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Qualitative Analysis of a Nicholson-Bailey Model in Patchy Environment

Rizwan Ahmed[1*] and Shehraz Akhtar[2]

[1]*Department of Mathematics, Air University Multan Campus, Multan, Pakistan*
[2]*Department of Mathematics, The Islamia University of Bahawalpur, Rahim Yar Khan Campus, Rahim Yar Khan, Pakistan*
[*]*Corresponding author*

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Abstract

We studied a host-parasite model qualitatively. The host-parasitoid model is obtained by modifying the Nicholson-Bailey model so that the number of hosts that parasitoids cannot attack is fixed. We investigate the effect of the presence of a refuge on the local stability and bifurcation of models. Topological classification of equilibria is achieved with the implementation of linearization. Furthermore, Neimark-Sacker bifurcation is explored using the bifurcation theory of normal forms at interior steady-state. The bifurcation in the model is controlled by implementing two control strategies. The theoretical studies are backed up by numerical simulations, which show the conclusions and their importance. A low rate of escaping of a host may lead to instability.

1. Introduction

The Nicholson-Bailey model [1] was proposed by Nicholson and Bailey in 1935 to model and study a biological system involving two insects: a host and a parasitoid. The parasitoid is a free-living adult parasite that lays eggs on the host larvae, and these eggs may survive to give birth to the next generation. The parasitoid hosts die, and the non-parasitoid hosts produce their offspring. There are some unnatural suppositions in the Nicholson-Bailey model, for instance, a homogeneous environment, a constant searching efficiency, and the reproductive rate of the host. These assumptions produce unstable positive fixed points for all the parametric values and lead to oscillations in the Nicholson-Bailey model at low parasitoid densities. By relaxing the homogeneous environment assumption and assuming a patchy environment, a proportion of the host population could hide away or refuge and be secure from the attack of parasitoids. Therefore, a modified Nicholson-Bailey model has been proposed by MP Hassell [2] and is given as

$$\begin{cases} H_{t+1} = r(1-\gamma)H_t + r\gamma H_t \exp(-aP_t), \\ P_{t+1} = e\gamma H_t(1-\exp(-aP_t)), \end{cases} \tag{1.1}$$

where H_t is the population size of the host in generation t and P_t is the parasitoid population size in generation t, r refers to the reproductive rate of the host, a to the efficiency with which the parasitoid searches for a host, and e represents the average number of viable eggs laid by a parasitoid on a single host, γ is the percentage of hosts that are vulnerable to parasitoids, and $1-\gamma$ shows how many are safe from parasities when they are in a refuge. It is evident to see that if we take $\gamma = 1$ in the system (1.1), then we retrieve the classical Nicholson-Bailey model

$$\begin{cases} H_{t+1} = rH_t \exp(-aP_t), \\ P_{t+1} = eH_t(1-\exp(-aP_t)), \end{cases} \tag{1.2}$$

where r, a, and e are positive constants. The parameters r, a, and e have the same biological interpretations as those in the previous model (1.2). Unfortunately, this classical model failed to produce a stable equilibrium. Several authors attempted to modify the model in order to achieve a more realistic and stable system.

Email addresses and ORCID numbers: **rizwanahmed488@gmail.com, 0000-0002-4900-0488 (R. Ahmed), shahraz5768@gmail.com, 0000-0002-9212-6887 (S. Akhtar)**
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Another way to model the effect of a refugee can be achieved by sheltering a certain quantity of hosts denoted as H_0, which are immune to being attacked by parasitoids, and another modification [3] in the Nicholson-Bailey model is given by

$$\begin{cases} H_{t+1} = rH_0 + r(H_t - H_0)\exp(-aP_t), \\ P_{t+1} = e(H_t - H_0)(1 - \exp(-aP_t)). \end{cases} \tag{1.3}$$

The mathematical modeling of population dynamics has been developed as a significant area of research within the last decade. The mathematical models described by exponential difference equations are extensively used to study population dynamics [16]. Nonlinear difference equations appear naturally in mathematical modeling as they provide a more flexible framework to model different biological systems' dynamics [4, 5]. These equations are the discrete-time counterparts of differential equations, which are used extensively in engineering and the biological sciences [6, 7]. The study of the consequences of the hiding behavior of host on the dynamics of host-parasitoid systems can be recognized as a major issue in applied mathematics and theoretical ecology. Some of the empirical and theoretical work have investigated the effect of host refuges and drawn a conclusion that the refuges used by host have a stabilizing effect on the considered interactions and host extinction can be prevented by the addition of refuges.
A complete examination of the qualitative behavior of models given by nonlinear difference equations, including local and global stability, bifurcation analysis, and chaos control, may be found in [8–15]. Q. Din [16] examined the qualitative behavior of the model (1.3). Specifically, the author examined the boundedness and persistence, the presence and uniqueness of steady-state, the local and global stability of the unique positive fixed point, and the rate of convergence of all solutions that converge to the fixed point for the model (1.3).
The motivation of our work is to study the impact of the refuge effect on the host population in the modified Nicholson-Bailey model. In this research, we investigate the qualitative behavior of the model (1.1) by identifying the unique positive fixed point, the parametric conditions for the local stability of the unique positive fixed point, and the presence of the Neimark-Sacker bifurcation at the positive fixed point, and by implementing the control strategies to control the Neimark-Sacker bifurcation in the model (1.1). In the end, some numerical examples are provided, followed by a necessary discussion on the qualitative behavior of the model (1.1).
The following describes the structure of the paper:
The derivation of a necessary and sufficient condition for the local asymptotic stability of the fixed point of the model (1.1) is given in Section 2. The Neimark-Sacker bifurcation at the unique positive fixed point is the subject of Section 3. In Section 4, two control techniques are employed to control the bifurcation in the model. The dependence of the model on the parameters γ and r is illustrated in Section 5. Section 6 has some final observations.

2. Local Stability of Positive Fixed Point

It is simple that $(0,0)$ and $(H_*, P_*) = \left(\frac{r\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right)}{ae(r-1)}, \frac{1}{a}\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right) \right)$ are the fixed points of the system (1.1). Also, for $r > 1$ and $\gamma > \frac{r-1}{r}$, (H_*, P_*) is the unique positive fixed point of system (1.1). The system will have to be linearized for stability analysis using the variational matrix at the fixed point (H_*, P_*). For the fixed point, $(H_*, P_*) = \left(\frac{r\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right)}{ae(r-1)}, \frac{1}{a}\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right) \right)$, the variational matrix is

$$J(H_*, P_*) = \begin{bmatrix} 1 & -\frac{r(1-r(1-\gamma))}{e(r-1)} \ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right) \\ \frac{e(r-1)}{r} & \frac{(1-r(1-\gamma))}{(r-1)} \ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right) \end{bmatrix}.$$

The characteristic polynomial of the variational matrix is given by

$$C(z) = z^2 - \left(1 + \frac{(1-r(1-\gamma))}{(r-1)} \ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right)\right) z + \frac{r(1-r(1-\gamma))}{(r-1)} \ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right). \tag{2.1}$$

The following lemma is very important for both the topological categorization of the fixed points and the determination of the criteria that are necessary as well as sufficient for the local stability of the fixed points.

Theorem 2.1 ([17]). *Let $C(z) = z^2 - A\lambda + B$, and $C(1) > 0$ with z_1, z_2 be the roots of $C(z) = 0$. Then the following results hold:*

(i) $|z_1| < 1$ and $|z_2| < 1$ iff $C(-1) > 0$ and $C(0) < 1$.
(ii) $|z| < 1$ and $|z| > 1$, or $|z| > 1$ and $|z| < 1$ iff $C(-1) < 0$.
(iii) $|z_1| > 1$ and $|z_2| > 1$ iff $C(-1) > 0$ and $C(0) > 1$.
(iv) $z_1 = -1$ and $z_2 \neq 1$ iff $C(-1) = 0$ and $C(0) \neq \pm 1$.
(v) z_1 and z_2 are complex and $|z_1| = 1$ and $|z_2| = 1$ iff $A^2 - 4B < 0$ and $C(0) = 1$.

By using simple computations, we have

$$C(1) = \left(\ln(\frac{r\gamma}{1-r(1-\gamma)})\right)(1-r(1-\gamma)),$$
$$C(-1) = 2 + \frac{\left(\ln(\frac{r\gamma}{1-r(1-\gamma)})\right)(1-r(1-\gamma))(r+1)}{(r-1)},$$
$$C(0) = \frac{r\left(\ln(\frac{r\gamma}{1-r(1-\gamma)})\right)(1-r(1-\gamma))}{(r-1)}.$$

Notice that for all $r>1$ and $\gamma>\frac{r-1}{r}$, we have $C(1)>0$ and $C(-1)>0$. Therefore, cases (ii) and (iv) of Theorem 2.1 are not possible. It means that (H_*,P_*) in the system (1.1) is not a saddle point because case (ii) of Theorem 2.1 is not true and period-doubling bifurcation is not possible because case (iv) of Theorem 2.1 is not true.

Theorem 2.2. *Suppose that $r>1$, and $\gamma>\frac{r-1}{r}$. The unique fixed point (H_*,P_*) of the system (1.1) is*

(i) stable iff

$$r\left(\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right)\right)(1-r(1-\gamma))<r-1,$$

(ii) unstable iff

$$r\left(\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right)\right)(1-r(1-\gamma))>r-1,$$

(iii) non-hyperbolic iff

$$r\left(\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right)\right)(1-r(1-\gamma))=r-1, \tag{2.2}$$

and

$$\left(1+\left(\frac{\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right)}{r-1}\right)(1-r(1-\gamma))\right)^2-4\left(\frac{r\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right)(1-r(1-\gamma))}{(r-1)}\right)<0. \tag{2.3}$$

3. Bifurcation Analysis

In this section, we use bifurcation theory to investigate the Neimark-Sacker bifurcation at (H_*,P_*), using γ as the bifurcation parameter in the system (1.1). The existence of the Neimark-Sacker bifurcation ensures that dynamically invariant closed curves are produced. We refer to [18–23] for the relevant literature concerning the bifurcation analysis of such types of discrete dynamical systems.
We are looking for conditions on the system (1.1) that will allow us to have a non-hyperbolic point $(H*,P_*)$ with a pair of complex conjugate eigenvalues that have modulus values that are equal to one for $J(H*,P*)$. The characteristic polynomial (2.1) has complex roots $z_{1,2}$ with $|z_{1,2}|=1$ in the following region

$$\Theta=\left\{(r,\gamma):r>1,\gamma>\frac{r-1}{r},\quad (2.2)\text{ and }(2.3)\text{ are satisfied}\right\}.$$

We select γ as a bifurcation parameter. When parameters vary in a local region of Θ, the system's unique positive fixed point (1.1) undergoes Neimark-Sacker bifurcation. We consider the following perturbation of the system (1.1):

$$\begin{bmatrix}H_{t+1}\\P_{t+1}\end{bmatrix}=\begin{bmatrix}r(1-\gamma-\delta)H_t+r(\gamma+\delta)H_t exp(-aP_t)\\ e(\gamma+\delta)H_t(1-exp(-aP_t))\end{bmatrix}, \tag{3.1}$$

where $|\delta|\ll 1$ is used as a small perturbation parameter.
We now consider the transformation $u_{t+1}=H_{t+1}-H_*,\ v_{t+1}=P_{t+1}-P_*$ to transfer the fixed point (H_*,P_*) of the system (1.1) to origin:

$$\begin{bmatrix}u_{t+1}\\v_{t+1}\end{bmatrix}=\begin{bmatrix}1 & -\frac{r(1-r(1-\gamma-\delta))}{e(r-1)}\ln\left(\frac{r(\gamma+\delta)}{1-r(1-\gamma-\delta)}\right)\\ \frac{e(r-1)}{r} & \frac{(1-r(1-\gamma-\delta))\ln\left(\frac{r(\gamma+\delta)}{1-r(1-\gamma-\delta)}\right)}{(r-1)}\end{bmatrix}\begin{bmatrix}u_t\\v_t\end{bmatrix}+\begin{bmatrix}f_1(u_t,v_t)\\f_2(u_t,v_t)\end{bmatrix}, \tag{3.2}$$

where

$$\begin{aligned}f_1(u_t,v_t)=&-(a(1-r(1-\gamma-\delta)))u_tv_t+\left(\frac{ar(1-r(1-\gamma-\delta))\ln\left(\frac{r(\gamma+\delta)}{1-r(1-\gamma-\delta)}\right)}{2e(r-1)}\right)v_t^2\\&+\frac{1}{2}(a^2(1-r(1-\gamma-\delta)))u_tv_t^2-\left(\frac{a^2r(1-r(1-\gamma-\delta))\ln\left(\frac{r(\gamma+\delta)}{1-r(1-\gamma-\delta)}\right)}{6e(r-1)}\right)v_t^3,\end{aligned}$$

and

$$\begin{aligned}f_2(u_t,v_t)=&\left(\frac{ae(1-r(1-\gamma-\delta))}{r}\right)u_tv_t-\left(\frac{a(1-r(1-\gamma-\delta))\ln\left(\frac{r(\gamma+\delta)}{1-r(1-\gamma-\delta)}\right)}{2(r-1)}\right)v_t^2\\&-\left(\frac{a^2e(1-r(1-\gamma-\delta))}{2r}\right)u_tv_t^2+\left(\frac{a^2(1-r(1-\gamma-\delta))\ln\left(\frac{r(\gamma+\delta)}{1-r(1-\gamma-\delta)}\right)}{6(r-1)}\right)v_t^3.\end{aligned}$$

The characteristic polynomial of the linearized part of (3.2) evaluated at the fixed point $(0,0)$ of (3.1) is given by

$$z^2-p(\delta)z+q(\delta)=0, \tag{3.3}$$

where

$$p(\delta)=1+\frac{(1-r(1-\gamma-\delta))}{(r-1)}\ln\left(\frac{r(\gamma+\delta)}{1-r(1-\gamma-\delta)}\right),$$
$$q(\delta)=\frac{r(1-r(1-\gamma-\delta))}{(r-1)}\ln\left(\frac{r(\gamma+\delta)}{1-r(1-\gamma-\delta)}\right).$$

The roots of (3.3) are

$$z_{1,2}=\frac{p(\delta)}{2}\pm\frac{i}{2}\sqrt{4q(\delta)-p^2(\delta)}$$

satisfying

$$|z_{1,2}|=\sqrt{q(\delta)},$$

and

$$\left(\frac{d|z_{1,2}|}{d\delta}\right)_{\tilde{\gamma}=0}=\frac{\sqrt{r}\left(1-r+r\gamma\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right)\right)}{2\gamma\sqrt{(r-1)(1-r(1-\gamma))\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right)}}>0.$$

We also have $p(0)=\left(1+\left(\frac{\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right)}{r-1}\right)(1-r(1-\gamma))\right)$ and $(r,\gamma)\in\Theta$ which means $p(0)\neq\pm2,0,1$. So $z_1^n,z_2^n\neq1$ for all $n=1,2,3,4$ at $\delta=0$. Thus the roots of equation (3.3) do not lie in the unit circle intersection with the coordinate axes when $\delta=0$. We use the following transformation to get the canonical form of the linearized part of (3.2) at $\delta=0$.

$$\begin{bmatrix}u_{t+1}\\v_{t+1}\end{bmatrix}=\begin{bmatrix}-\frac{1}{e} & 0\\ \frac{1-r}{2r} & -\frac{\sqrt{4-(1+\frac{1}{r})^2}}{2}\end{bmatrix}\begin{bmatrix}x_{t+1}\\y_{t+1}\end{bmatrix}. \tag{3.4}$$

Under the transformation (3.4), the system (3.2) becomes

$$\begin{bmatrix}x_{t+1}\\y_{t+1}\end{bmatrix}=\begin{bmatrix}\frac{1+r}{2r} & -\frac{\sqrt{4-(1+\frac{1}{r})^2}}{2}\\ \frac{\sqrt{4-(1+\frac{1}{r})^2}}{2} & \frac{1+r}{2r}\end{bmatrix}\begin{bmatrix}x_t\\y_t\end{bmatrix}+\begin{bmatrix}F(x_t,y_t)\\G(x_t,y_t)\end{bmatrix}, \tag{3.5}$$

where

$$\begin{aligned}F(x,y)=&\frac{a(r-1)(1+3r+4r^2(-1+\gamma))}{8r^2}x^2-\frac{a(4-(1+\frac{1}{r})^2)}{8}y^2\\&+\frac{a(1+r+2r^2(-1+\gamma))\sqrt{-1-2r+3r^2}}{4r^2}xy+\frac{a^2(-1+r)^2(1+5r+6r^2(-1+\gamma))}{48r^3}x^3\\&-\frac{a^2}{48}(4-(1+\frac{1}{r})^2)^{3/2}y^3+\frac{a^2(-1-2r+3r^2)(1+r+2r^2(-1+\gamma)))}{16r^3}xy^2\\&+\frac{a^2(-1+r)(1+3r+4r^2(-1+\gamma))\sqrt{-1-2r+3r^2}}{16r^3}x^2y+O((|x|+|y|)^4),\end{aligned}$$

and

$$\begin{aligned}G(x,y)=&-\frac{a(-1+r)(1+r)(1+3r+4r^2(-1+\gamma))}{8r^2\sqrt{-1-2r+3r^2}}x^2+\frac{a(1+r)\sqrt{-1-2r+3r^2}}{8r^2}y^2\\&-\frac{a(1+r)(1+r+2r^2(-1+\gamma))}{4r^2}xy-\frac{a^2(-1+r)^2(1+r)(1+5r+6r^2(-1+\gamma))}{48r^3\sqrt{-1-2r+3r^2}}x^3\\&+\frac{a^2(-1-3r+r^2+3r^3)}{48r^3}y^3-\frac{a^2(-1-3r+r^2+3r^3)(1+r+2r^2(-1+\gamma))}{16r^3\sqrt{-1-2r+3r^2}}xy^2\\&-\frac{a^2(-1+r)(1+r)(1+3r+4r^2(-1+\gamma))}{16r^3}x^2y+O((|x|+|y|)^4).\end{aligned}$$

We define the real number L, which analyzes the direction of the closed invariant curve in a system undergoing Neimark-Sacker bifurcation [24].

$$L=\left(\left[-Re\left(\frac{(1-2z_1)z_2^2}{1-z_1}\eta_{20}\eta_{11}\right)-\frac{1}{2}|\eta_{11}|^2-|\eta_{02}|^2+Re(z_2\eta_{21})\right]\right)_{\delta=0},$$

where

$$\eta_{20} = \frac{1}{8}\left[F_{xx} - F_{yy} + 2G_{xy} + i(G_{xx} - G_{yy} - 2F_{xy})\right],$$
$$\eta_{11} = \frac{1}{4}\left[F_{xx} + F_{yy} + i(G_{xx} + G_{yy})\right],$$
$$\eta_{02} = \frac{1}{8}\left[F_{xx} - F_{yy} - 2G_{xy} + i(G_{xx} - G_{yy} + 2F_{xy})\right],$$
$$\eta_{21} = \frac{1}{16}\left[F_{xxx} + F_{xyy} + G_{xxy} + G_{yyy} + i(G_{xxx} + G_{xyy} - F_{xxy} - F_{yyy})\right],$$

and

$$F_{xx} = \frac{a(-1+r)(1+3r+4r^2(-1+\gamma))}{4r^2}, \quad F_{yy} = -\frac{1}{4}a\left(4-\left(1+\frac{1}{r}\right)^2\right),$$
$$F_{xy} = \frac{a(1+r+2r^2(-1+\gamma))\sqrt{-1-2r+3r^2}}{4r^2}, \quad F_{xxx} = \frac{a^2(-1+r)^2(1+5r+6r^2(-1+\gamma))}{8r^3},$$
$$F_{yyy} = -\frac{1}{8}a^2\left(4-\left(1+\frac{1}{r}\right)^2\right)^{\frac{3}{2}}, \quad F_{xyy} = \frac{a^2(-1-2r+3r^2)(1+r+2r^2(-1+\gamma))}{8r^3},$$
$$F_{xxy} = \frac{a^2(-1+r)(1+3r+4r^2(-1+\gamma))\sqrt{-1-2r+3r^2}}{8r^3}, \quad G_{xx} = -\frac{a(-1+r)(1+r)(1+3r+4r^2(-1+\gamma))}{4r^2\sqrt{-1-2r+3r^2}},$$
$$G_{yy} = \frac{a(1+r)\sqrt{-1-2r+3r^2}}{4r^2}, \quad G_{xy} = -\frac{a(1+r)(1+r+2r^2(-1+\gamma))}{4r^2},$$
$$G_{xxx} = -\frac{a^2(-1+r)^2(1+r)(1+5r+6r^2(-1+\gamma))}{8r^3\sqrt{-1-2r+3r^2}}, \quad G_{yyy} = \frac{a^2(-1-3r+r^2+3r^3)}{8r^3},$$
$$G_{xyy} = -\frac{a^2(-1-3r+r^2+3r^3)(1+r+2r^2(-1+\gamma))}{8r^3\sqrt{-1-2r+3r^2}}, \quad G_{xxy} = -\frac{a^2(-1+r)(1+r)(1+3r+4r^2(-1+\gamma))}{8r^3}.$$

Due to the above calculations, we have the following theorem for the existence and direction of Neimark-Sacker bifurcation.

Theorem 3.1. *Assume that $(r,\gamma)\in\Theta$. If $L\neq 0$, then the system (1.1) experiences Neimark-Sacker bifurcation at the unique positive fixed point (H_*,P_*) when the parameter γ differs in a small neighborhood of Θ. Moreover, if $L<0$, then an attracting closed invariant curve bifurcates from the fixed point (H_*,P_*), and if $L>0$, then a repelling closed invariant curve bifurcates from the fixed point (H_*,P_*).*

4. Chaos Control

Controlling bifurcation in discrete models has recently fascinated the interest of many researchers, and practical approaches are being used in a variety of fields, including cardiology, physics laboratories, laser and plasma systems, biochemistry, turbulence, communications, mechanical and chemical engineering [25, 26].

We use the state feedback control technique [7, 19, 27–29] to stabilize the unstable fixed point of the system (1.1). We consider the controlled system in compliance with (1.1) as follows:

$$\begin{cases} H_{t+1} = r(1-\gamma)H_t + r\gamma H_t\exp(-aP_t) - U_t, \\ P_{t+1} = e\gamma H_t(1-\exp(-aP_t)), \end{cases} \tag{4.1}$$

where $U_t = h(H_t - H_*) + p(P_t - P_*)$ is the feedback control and p, h are feedback gains. The variational matrix of the system (4.1) evaluated at (H_*,P_*) is given by

$$J_C(H_*,P_*) = \begin{bmatrix} 1-h & -p-\frac{r(1-r(1-\gamma))}{e(r-1)}\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right) \\ \frac{e(r-1)}{r} & \frac{(1-r(1-\gamma))}{(r-1)}\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right) \end{bmatrix}.$$

The characteristic equation corresponding to $J_C(H_*,P_*)$ is given by

$$z^2 - \left(1-h+\frac{(1-r(1-\gamma))}{(r-1)}\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right)\right)z + \frac{pe(r-1)}{r} - \left(\frac{(h-r)(1-r(1-\gamma))}{(r-1)}\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right)\right) = 0. \tag{4.2}$$

If z_1 and z_2 are roots of the system (4.2), then we have

$$z_1 + z_2 = 1-h+\frac{(1-r(1-\gamma))}{(r-1)}\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right), \tag{4.3}$$

and

$$z_1 z_2 = \frac{pe(r-1)}{r} - \left(\frac{(h-r)(1-r(1-\gamma))}{(r-1)}\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right)\right). \tag{4.4}$$

To get marginal lines of stability, we assume $z_1 = \pm 1$ and $z_1 z_2 = 1$ which implies $|z_{1,2}| \leq 1$. If we assume that $z_1 z_2 = 1$ then (4.4) gives

$$L_1 : \left(\frac{(1-r(1-\gamma))}{(r-1)} \ln \left(\frac{r\gamma}{1-r(1-\gamma)} \right) \right) h - \left(\frac{e(r-1)}{r} \right) p = \frac{r(1-r(1-\gamma))}{(r-1)} \ln \left(\frac{r\gamma}{1-r(1-\gamma)} \right) - 1. \tag{4.5}$$

Next, if we assume that $z_1 = 1$ then (4.3) and (4.4) implies

$$L_2 : \left(1 - \frac{(1-r(1-\gamma))}{(r-1)} \ln \left(\frac{r\gamma}{1-r(1-\gamma)} \right) \right) h + \left(\frac{e(r-1)}{r} \right) p = -(1-r(1-\gamma)) \ln \left(\frac{r\gamma}{1-r(1-\gamma)} \right). \tag{4.6}$$

If we assume that $z_1 = -1$ then (4.3) and (4.4) yields

$$L_3 : \left(1 + \frac{(1-r(1-\gamma))}{(r-1)} \ln \left(\frac{r\gamma}{1-r(1-\gamma)} \right) \right) h - \left(\frac{e(r-1)}{r} \right) p = 2 + \frac{(r+1)(1-r(1-\gamma))}{(r-1)} \ln \left(\frac{r\gamma}{1-r(1-\gamma)} \right). \tag{4.7}$$

It is easy to see that the triangular area bounded by the straight lines L_1, L_2, and L_3 have stable eigenvalues.
Next, we use a hybrid control technique [22, 30–33] to control the chaotic behavior of (1.1) at fixed point (H_*, P_*) due to Neimark-Sacker bifurcation. We consider the following controlled system associated with the system (1.1):

$$\begin{cases} H_{t+1} = \alpha(r(1-\gamma)H_t + r\gamma H_t \exp(-aP_t)) + (1-\alpha)H_t, \\ P_{t+1} = \alpha e\gamma H_t (1-\exp(-aP_t)) + (1-\alpha)P_t, \end{cases} \tag{4.8}$$

where $0 < \alpha \leq 1$. The fixed points of the controlled system (4.8) and the original system (1.1) are the same.
By using theorem (2.1), we have the following result for local asymptotic stability of fixed point (H_*, P_*) of the controlled system (4.8).

Theorem 4.1. *Let $r > 1$ and $\gamma > \frac{r-1}{r}$. The unique positive fixed point (H_*, P_*) of the controlled system (4.8) is locally asymptotically stable iff*

$$\frac{(1-\alpha)(r-1) + \alpha(1+\alpha(r-1))(1-r(1-\gamma)) \ln \left(\frac{r\gamma}{1-r(1-\gamma)} \right)}{(r-1)} < 1.$$

Proof. The variational matrix of the system (4.8) at the fixed point (H_*, P_*) is

$$J_C(H_*, P_*) = \begin{bmatrix} 1 & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

where

$$J_{12} = -\frac{\alpha r(1-r(1-\gamma)) \ln \left(\frac{r\gamma}{1-r(1-\gamma)} \right)}{e(r-1)},$$

$$J_{21} = \frac{\alpha e(r-1)}{r},$$

$$J_{22} = \frac{r-1+\alpha(1-r) + (\alpha + \alpha r(\gamma-1)) \ln \left(\frac{r\gamma}{1-r(1-\gamma)} \right)}{r-1}.$$

The characteristic polynomial of $J_C(H_*, P_*)$ is

$$F_C(z) = z^2 - \left(2 - \alpha + \frac{(\alpha + \alpha r(\gamma-1)) \ln \left(\frac{r\gamma}{1-r(1-\gamma)} \right)}{r-1} \right) z + K,$$

where

$$K = \frac{r-1+\alpha(1-r) + \alpha(1+\alpha(r-1))(1-r(1-\gamma)) \ln \left(\frac{r\gamma}{1-r(1-\gamma)} \right)}{r-1}.$$

By simple computations,

$$F_C(1) = \alpha^2 (1-r(1-\gamma)) \ln \left(\frac{r\gamma}{1-r(1-\gamma)} \right) > 0,$$

$$F_C(-1) = 4 - 2\alpha + \frac{\alpha(2+\alpha(r-1))(1-r(1-\gamma)) \ln \left(\frac{r\gamma}{1-r(1-\gamma)} \right)}{r-1} > 0,$$

and

$$F_C(0) = \frac{(1-\alpha)(r-1) + \alpha(1+\alpha(r-1))(1-r(1-\gamma)) \ln \left(\frac{r\gamma}{1-r(1-\gamma)} \right)}{(r-1)}.$$

□

5. Numerical Examples

Some interesting numerical examples are provided in this section to strengthen our theoretical findings on different qualitative characteristics of the model (1.1).

5.1. Neimark-Sacker bifurcation by using γ as bifurcation parameter

Setting the parameters $r = 2, a = 4, e = 1$ and initial condition $H_0 = 0.5, P_0 = 0.2$ for the system (1.1), the bifurcation value is $\gamma \approx 0.698976$ and the fixed point is $(H_*, P_*) \approx (0.628216, 0.314108)$. The eigenvalues of $J(H_*, P_*)$ are $z_{1,2} = .75 \pm 0.661438i$ having $|z_{1,2}| = 1$ which confirms that the system (1.1) undergoes Neimark-Sacker bifurcation at (H_*, P_*). It is observed that the fixed point is locally asymptotically stable for $\gamma < 0.698976$, and the fixed point is unstable for $\gamma \geq 0.698976$ due to the Neimark-Sacker bifurcation as shown in Figure 5.1.

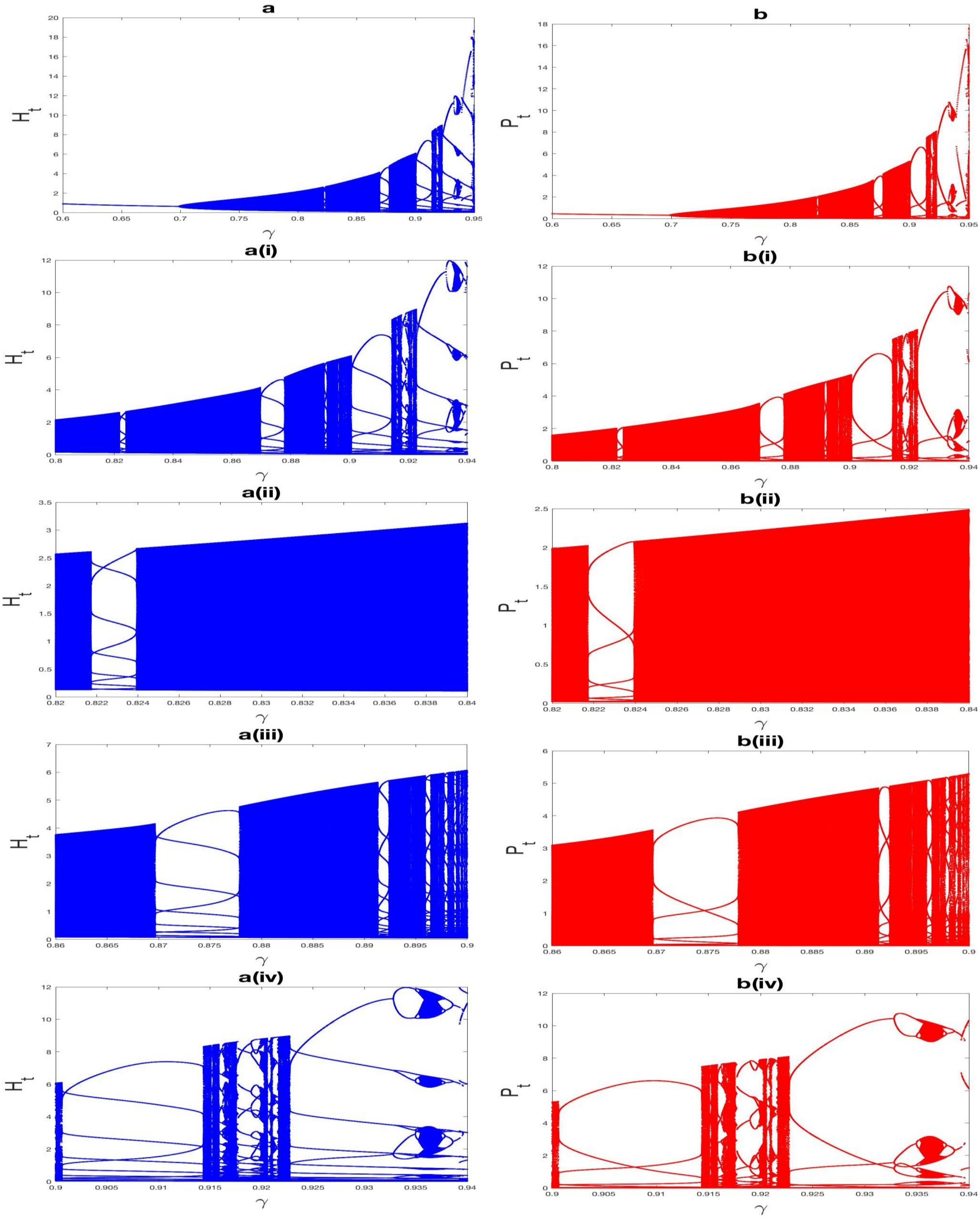

Figure 5.1: Bifurcation diagrams for system (1.1) and their amplifications.

The closed invariant curves and periodic orbits are observed for $\gamma \geq 0.698976$ as shown in Figure 5.2.

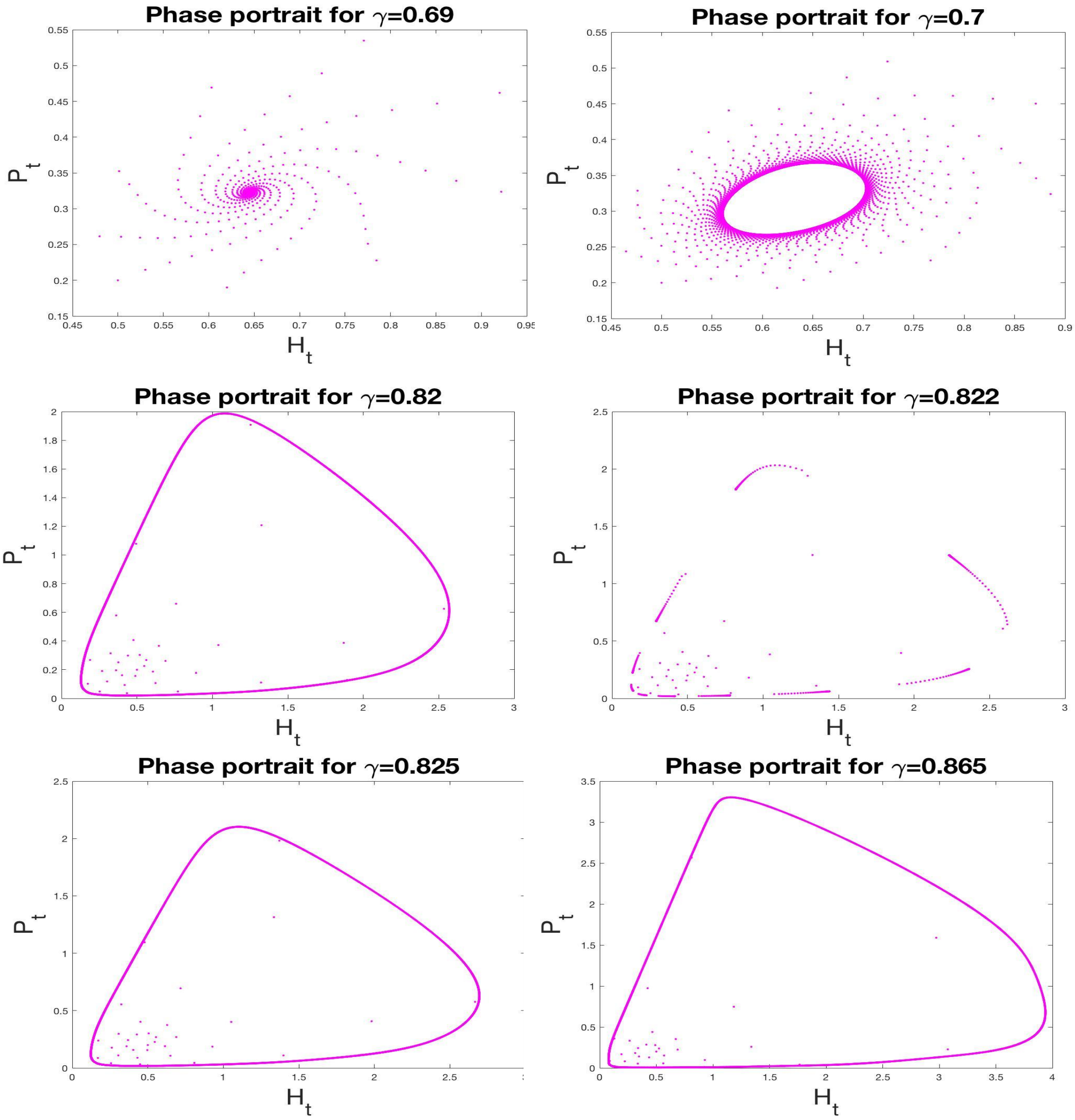

Figure 5.2: Phase portraits for system (1.1) for different values of γ.

Figure 5.3 displays the maximum Lyapunov exponent which affirms the stability and bifurcation regions obtained for the system (1.1).

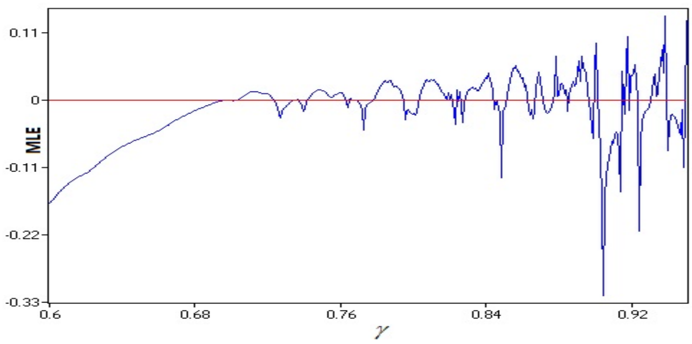

Figure 5.3: Maximum Lyapunov exponent for system (1.1).

5.2. Feedback control method

Setting the parameters $r=2, a=4, e=1, \gamma=0.7$ and initial condition $H_0=0.5, P_0=0.2$ for the system (4.1), the unique positive fixed point of the system (1.1) is unstable and the marginal stability lines for the controlled system (4.1) are

$$L_1 : h = 0.9977945005p - 0.9955890011,$$

$$L_2 : h = -1.002215271p - 1.004430524,$$

and

$$L_3 : h = 0.3330879170p + 2.333824167.$$

Figure 5.4 depicts the stable triangular area bounded by the marginal lines L_1, L_2, and L_3 for the controlled system (4.1).

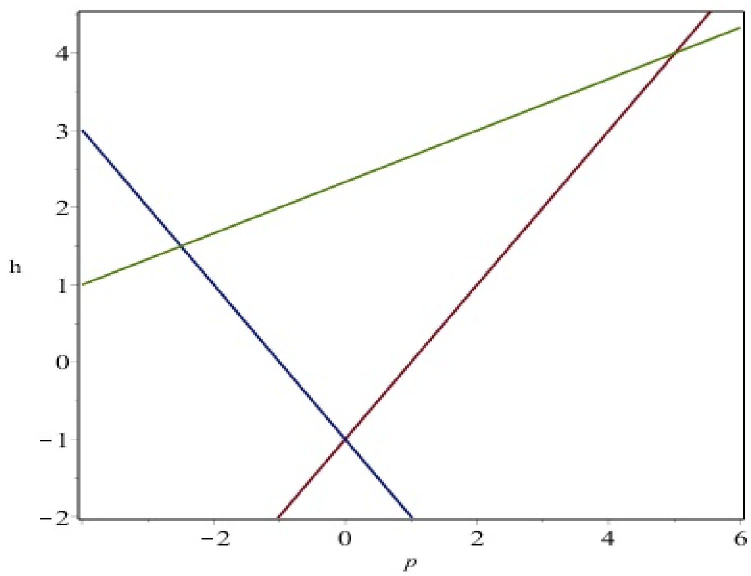

Figure 5.4: Stability region for controlled system (4.1).

5.3. Hybrid control method

Setting the parameters $r=2, a=4, e=1$ and initial condition $H_0=0.5, P_0=0.2$ for the system (4.8), the bifurcation diagrams for H_t are displayed against the bifurcation parameter α in Figure 5.5, for different values of γ. These graphs show that the fixed point (H_*, P_*) of the controlled system (4.8) is locally asymptotically stable for a wide range of the control parameter α.

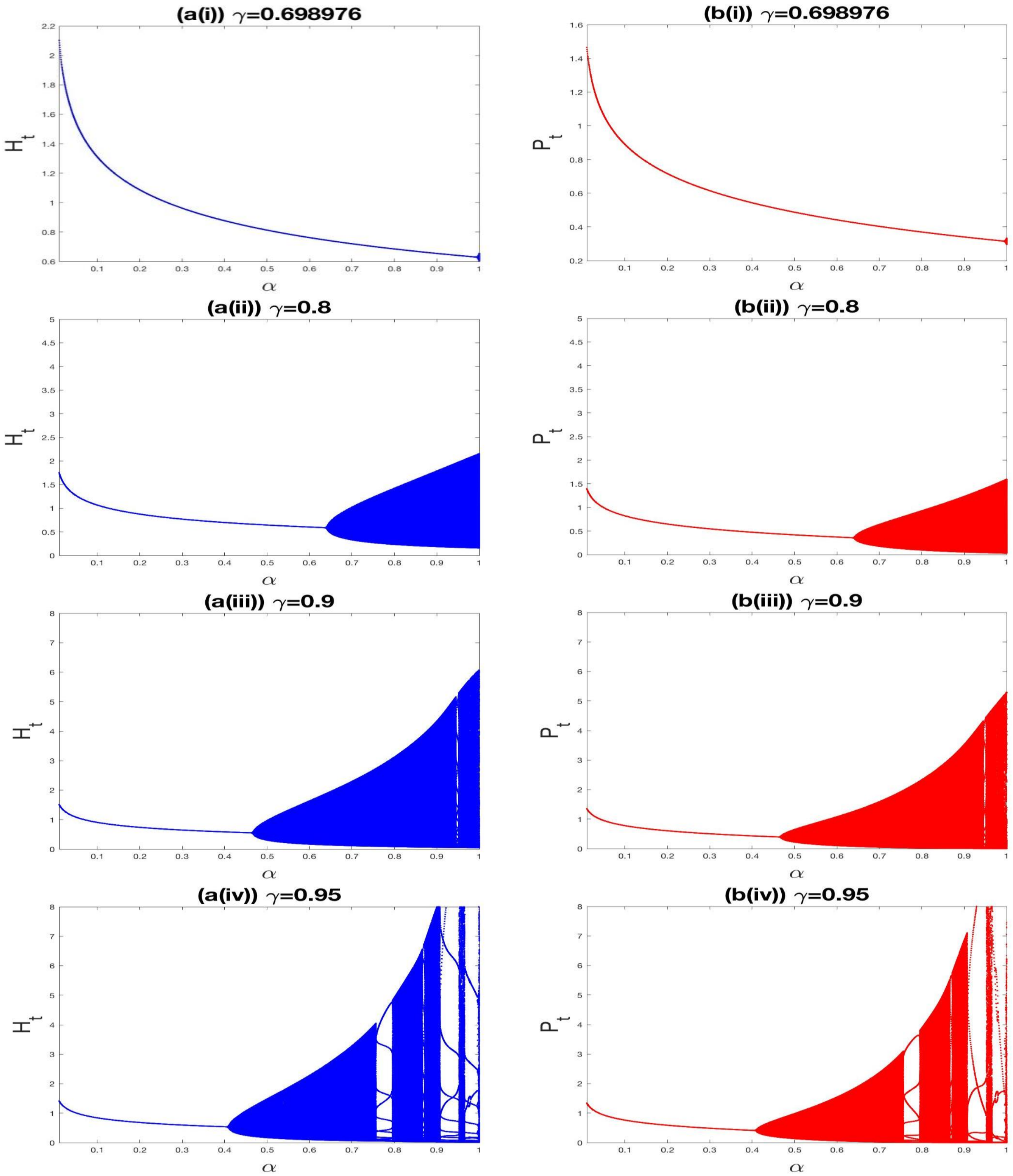

Figure 5.5: Bifurcation diagrams for controlled system (4.8) for different values of γ.

Furthermore, bifurcation diagrams for H_t are displayed against the bifurcation parameter γ in Figure 5.6 for different values of α. These graphs confirm that the bifurcation is delayed in the controlled system (4.8) compared to the original system (1.1).

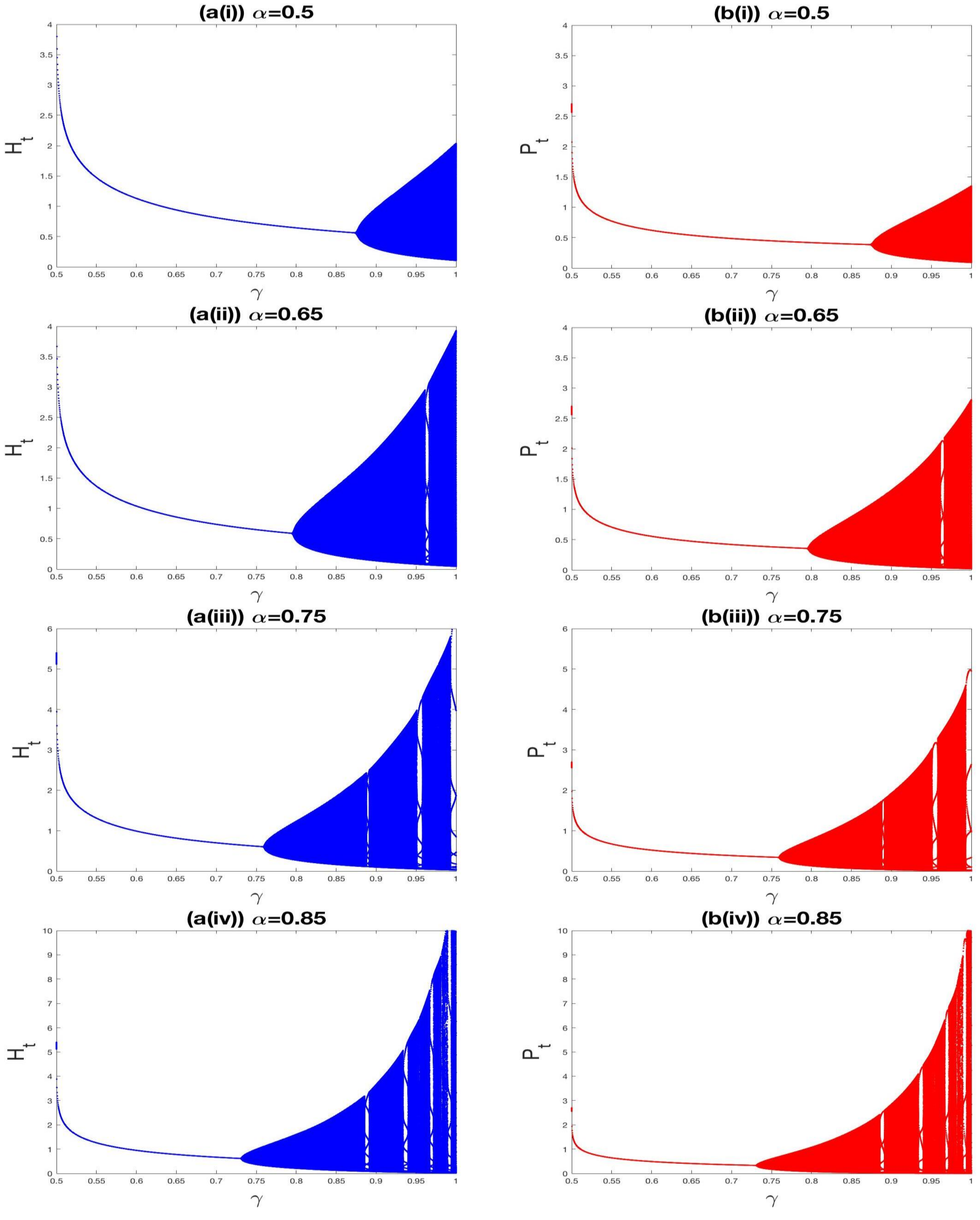

Figure 5.6: Bifurcation diagrams for controlled system (4.8) for different values of α.

5.4. Sensitive dependence on the initial conditions

Figure 5.7 shows two perturbed trajectories in blue and red colors to highlight the sensitivity of the system (1.1) to initial conditions. The two trajectories are initially overlapping and indistinguishable, but after a few iterations, the difference between them grows fast. With initial values $(H_0, P_0) = (0.5, 0.2)$ and $(H_0, P_0) = (0.50001, 0.20001)$, Figure 5.7 illustrates a sensitive dependence on the initial conditions for the system (1.1).

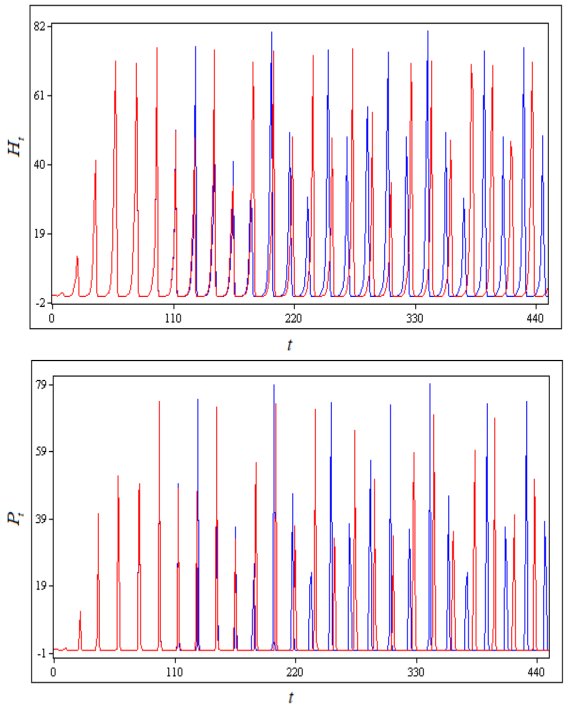

Figure 5.7: Sensitivity to initial conditions of the system (1.1).

6. Conclusion

The qualitative analysis of a host-parasitoid system (1.1) is carried out. The system (1.1) is a modification in the classical Nicholson-Bailey model, which is achieved by relaxing the uniform environment assumption with the patchy environment in which the number of hosts safe from attack by the parasitoid is fixed. The unique positive steady state of the system (1.1) is found to be

$$(H_*, P_*) = \left(\frac{r \ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right)}{ae(r-1)}, \frac{1}{a} \ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right) \right).$$

The unique positive steady-state (H_*, P_*) is topologically classified by linearization. The local stability of the steady-state (H_*, P_*) is characterized by the following set of inequalities;

$$r\left(\ln\left(\frac{r\gamma}{1-r(1-\gamma)}\right)\right)(1-r(1-\gamma)) < r-1, \quad r>1, \quad \gamma > \frac{r-1}{r}.$$

The necessary and sufficient parametric conditions are derived for the local stability of the steady-state (H_*, P_*). In addition, sufficient conditions (2.2), (2.3) and $r > 1$, $\gamma > \frac{r-1}{r}$ are derived for the steady-state (H_*, P_*) to be non-hyperbolic. The Neimark-Sacker bifurcation is carried out using the theory of normal forms by taking γ as a bifurcation parameter. The state feedback control and hybrid control strategies are used to stabilize the unstable steady state of the system. Finally, numerous numerical examples have been presented to illustrate the significance of the bifurcation parameter γ and the reproductive rate r of the host in the model (1.1). We show that the presence of a safe refuge, where a portion of the host is in a safe refuge from predation, has a stabilizing effect on the model. It is clear, therefore, that γ, the percentage of hosts that are vulnerable to parasitoids, can have a crucial impact on the stability of a host-parasitoid interaction. A small rate of escaping of a host, $1-\gamma$, may lead to instability.

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