Turk. J. Math. Comput. Sci.
15(2)(2023) 212–217
© MatDer
DOI : 10.47000/tjmcs.1167848

Some New Identities for Fuzzy Fibonacci Number

Merve Güney Duman

Fundamental Sciences of Engineering, Faculty of Technology, Sakarya University of Applied Sciences, Sakarya, Türkiye.

Received: 28-08-2022 • Accepted: 26-12-2022

Abstract. With fuzzy logic, the concept of belonging to the sets has been replaced by the concept of how much they belong to the set. The subject of fuzzy sets was discussed and fuzzy logic studies have been done in many fields. Recently, the Fuzzy Fibonacci number sequence was defined using Fibonacci number sequences. In this study, we give some new properties related to fuzzy Fibonacci numbers.

2010 AMS Classification: 11B39, 11B83, 03E72, 94D05

Keywords: Fibonacci number, fuzzy number, fuzzy set, fuzzy Fibonacci number.

1. Introduction

In a complex problem, the use of classical set is not sufficient. In this set, a given element x is either a member of the set or not an element of the set, that is, its membership function is either 1 or 0, respectively. In classic sets, since the membership functions of classical sets take only 0 and 1 values, the studies were limited. In 1965, a work on fuzzy sets was published by Zadeh and so membership degree could take values in the range $[0, 1]$ to indicate 0 lack of membership and 1 full membership [24]. As the membership degree gets closer to 1, the membership of x to set A increases [24]. In other words, if the value of a set is in the range of $[0, 1]$, the set A is called a fuzzy set and is denoted by $\tilde{A}$. Thus, it became possible to operate with concepts such as few members or many members. With the fuzzy set theory put forward, a more flexible and regular structure has emerged. Many applications have been made on fuzzy sets. See [1–4, 6–13, 15–17, 19–24] for fuzzy set and its applications.

The membership function and membership degree of a set $\tilde{A}$ is denoted, respectively, by $\mu_{\tilde{A}}(x)$ or $\mu_{\tilde{A}}$, and $\mu_{\tilde{A}}(x) : A \to [0, 1]$. We can show a fuzzy set in the list form as follows:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in A\},$$

where an object x and its membership degree $\mu_{\tilde{A}} \in [0, 1]$ in a set $\tilde{A}$.

There are many fuzzy membership function (MF) types in use today. The most commonly used fuzzy membership function is triangular MF, trapezoidal MF, Gaussian MF, Generalized Bell MF. We will use the triangular membership function in this study. Moreover, fuzzy operations on fuzzy sets are defined as the crisp operations performed on crisp sets. Operations on fuzzy sets are done using fuzzy membership functions. Operations such as addition, subtraction, multiplication, and division are defined on fuzzy set.

When fuzzy set operations are applied to a set, the result is a fuzzy set. But these sets need to be converted to a real number, that is, an inference must be made. This process is called defuzzification, which means the inversion of fuzzification [19]. There are many different methods used for defuzzification. For example, maximum-membership

Email address: merveduman@subu.edu.tr (M. Güney Duman)

principle, centroid method, weighted average method, mean-maximum membership, centre of sums etc. For more information on fuzzy or defuzzy, see [1, 15, 17] etc.. We will use the maximum-membership principle in this study. The defuzzification crisp value found by this method is the fuzzy output with the highest value membership function in the fuzzy set. To use this method, the fuzzy set must have peak values. If a fuzzy set has a single peak value, it is the most used defuzzification method. In [15], the defuzzified value z^* according to the maximum-membership principle is given by

$$\mu(z^*) \geq \mu(z) \text{ for all } z.$$

In this study, we will give some new identities for Fibonacci fuzzy numbers. Firstly, we will give some basic properties of fuzzy numbers. Later, we will give the known properties and definition of Fuzzy Fibonacci numbers. In recent years, fuzzy numbers have been defined to express numbers such as "*about 5, more than 5, less than 5*". In [7], Dubois and Prade defined fuzzy number which is a fuzzy subset of the real line. Later, in [6, 8, 9], the same authors developed fuzzy numbers and introduced the LR model and some formulas for fuzzy operations. In [10], Gao et al have given some arithmetic operations on triangular fuzzy numbers by the α-cuts which are intervals. Arithmetic operations on fuzzy numbers are usually done with the direct use of the membership function or the use of the α-cut function. With the α-cut approach, a much larger family of fuzzy numbers is obtained than with the standard LR model.

2. Preliminaries

In this section, we will give definitions for Fuzzy numbers, Fuzzy Fibonacci numbers and operations on these numbers.

If a fuzzy set is convex and normalized, and its membership function is defined in R and piecewise continuous, then it is called a fuzzy number. There are various shapes of fuzzy numbers, and the most popular of them is the triangular fuzzy number. It is a fuzzy number represented by three dots. The triangular membership function with $\widetilde{A} = (a_1, a_{2,}a_3)$ is as follows:

$$\mu_{\widetilde{A}}(x) = \begin{cases} 0 & x \leq a_1, \\ \frac{x-a_1}{a_2-a_1} & a_1 < x \leq a_2, \\ \frac{a_3-x}{a_3-a_2} & a_2 < x \leq a_3, \\ 0 & x > a_3. \end{cases}$$

If the α-cut operation is applied to the triangular fuzzy number $\widetilde{A} = (a_1, a_{2,}a_3)$, then the form indicated by

$$A^\alpha = [a_1^\alpha, a_3^\alpha] = [a_1 + \alpha(a_2 - a_1), a_3 - \alpha(a_3 - a_2)]$$

is obtained with $\alpha = [0, 1]$ and $a_1^\alpha, a_2^\alpha, a_3^\alpha \in \mathbb{R}$. Let $\widetilde{A} = (a_1, a_{2,}a_3)$ and $\widetilde{B} = (b_1, b_{2,}b_3)$ be the triangular fuzzy numbers and $A^\alpha = [a_1^\alpha, a_3^\alpha]$ and $B^\alpha = [b_1^\alpha, b_3^\alpha]$ be the $\alpha-$cuts obtained from these numbers. Then,

$$\begin{aligned} A^\alpha + B^\alpha &= [a_1^\alpha + b_1^\alpha, a_3^\alpha + b_3^\alpha], \\ A^\alpha - B^\alpha &= [a_1^\alpha - b_3^\alpha, a_3^\alpha - b_1^\alpha], && (2.1) \\ A^\alpha \cdot B^\alpha &= [\min\{a_1^\alpha \cdot b_1^\alpha, a_3^\alpha \cdot b_3^\alpha, a_3^\alpha \cdot b_1^\alpha, a_1^\alpha \cdot b_3^\alpha\}, \max\{a_1^\alpha \cdot b_1^\alpha, a_3^\alpha \cdot b_3^\alpha, a_3^\alpha \cdot b_1^\alpha, a_1^\alpha \cdot b_3^\alpha\}], && (2.2) \\ k \cdot A^\alpha &= [k \cdot a_1^\alpha, k \cdot a_3^\alpha] \text{ where } k \in \mathbb{R}^+. \end{aligned}$$

Let $k, n \in Z$. The sequences of Fibonacci numbers denoted by $(F_k)_{k\geq 0}$ is given by

$$F_0 = 0,\ F_1 = 1;\ F_k = F_{k-1} + F_{k-2} \text{ for } k \geq 2. \tag{2.3}$$

The following identities are well known (see [18]).

$$\begin{aligned} F_n &= F_{n+2} - F_{n+1}, && (2.4) \\ 3F_{n+2} &= F_{n+4} + F_n, \\ F_{n+10} &= 11F_{n+5} + F_n, \\ F_{n+1} \cdot F_{n-1} - F_n^2 &= (-1)^n. && (2.5) \end{aligned}$$

The triangular membership function of $\widetilde{F_n} = (F_{n-1}, F_n, F_{n+1})$ is

$$\mu_{\widetilde{F_n}}(x) = \begin{cases} 0 & x \leq F_{n-1}, \\ \frac{x-F_{n-1}}{F_n-F_{n-1}} & F_{n-1} < x \leq F_n, \\ \frac{F_{n+1}-x}{F_{n+1}-F_n} & F_n < x \leq F_{n+1}, \\ 0 & x > F_{n+1}, \end{cases}$$

where $n > 2$. Then,

$$\begin{aligned} F_n^\alpha &= [F_{n-1}^\alpha, F_{n+1}^\alpha] = [F_{n-1} + \alpha(F_n - F_{n-1}), F_{n+1} - \alpha(F_{n+1} - F_n)] \\ &= [F_{n-1} + \alpha F_{n-2}, F_{n+1} - \alpha F_{n-1}] \end{aligned} \tag{2.6}$$

is obtained with $\alpha = [0, 1]$.

From now on, we will assume $n > 2, n \in \mathbb{Z}^+$, $\widetilde{F_n} = (F_{n-1}, F_n, F_{n+1})$ and $F_n^\alpha = [F_{n-1}^\alpha, F_{n+1}^\alpha]$. Similar to the Definition 2.1 in [14], the following definition can be given for $\widetilde{F_n}$.

Definition 2.1. Fuzzy Fibonacci numbers are defined as

$$F_0^\alpha = [1-\alpha, 1-\alpha],\ F_1^\alpha = [\alpha, 1],\ F_{n+1}^\alpha = F_n^\alpha + F_{n-1}^\alpha \text{ for } n \geq 2 \tag{2.7}$$

where $\widetilde{F_n} = (F_{n-1}, F_n, F_{n+1})$.

It is true that

$$\begin{aligned} F_n^\alpha + F_{n-1}^\alpha &= [F_{n-1} + \alpha F_{n-2}, F_{n+1} - \alpha F_{n-1}] + [F_{n-2} + \alpha F_{n-3}, F_n - \alpha F_{n-2}] \\ &= [F_{n-1} + F_{n-2} + \alpha(F_{n-2} + F_{n-3}), F_{n+1} + F_n - \alpha(F_{n-1} + F_{n-2})] \\ &= [F_n + \alpha F_{n-1}, F_{n+2} - \alpha F_{n+1}] \\ &= F_{n+1}^\alpha. \end{aligned}$$

For more information on fuzzy numbers and Fibonacci numbers, see [5, 22].

3. Main Theorems

In this section, we will give some new identities and the Cassini formula for triangular fuzzy Fibonacci numbers. Since the recurrence relation is not provided according to the difference operator, the equations are written according to the sum operator. Additionally, although $F_n = F_{n+2} - F_{n+1}$ is true, $F_n^\alpha = F_{n+2}^\alpha - F_{n+1}^\alpha$ is not true.

Theorem 3.1. $F_{n+4}^\alpha + F_n^\alpha = 3F_{n+2}^\alpha$.

Proof.

$$\begin{aligned} F_{n+4}^\alpha + F_n^\alpha &= (F_{n+3}^\alpha + F_{n+2}^\alpha) + F_n^\alpha = (F_{n+2}^\alpha + F_{n+1}^\alpha) + F_{n+2}^\alpha + F_n^\alpha \\ &= 2F_{n+2}^\alpha + (F_{n+1}^\alpha + F_n^\alpha) = 3F_{n+2}^\alpha \end{aligned}$$

by using (2.7). □

Theorem 3.2. $F_{n+10}^\alpha = 11F_{n+5}^\alpha + F_n^\alpha$.

Proof.

$$\begin{aligned}
F^{\alpha}_{n+10} &= F^{\alpha}_{n+9} + F^{\alpha}_{n+8} = 2F^{\alpha}_{n+8} + F^{\alpha}_{n+7} = 3F^{\alpha}_{n+7} + 2F^{\alpha}_{n+6}\\
&= 5F^{\alpha}_{n+6} + 3F^{\alpha}_{n+5} = 8F^{\alpha}_{n+5} + 5F^{\alpha}_{n+4}\\
&= 8F^{\alpha}_{n+5} + 3F^{\alpha}_{n+4} + 2F^{\alpha}_{n+4}\\
&= 8F^{\alpha}_{n+5} + 3F^{\alpha}_{n+4} + (2F^{\alpha}_{n+3} + 2F^{\alpha}_{n+2})\\
&= 8F^{\alpha}_{n+5} + 3F^{\alpha}_{n+4} + 4F^{\alpha}_{n+2} + 2F^{\alpha}_{n+1}\\
&= 8F^{\alpha}_{n+5} + 3F^{\alpha}_{n+4} + 3F^{\alpha}_{n+2} + F^{\alpha}_{n+2} + 2F^{\alpha}_{n+1}\\
&= 8F^{\alpha}_{n+5} + 3F^{\alpha}_{n+4} + 3F^{\alpha}_{n+2} + F^{\alpha}_{n+1} + F^{\alpha}_{n} + 2F^{\alpha}_{n+1}\\
&= 8F^{\alpha}_{n+5} + 3F^{\alpha}_{n+4} + 3F^{\alpha}_{n+2} + 3F^{\alpha}_{n+1} + F^{\alpha}_{n}\\
&= 8F^{\alpha}_{n+5} + 3F^{\alpha}_{n+4} + 3F^{\alpha}_{n+3} + F^{\alpha}_{n}\\
&= 8F^{\alpha}_{n+5} + 3F^{\alpha}_{n+5} + F^{\alpha}_{n}\\
&= 11F^{\alpha}_{n+5} + F^{\alpha}_{n}
\end{aligned}$$

by using (2.7). □

Theorem 3.3. $F^{\alpha}_{n+2} - F^{\alpha}_{n+1} = (-F_n, F_n, 2F_{n+1})$, $(F^{\alpha}_n)^* = (F^{\alpha}_{n+2} - F^{\alpha}_{n+1})^*$ *and* $\mu_{(F^{\alpha}_{n+2} - F^{\alpha}_{n+1})}(x) = \mu_{(F^{\alpha}_{n+2} - F^{\alpha}_{n+1})}(x; -F_n, F_n, 2F_{n+1})$.

Proof.

$$\begin{aligned}
F^{\alpha}_{n+2} - F^{\alpha}_{n+1} &= [F_{n+1} + \alpha F_n, F_{n+3} - \alpha F_{n+1}] - [F_n + \alpha F_{n-1}, F_{n+2} - \alpha F_n]\\
&= [F_{n+1} - F_{n+2} + 2\alpha F_n, F_{n+3} - F_n - \alpha(F_{n+1} + F_{n-1})]\\
&= [-F_n + 2\alpha F_n, 2F_{n+1} - \alpha(F_{n+1} + F_{n-1})]
\end{aligned}$$

by using (2.1), (2.3), and (2.6). Then, from (2.6), we find

$$F^{0}_{n+2} - F^{0}_{n+1} = [-F_n, 2F_{n+1}]$$

for $\alpha = 0$ and

$$F^{1}_{n+2} - F^{1}_{n+1} = [F_n, F_n]$$

for $\alpha = 1$. So, it follows that

$$\mu_{(F^{\alpha}_{n+2} - F^{\alpha}_{n+1})}(x) = \mu_{(F^{\alpha}_{n+2} - F^{\alpha}_{n+1})}(x; -F_n, F_n, 2F_{n+1}).$$

This shows that

$$(F^{\alpha}_{n+2} - F^{\alpha}_{n+1})^* = F_n \tag{3.1}$$

by using maximum-membership principle. Similarly, it is seen that

$$F^{\alpha *}_{n} = F_n \tag{3.2}$$

by using maximum-membership principle since

$$\mu_{F^{\alpha}_n}(x) = \mu_{F^{\alpha}_n}(x; F_{n-1}, F_n, F_{n+1}].$$

Then, we can write

$$(F^{\alpha}_n)^* = (F^{\alpha}_{n+2} - F^{\alpha}_{n+1})^*$$

by (3.1) and (3.2). □

Note that $F^{\alpha}_{n+2} - F^{\alpha}_{n+1} = (-F_n, F_n, 2F_{n+1}) \neq F^{\alpha}_n = (F_{n-1}, F_n, F_{n+1})$.

Theorem 3.4. *If n is an even integer, then*

$$F^{\alpha}_{n+1} \cdot F^{\alpha}_{n-1} + F^{\alpha}_0 \cdot F^{\alpha}_2 = (F^{\alpha}_n)^2 + (F^{\alpha}_1)^2$$

and if n is an odd integer, then

$$F^{\alpha}_{n+1} \cdot F^{\alpha}_{n-1} + (F^{\alpha}_1)^2 = (F^{\alpha}_n)^2 + F^{\alpha}_0 \cdot F^{\alpha}_2.$$

Proof. Firstly, let n be an odd integer. Then, by using (2.1)-(2.7), we write

$$\begin{aligned}
F^{\alpha}_{n+1}\cdot F^{\alpha}_{n-1}+(F^{\alpha}_{1})^2 &= [F_n+\alpha F_{n-1}, F_{n+2}-\alpha F_n]\cdot[F_{n-2}+\alpha F_{n-3}, F_n-\alpha F_{n-2}]+[\alpha,1]\cdot[\alpha,1]\\
&= [F_n\cdot F_{n-2}+\alpha(F_n\cdot F_{n-3}+F_{n-1}\cdot F_{n-2})+\alpha^2 F_{n-1}\cdot F_{n-3},\\
&\quad F_{n+2}\cdot F_n-\alpha(F_{n+2}\cdot F_{n-2}+F_n\cdot F_n)+\alpha^2 F_n\cdot F_{n-2}]+[\alpha^2,1]\\
&= [F_n\cdot F_{n-2}+\alpha(F_{n-3}\cdot(F_{n-1}+F_{n-2})+F_{n-1}\cdot F_{n-2})\\
&\quad +\alpha^2(F_{n-1}\cdot F_{n-3}+1),\\
&\quad F_{n+2}\cdot F_n+1-\alpha((F_{n+1}+F_n)\cdot(F_n-F_{n-1})+(F_n)^2)+\alpha^2 F_n\cdot F_{n-2}]\\
&= [1+F^2_{n-1}+\alpha(F_{n-3}\cdot F_{n-1}+F_{n-2}\cdot F_{n-3}+F_{n-1}\cdot F_{n-2})+\alpha^2\cdot F^2_{n-2},\\
&\quad F^2_{n+1}+2-\alpha((F_{n+1}\cdot F_n-F_{n-1}\cdot F_{n+1}+F^2_n-F_n\cdot F_{n-1})+(F_n)^2)\\
&\quad +\alpha^2(1+F^2_{n-1})]\\
&= [1+F^2_{n-1}+\alpha((-1)^{n-2}+F^2_{n-2}+F_{n-2}\cdot F_{n-3}+F_{n-1}\cdot F_{n-2})\\
&\quad +\alpha^2\cdot F^2_{n-2},\\
&\quad F^2_{n+1}+2-\alpha(F_n(F_{n+1}-F_{n-1})+2\cdot F^2_n-F_{n+1}\cdot F_{n-1})\\
&\quad +\alpha^2(1+F^2_{n-1})]\\
&= [1+F^2_{n-1}+\alpha(-1+F_{n-2}(F_{n-2}+F_{n-3})+F_{n-1}\cdot F_{n-2})+\alpha^2\cdot F^2_{n-2},\\
&\quad F^2_{n+1}+2-\alpha(3F^2_n-F_{n+1}\cdot F_{n-1})+\alpha^2(1+F^2_{n-1})]\\
&= [1+F^2_{n-1}+\alpha(-1+2\cdot F_{n-1}\cdot F_{n-2})+\alpha^2\cdot F^2_{n-2},\\
&\quad F^2_{n+1}+2-\alpha(3(F_{n+1}\cdot F_{n-1}-(-1)^n)-F_{n+1}\cdot F_{n-1})\\
&\quad +\alpha^2(1+F^2_{n-1})]\\
&= [1+F^2_{n-1}+\alpha(-1+2\cdot F_{n-1}\cdot F_{n-2})+\alpha^2\cdot F^2_{n-2},\\
&\quad F^2_{n+1}+2-\alpha(3+2F_{n+1}\cdot F_{n-1})+\alpha^2(1+F^2_{n-1})].
\end{aligned}\tag{3.3}$$

By using (2.1)- (2.7), on the other hand, we obtain

$$\begin{aligned}
(F^{\alpha}_n)^2+F^{\alpha}_0\cdot F^{\alpha}_2 &= [F_{n-1}+\alpha F_{n-2}, F_{n+1}-\alpha F_{n-1}]\cdot[F_{n-1}+\alpha F_{n-2}, F_{n+1}-\alpha F_{n-1}]\\
&\quad +[1-\alpha,1-\alpha]\cdot[1,2-\alpha]\\
&= [F^2_{n-1}+2\cdot\alpha\cdot F_{n-2}\cdot F_{n-1}+\alpha^2F^2_{n-2}, F^2_{n+1}-2\cdot\alpha\cdot F_{n-1}\cdot F_{n+1}+\alpha^2F^2_{n-1}]\\
&\quad +[1-\alpha,2-3\alpha+\alpha^2]\\
&= [1+F^2_{n-1}+\alpha(2\cdot F_{n-2}\cdot F_{n-1}-1)+\alpha^2F^2_{n-2},\\
&\quad F^2_{n+1}+2-\alpha(2\cdot F_{n-1}\cdot F_{n+1}+3)+\alpha^2(F^2_{n-1}+1)].
\end{aligned}\tag{3.4}$$

From (3.3) and (3.4), we have $F^{\alpha}_{n+1}\cdot F^{\alpha}_{n-1}+(F^{\alpha}_1)^2=(F^{\alpha}_n)^2+F^{\alpha}_0\cdot F^{\alpha}_2$. Now, let n be an even integer. Then, from (2.1)-(2.7), we get

$$\begin{aligned}
F^{\alpha}_{n+1}\cdot F^{\alpha}_{n-1}+F^{\alpha}_0\cdot F^{\alpha}_2 &= [F_n+\alpha F_{n-1}, F_{n+2}-\alpha F_n]\cdot[F_{n-2}+\alpha F_{n-3}, F_n-\alpha F_{n-2}]+[1-\alpha,1-\alpha]\cdot[1,2-\alpha]\\
&= [F_n\cdot F_{n-2}+\alpha(F_n\cdot F_{n-3}+F_{n-1}\cdot F_{n-2})+\alpha^2F_{n-1}\cdot F_{n-3},\\
&\quad F_{n+2}\cdot F_n-\alpha(F_{n+2}\cdot F_{n-2}+F_n\cdot F_n)+\alpha^2F_n\cdot F_{n-2}]+[1-\alpha,2-3\alpha+\alpha^2]\\
&= [F_n\cdot F_{n-2}+1+\alpha(F_{n-3}\cdot(F_{n-1}+F_{n-2})+F_{n-1}\cdot F_{n-2}-1)\\
&\quad +\alpha^2(F_{n-1}\cdot F_{n-3}), F_{n+2}\cdot F_n+2-\alpha((F_{n+1}+F_n)\cdot(F_n-F_{n-1})\\
&\quad +(F_n)^2+3)+\alpha^2(1+F_n\cdot F_{n-2})]\\
&= [F^2_{n-1}+\alpha(F_{n-3}\cdot F_{n-1}+F_{n-2}\cdot F_{n-3}+F_{n-1}\cdot F_{n-2}-1)+\alpha^2\cdot(F^2_{n-2}+1),\\
&\quad F^2_{n+1}+1-\alpha((F_{n+1}\cdot F_n-F_{n-1}\cdot F_{n+1}+F^2_n-F_n\cdot F_{n-1})+3+(F_n)^2)+\alpha^2\cdot F^2_{n-1}]
\end{aligned}$$

$$\begin{aligned}
&= [F_{n-1}^2 + \alpha((-1)^{n-2} + F_{n-2}^2 + F_{n-2}\cdot F_{n-3} + F_{n-1}\cdot F_{n-2} - 1) + \alpha^2\cdot(F_{n-2}^2+1),\\
&\quad F_{n+1}^2 + 1 - \alpha(F_n(F_{n+1} - F_{n-1}) + 2\cdot F_n^2 - F_{n+1}\cdot F_{n-1} + 3) + \alpha^2\cdot F_{n-1}^2]\\
&= [F_{n-1}^2 + \alpha(F_{n-2}(F_{n-2} + F_{n-3}) + F_{n-1}\cdot F_{n-2}) + \alpha^2\cdot(1+F_{n-2}^2),\\
&\quad F_{n+1}^2 + 1 - \alpha(3F_n^2 - F_{n+1}\cdot F_{n-1} + 3) + \alpha^2\cdot F_{n-1}^2]\\
&= [F_{n-1}^2 + \alpha(2\cdot F_{n-1}\cdot F_{n-2}) + \alpha^2\cdot(1+F_{n-2}^2),\\
&\quad F_{n+1}^2 + 1 - \alpha(3(F_{n+1}\cdot F_{n-1} - (-1)^n) - F_{n+1}\cdot F_{n-1} + 3) + \alpha^2\cdot F_{n-1}^2]\\
&= [F_{n-1}^2 + 2\cdot\alpha\cdot F_{n-1}\cdot F_{n-2}) + \alpha^2\cdot(1+F_{n-2}^2),\\
&\quad F_{n+1}^2 + 1 - 2\cdot\alpha\cdot F_{n+1}\cdot F_{n-1} + \alpha^2 F_{n-1}^2]
\end{aligned} \tag{3.5}$$

By using (2.1)-(2.7), on the other hand, we obtain

$$\begin{aligned}
(F_n^\alpha)^2 + (F_1^\alpha)^2 &= [F_{n-1} + \alpha F_{n-2}, F_{n+1} - \alpha F_{n-1}]\cdot[F_{n-1} + \alpha F_{n-2}, F_{n+1} - \alpha F_{n-1}] + [\alpha, 1]\cdot[\alpha, 1]\\
&= [F_{n-1}^2 + 2\cdot\alpha\cdot F_{n-2}\cdot F_{n-1} + \alpha^2 F_{n-2}^2, F_{n+1}^2 - 2\cdot\alpha\cdot F_{n-1}\cdot F_{n+1} + \alpha^2 F_{n-1}^2] + [\alpha^2, 1]\\
&= [F_{n-1}^2 + 2\cdot\alpha\cdot F_{n-2}\cdot F_{n-1} + \alpha^2(1+F_{n-2}^2), F_{n+1}^2 + 1 - 2\cdot\alpha\cdot F_{n-1}\cdot F_{n+1} + \alpha^2 F_{n-1}^2].
\end{aligned} \tag{3.6}$$

From (3.5) and (3.6), we have $F_{n+1}^\alpha \cdot F_{n-1}^\alpha + F_0^\alpha \cdot F_2^\alpha = (F_n^\alpha)^2 + (F_1^\alpha)^2$. □

Conflicts of Interest

The author declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author have read and agreed to the published version of the manuscript.

References

[1] Abraham, A.M., Fuzzy Logic for Embedded System Application, Ph.D. senior member, IEEE, 2004.
[2] Bezdek, J., Keller, J., Krishnapuram, R., Pal, N. R., Fuzzy Models and Algorithms for Pattern Recognition and Image Processing, The Handbooks of Fuzzy Sets Series, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1999.
[3] Buckley, J.J., Eslami, E., Introduction to Fuzzy Logic and Fuzzy Sets, Physica-Verlag, Heidelberg, Germany, 2002.
[4] Chalam, G.A., *Fuzzy goal programming (FGP) approach to a stochastic transportation problem under budgetary constraint*, Fuzzy Sets and Systems, **66**(1994), 293–299.
[5] Dijkman, J.G., Van Haeringen, H., De Lange, S.T., *Fuzzy numbers*, J. Math. Anal. Appl., **92**(1983), 301–341.
[6] Dubois, D., Prade, H., *Operations on fuzzy numbers*, International Journal of Systems Science, **6**(9)(1978), 613–626.
[7] Dubois, D., Prade, H., Fuzzy Sets and Systems: Theory and Applications, Academic Press, New York, 1980.
[8] Dubois, D., Prade, H., *Ranking fuzzy numbers in a setting of possibility theory*, Inf. Sci., **30**(1983), 183–224.
[9] Dubois, D., Prade, H., Possibility Theory, An Approach to Computerized Processing of Uncertainty, Plenum Press, New York, 1988.
[10] Gao, S., Zhang, Z., Cao, C., *Multiplication operation on fuzzy numbers*, Journal of Software, **4**(4)(2009), 331–338.
[11] Gegov, A., Fuzzy Networks for Complex Systems: A Modular Rule Base Approach, Springer, Berlin, 2010.
[12] Gegov, A, Sanders, D, Vatchova, B, *Complexity management methodology for fuzzy systems with feedforward rule bases*, Knowledge Based and Intelligent Engineering Systems, **19**, 83–95.
[13] Hirota, K., Industrial Applications of Fuzzy Technology, Springer, Tokyo, 1993.
[14] Irmak, N., Demirtaş, N.,*Fuzzy Fibonacci and fuzzy Lucas numbers with their properties,* Mathematical Sciences and Applications E-Notes, **7**(2)(2019), 218–224.
[15] Kecman, V., Learning and Soft Computing Support Vector Machines, Neural Networks, and Fuzzy Logic Models, A Bradford Book, the MIT Press, 2001.
[16] Klir, G. J., Yuan, B., Fuzzy Sets and Fuzzy Logic: Theory and Applications, Prentice Hall, Upper Saddle River, N.J., 1995.
[17] Paksoy, P., Pehlivan, N.Y., Özceylan, E., Bulanık Küme Teorisi, Nobel Yayınları, Ankara, 2013.
[18] Ribenboim, P., My Numbers, My Friends: Popular Lectures on Number Theory, Springer-Verlag New York, Inc., 2000.
[19] Roychowdhury, S., Pedrycz, W., *A survey of defuzzification strategies*, International Journal of Intelligent Systems, **16**(6)(2001), 679–695.
[20] Sakawa, M., Yano, H.,*An interactive fuzzy satisfying method using augmented minimax problems and its application to environmental systems*, IEEE Trans. Syst. Man. Cybern, **15**(1985), 720–729.
[21] Terano, T., Asai, K., Sugeno, M., Applied Fuzzy Systems, Academic Press Professional, Boston, 1994.
[22] Zimmermann, H.J., Fuzzy Sets, Decision Making, and Expert Systems, Vol. 10, Springer Science & Business Media, 1987.
[23] Zimmermann, H.J., Fuzzy Set Theory and Its Applications, Kluwer Academic Publishers, Boston, 1991.
[24] Zadeh, L.A. , *Fuzzy sets,* Information and Control, **8**(3)(1965).