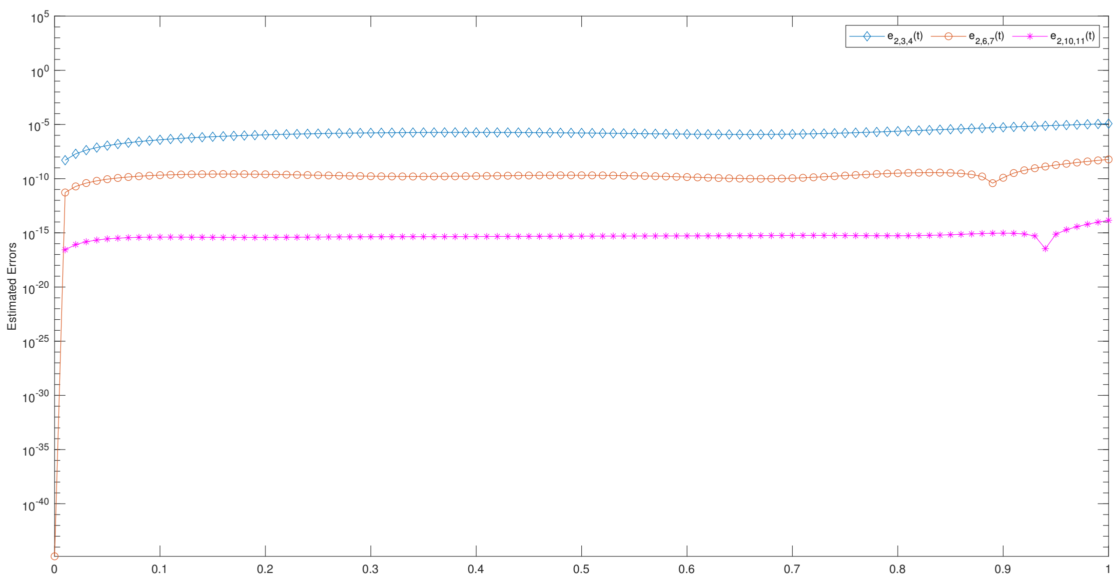

FIGURE 9. Comparison of the estimated error functions $e_{2,N,M}(t)$ for $(N,M) = (3,4)$, $(N,M) = (6,7)$, $(N,M) = (10,11)$

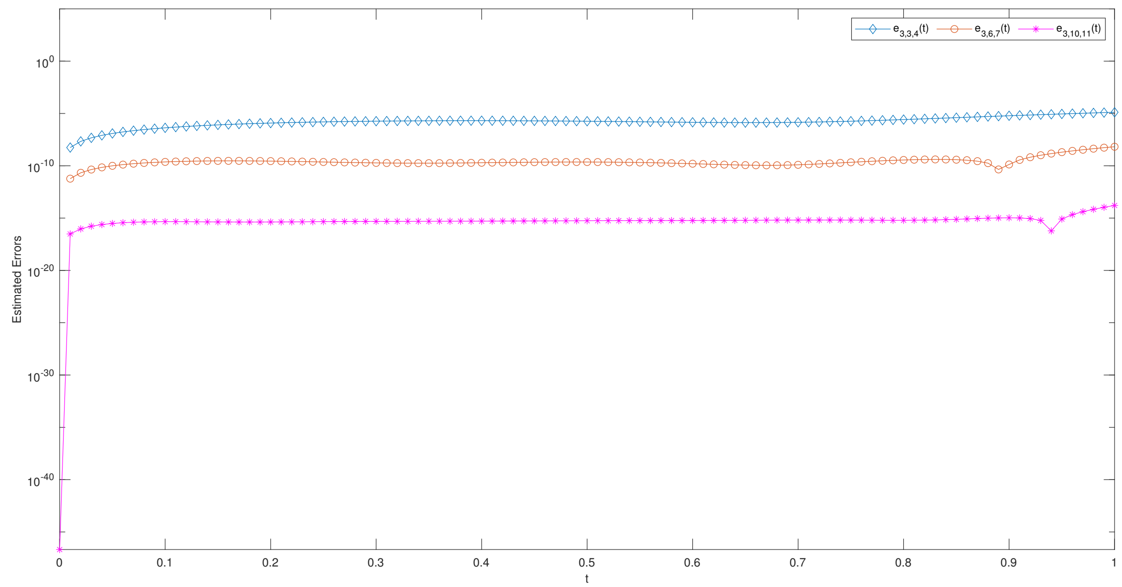

Figure 10. Comparison of the estimated error functions $e_{3,N,M}(t)$ for $(N, M) = (3, 4)$, $(N, M) = (6, 7)$, $(N, M) = (10, 11)$

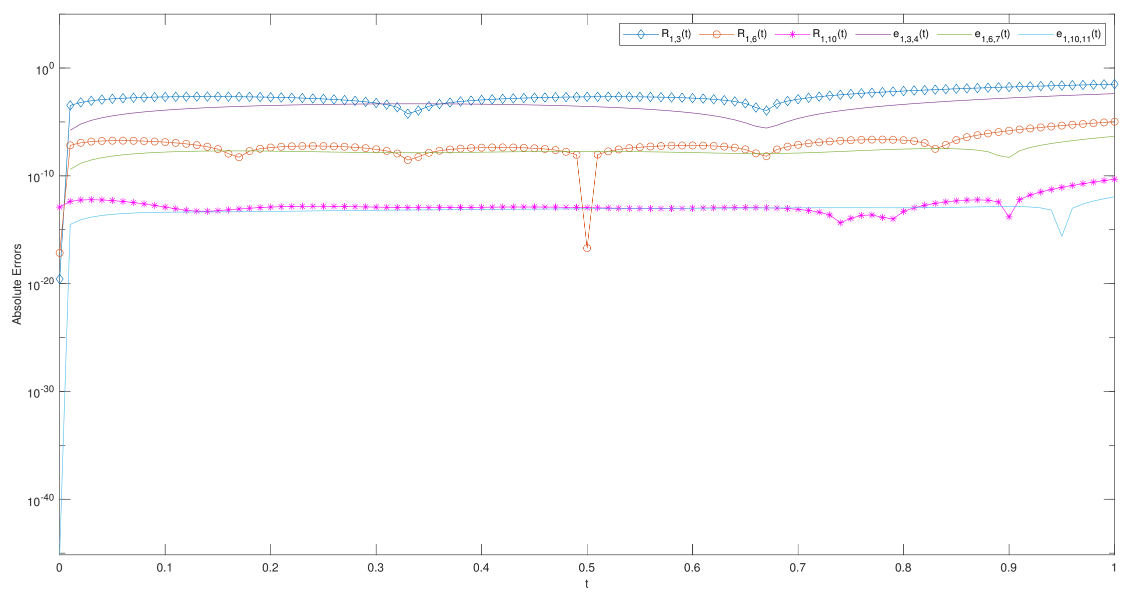

Figure 11. Comparison of the estimated error functions $e_{1,N,M}(t)$ with the residual error functions $R_{1,N}(t)$ for $(N, M) = (3, 4)$, $(N, M) = (6, 7)$, $(N, M) = (10, 11)$

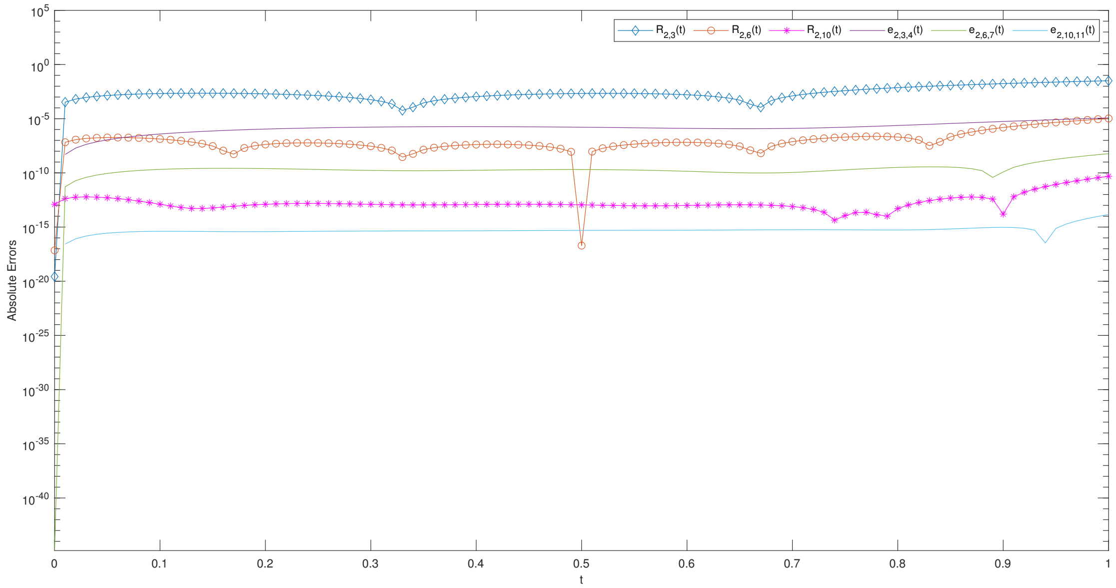

FIGURE 12. Comparison of the estimated error functions $e_{2,N,M}(t)$ with the residual error functions $R_{2,N}(t)$ for $(N, M) = (3, 4)$, $(N, M) = (6, 7)$, $(N, M) = (10, 11)$

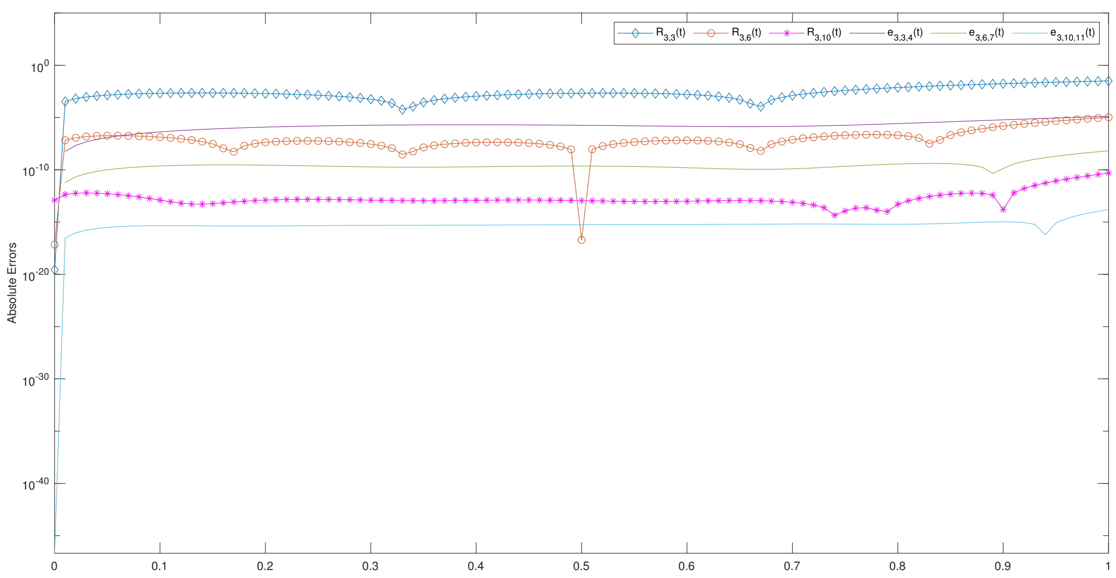

FIGURE 13. Comparison of the estimated error functions $e_{3,N,M}(t)$ with the residual error functions $R_{3,N}(t)$ for $(N, M) = (3, 4)$, $(N, M) = (6, 7)$, $(N, M) = (10, 11)$

In Table 1, some values of $|R_{1,N}(t)|, |R_{2,N}(t)|, |R_{3,N}(t)|$ of the model (4.1) are given for $N = 3, N = 6, N = 10$. In Table 2, some values of $|e_{1,N,M}(t)|, |e_{2,N,M}(t)|, |e_{3,N,M}(t)|$ of the model (4.1) are given for $(N, M) = (3, 4)$, $(N, M) = (6, 7)$, $(N, M) = (10, 11)$.

The estimated error functions $e_{1,N,M}(t), e_{2,N,M}(t)$ and $e_{3,N,M}(t)$ of the model (4.1) are, respectively, visualized for $(N, M) = (3, 4)$, $(N, M) = (6, 7)$, $(N, M) = (10, 11)$ in Figures 8,9 and 10.

In Figures 11,12 and 13 the residual error functions $R_{1,N}(t), R_{2,N}(t)$ and $R_{3,N}(t)$ and the estimated error functions $e_{1,N,M}(t), e_{2,N,M}(t)$ and $e_{3,N,M}(t)$ of the model (4.1) are, respectively, compared for $(N, M) = (3, 4)$, $(N, M) = (6, 7)$, $(N, M) = (10, 11)$.

According to Figures 2, 3, 4, it can be said that the Boubaker polynomial solutions $y_{1,N}(t), y_{2,N}(t), y_{3,N}(t)$ of model (4.1) give similar results for $N = 3, N = 6, N = 10$. It is said that the pollution in the lakes increases when the Boubaker polynomial solutions $y_{1,N}(t), y_{2,N}(t), y_{3,N}(t)$ are examined. Also, while Lake 3 gets the most pollution, the least pollution happens in Lake 1.

According to the comparison made for $R_{1,N}(t)$ in Figure 5, similar results are obtained with the Bessel collocation method [31] for $N = 3$ and $N = 6$ but it can be said that the Bessel collocation method [31] for $N = 10$ is more successful at some points. However, for $N = 3, N = 6$ and $N = 10$ in Figure 6 and Figure 7, the present method is more accurate than the Bessel collocation method [31]. Thus, in general, it is said that our method is more successful.

According to Table 1, it is observed that the residual absolute errors $|R_{1,N}(t)|, |R_{2,N}(t)|, |R_{3,N}(t)|$ decrease as the value of N increases. According to Table 2, it is commented that the estimated absolute errors $|e_{1,N,M}(t)|$, $|e_{2,N,M}(t)|$, $|e_{3,N,M}(t)|$ decrease as the values of (N,M) increases. This result is also seen from Figures 8, 9, 10.

According to Figures 11, 12, 13, it is observed that the estimated errors $e_{1,N,M}(t)$, $e_{2,N,M}(t)$, $e_{3,N,M}(t)$ of model (4.1) give better results than the residual errors for $(N, M) = (3, 4)$, $(N, M) = (6, 7)$, $(N, M) = (10, 11)$.

5. Conclusion

In this paper, a collocation method based on the Boubaker polynomials is presented for the approximate solutions of problem of pollutions of three lakes with interconnecting channels. Parameter values and known function $p(t)$ in the method are chosen as $p(t) = 1 + sin(t), F_{13} = 38mi^3/year, F_{21} = 18mi^3/year, F_{31} = 20mi^3/year, F_{32} = 18mi^3/year, V_1 = 2900mi^3, V_2 = 850mi^3, V_3 = 1180mi^3, y_1(0) = 0, y_2(0) = 0, y_3(0) = 0$. The application of the method is made by using a program written in MATLAB. The comparisons are also made with the Bessel collocation method [31] in the literature. Application results are shown in tables and graphs. According to the results, it can be said that the present method is more successful than the Bessel collocation method [31]. In addition, according to the method, it is seen that fewer errors are obtained as the values of (N,M) increase. According to Figures 11, 12, 13, it is seen that the error estimation method is successful. According to Figures Figures 2, 3, 4, it is observed that the pollution in the lakes is increased. Besides, while Lake 3 gets the most pollution, the least pollution happens in Lake 1. It can also be extended to nonlinear problems of similar type in a future work.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The authors have read and agreed to the published version of the manuscript.

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