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SOME RESULTS ON THE STUDY OF Ξ-HILFER TYPE FUZZY FRACTIONAL DIFFERENTIAL EQUATIONS WITH TIME DELAY

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Abstract. This paper is concerned with the finite-time stability of Ξ-Hilfer type fuzzy fractional differential equations (FFDEs) with time delay. By applying standard theorems and a hypothetical condition, we explore the existence of solution and stabilty results.

1. Introduction

In this manuscript, we will explore the existence and stabilty of the following Ξ-Hilfer type FFDE with time delay

$$\begin{cases} \mathscr{D}_{0^+}^{\zeta_1,\zeta_2,\Xi} w(t) = g(t, w_t), \quad t \in (0, b], \\ \mathscr{I}_{0^+}^{1-\gamma,\Xi} w(0^+) = w_0, \quad \gamma = \zeta_1 + \zeta_2 - \zeta_1\zeta_2, \\ w(t) = \chi(t), \quad t \in [-\tau, 0], \end{cases} \tag{1.1}$$

where $w \in \mathbb{R}^c$, $g : [0, b] \times C([-\tau, b], E_c) \to E_c$ is fuzzy function, where $\chi \in C([-\tau, 0], E_c)$ and E_c is the space of fuzzy sets. Moreover $\mathscr{I}_{0^+}^{1-\gamma,\Xi}$, $\mathscr{D}_{0^+}^{\zeta_1,\zeta_2,\Xi}$ denotes the Ξ-Hilfer fractional integral and derivative of order $\zeta_1 \in (0, 1)$ and type $\zeta_2 \in [0, 1]$. Compared to the literature [1] to [35], the main contributions and novality of this paper are reflected in the following aspects:

(i) The system (1.1) has delay terms, which can be truly reflected the object process of change.

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(ii) In view of different systems, although the method used to study the existence and stability, but there are many differences in the processes of proof.

(iii) We are able to prove time-stability by using new assumptions.

2. Elementary results

This section of research paper is devoted to basic results and definitions that we need for investigation of the main results.

Let us take $\mathfrak{J} = [0, b]$. Let $C([-\tau, b], E_c)$ be the family of all continuous fuzzy functions from $[-\tau, b]$ into E_c, which endowed with the supremum metric: $D_{[-\tau,b]}[w, \widehat{0}] = \sup_{t\in[-\tau,b]} D_0[w(t), \widehat{0}]$ and $AC(\mathfrak{J}, E_c)$ be the family of all absolutely continuous fuzzy functions on the interval $\mathfrak{J}$ with the values in E_c. Let $\gamma \in (0, 1)$, by $C_{\gamma,\Xi}(\mathfrak{J}, E_c)$. We denote the family of continuous functions defined by $C_{\gamma,\Xi}(\mathfrak{J}, E_c) = \{w : (0, b] \to E_c|(\Xi(t) - \Xi(0))^{1-\gamma} w(t) \in C(\mathfrak{J}, E_c)\}$.

Let E_c denote the space of all fuzzy numbers on $\mathbb{R}^c$, if $w : \mathbb{R}^c \to [0, 1]$ satisfies normal, convex, upper semicontinuous and compactly supported.
The q-level set of w is defined by

$$[w]^q = \{t \in \mathbb{R}^c : w(t) \geq q\}, \quad q \in [0, 1] \quad \text{and}$$
$$[w]^0 = \{t \in \mathbb{R}^c | w(t) > 0\}.$$

It follows that the q-level set of $w \in E_c$, $[w]^q$ is a nonempty compact interval, for any $q \in [0, 1]$. We denote by $[\underline{w}(q), \overline{w}(q)]$ the q-level of a fuzzy number w.

Definition 2.1. [12] *Ler w_1 and w_2 be two fuzzy sets defined on E_c and $\mu \in \mathbb{R}^c$. Due to Zadeh's extension principle, $w_1 + w_2$ and μw_1 are in E_c and defined as*

$$[w_1 + w_2]^q = [w_1]^q + [w_2]^q,$$
$$[\mu w]^q = \mu[w]^q, \quad \textit{for all} \quad q \in [0, 1],$$

where $[w_1]^q + [w_2]^q$ represents the usual addition of two intervals of $\mathbb{R}^c$ and $\mu[w_1]^q$ represents the usual scalar product between μ and an real interval.

Definition 2.2. [12] *The distance $D_0[w_1, w_2]$ between two fuzzy numbers is defined by*

$$D_0[w_1, w_2] = \sup_{0\leq q\leq 1} H([w_1]^q, [w_2]^q) \quad \textit{for all} \quad w_1, w_2 \in E_c, \tag{2.1}$$

where $H([w_1]^q, [w_2]^q) = max\{|\underline{w_1}(q) - \underline{w_2}(q)|, |\overline{w_1}(q) - \overline{w_2}(q)|\}$ is the Hausdroff distance between $[w_1]^q$ and $[w_2]^q$.

Definition 2.3. [12] *Let $w_1, w_2 \in E_c$. There exists $w_3 \in E_c$ such that $w_1 = w_2+w_3$, that is., $w_3 = w_1 \ominus w_2$, where w_3 is Hukuhara difference of w_1 and w_2.*
The generalized Hukuhara difference of two fuzzy numbers $w_1, w_2 \in E_c$ [gH-difference] is defined as

$$w_1 \ominus_{gH} w_2 = w_3 \Leftrightarrow \begin{cases} (i) w_1 = w_2 + w_3, \quad or \\ (ii) w_2 = w_1 + (-1)w_3, \end{cases} \tag{2.2}$$

where $w_1 \ominus_{gH} w_2$ is called as gH-difference of w_1 and w_2 in E_c.

In the q-levels, we have that for all $q \in [0, 1]$,

$$[w_1 \ominus_{gH} w_2]^q = [min\{\underline{w_1}(s) - \underline{w_2}(s), \overline{w_1}(s) - \overline{w_2}(s)\}, \tag{2.3}$$
$$max\{\underline{w_1}(s) - \underline{w_2}(s), \overline{w_1}(s) - \overline{w_2}(s)\}]. \tag{2.4}$$

Also, the condition for the existence of $w_1 \ominus_{gH} w_2$ *in the case(i) is* $d([w_1]^q) \geq d([w_2]^q)$*, and the condition for the existence of* $w_1 \ominus_{gH} w_2$ *in the case(ii) is* $d([w_2]^q) \geq d([w_1]^q)$*.*

Definition 2.4. [12] *A function* $w : [0,b] \to E_c$ *is said to be d-increasing (d-decreasing) on* $[0,b]$ *if for every* $q \in [0,1]$ *the function* $t \mapsto d[w(t)]^q$ *is nondecreasing (nonincreasing) on* $[0,b]$*. Let* w *be a d-increasing or d-decreasing on* $[0,b]$*, then we say that* w *is d-monotone on* $[0,b]$*.*

Definition 2.5. [12] *The generalized Hukuhara derivative of a fuzzy-valued function* $w : (0,b) \to E_c$ *at* t *is defined as*

$$w'_{gH}(t) = \lim_{h\to 0} \frac{w(t+h) \ominus_{gH} w(t)}{h},$$

if $w'_{gH}(t) \in E_c$*, we say that* w *is generalized Hukuhara differentiable (gH-differentiable) at* t*.*

Moreover, we say that w *is* $[(i) - gH]$*-differentiable at* t *if*

$$\begin{aligned}[w'_{gh}(t)]^q &= \left[\left[\lim_{h\to 0} \frac{\underline{w}(t+h) \ominus_{gH} \underline{w}(t)}{h}\right]^q, \left[\lim_{h\to 0} \frac{\overline{w}(t+h) \ominus_{gH} \overline{w}(t)}{h}\right]^q\right],\\ &= [(\underline{w})'(q,t), (\overline{w})'(q,t)], \end{aligned} \tag{2.5}$$

and that w *is* $[(ii) - gH]$*-differentiable at* t *if*

$$[w'_{gH}(t)]^q = [(\overline{w})'(q,t), (\underline{w})'(q,t)]. \tag{2.6}$$

Definition 2.6. [12] *Let us consider* $w \in \mathscr{L}(\mathfrak{J}, E_c)$ *as a fuzzy function and* $\zeta_1 \in (0,1)$*, then the fuzzy* Ξ*-type Riemann-Liouville integral of fuzzy-valued function* w *is defined as follows:*

$$(\mathscr{I}_{0^+}^{\zeta_1,\Xi} w)(t) = \frac{1}{\Gamma(\zeta_1)} \int_0^t \Xi'(s)(\Xi(t) - \Xi(s))^{\zeta_1 - 1} w(s)ds, \quad \text{for all} \quad t \in \mathfrak{J}, \tag{2.7}$$

where $\Gamma(\zeta_1)$ *is the Gamma function.*

Definition 2.7. [12] *Let* $w : \mathfrak{J} \to E_c$ *be a continuous fuzzy mapping. The fuzzy* Ξ*-type Riemann-Liouville fractional derivative of order* $n-1 < \alpha < n$ *for fuzzy-valued function* w *is defined by*

$$(\mathscr{D}_{0^+}^{\zeta_1,\Xi} w)(t) = \frac{1}{\Gamma(n-\zeta_1)} \left(\frac{1}{\Xi'(t)} \frac{d}{dt}\right)^n \int_0^t \Xi'(s)(\Xi(t) - \Xi(s))^{n-\zeta_1-1} w(s)ds, \forall \quad t \in \mathfrak{J}. \tag{2.8}$$

If $w \in C(\mathfrak{J}, E_c)$*, then the* Ξ*-Hilfer fractional integral of order* ζ_1 *of the fuzzy-valued function* w *is defined as follows:*

$$w_{\zeta_1,\Xi}(t) = (\mathscr{I}_{0^+}^{\zeta_1,\Xi} w)(t) = \frac{1}{\Gamma(\zeta_1)} \int_0^t \Xi'(t)(\Xi(t) - \Xi(s))^{\zeta_1 - 1} w(s)ds.$$

Since $[w(t)]^q = [\underline{w}(q,t), \overline{w}(q,t)]$ *and* $0 < \zeta_1 < 1$*, let us consider the fuzzy* Ξ*-fractional integral of the fuzzy-valued function* w *based on lower and upper functions, that is,*

$$[(\mathscr{I}_{0^+}^{\zeta_1,\Xi} w)(t)]^q = [(\mathscr{I}_{0^+}^{\zeta_1,\Xi} \underline{w})(q,t), (\mathscr{I}_{0^+}^{\zeta_1,\Xi} \overline{w})(q,t)],$$

where

$\left(\mathscr{I}_{0^+}^{\zeta_1,\Xi}\underline{w}\right)(q,t) = \frac{1}{\Gamma(\zeta_1)}\int_0^t \Xi^{'}(s)(\Xi(t)-\Xi(s))^{\zeta_1-1}\underline{w}(q,s)(s)ds,$

and

$\left(\mathscr{I}_{0^+}^{\zeta_1,\Xi}\overline{w}\right)(q,t) = \frac{1}{\Gamma(\zeta_1)}\int_0^t \Xi^{'}(s)(\Xi(t)-\Xi(s))^{\zeta_1-1}\overline{w}(q,s)(s)ds.$

In addition, it follows that the opeartor $w_{\zeta_1,\Xi}(t)$ is linear and bounded from $C([\mathfrak{J},E_c)$ to $C(\mathfrak{J},E_c)$. Indeed, we have

$$c \leq \|w\|_0 \frac{1}{\Gamma(\zeta_1)}\int_0^t \Xi^{'}(s)(\Xi(t)-\Xi(s))^{\zeta_1-1}ds = \frac{\|w\|_0}{\Gamma(\zeta_1+1)}\left(\Xi(t)-\Xi(0)\right)^{\zeta_1},$$

where $\|w\|_0 = \sup_{t\in\mathfrak{J}} D_0[w(t),\widehat{0}]$.

Definition 2.8. [12] *Let order ζ_1 and type ζ_2 satisfy $n-1<\zeta_1\leq n$ and $0\leq\zeta_2\leq 1$, with $n\in N$. The fuzzy Ξ-Hilfer generalized Hukuhara fractional derivative(or Ξ-Hilfer gH-fractional derivative) (left-sided/right-sided), with respect to t, with a function $w\in C_{1-\gamma,\Xi}(\mathfrak{J},E_c)$, is defined as follows:*

$$\begin{aligned}\left(\mathscr{D}_{0^+}^{\zeta_1,\zeta_2,\Xi}w\right)(t) =& \left(\mathscr{I}_{0^+}^{\zeta_2(1-\zeta_1),\Xi}\right)\left(\frac{1}{\Xi'(t)}\frac{d}{dt}\right)\left(\mathscr{I}_{0^+}^{(1-\zeta_2)(1-\zeta_1),\Xi}w\right)(t)\\ =& \left(\mathscr{I}_{0^+}^{\zeta_2(1-\zeta_1),\Xi}f^{\Xi}\mathscr{I}_{0^+}^{(1-\zeta_2)(1-\zeta_1),\Xi}w\right)(t),\end{aligned}$$

if the gH-derivative $w^{'}_{(1-\zeta_1),\Xi}(t)$ exists for $t\in\mathfrak{J}$, where

$$w_{(1-\zeta_1),\Xi}(t) := \left(\mathscr{I}_{0^+}^{(1-\zeta_1),\Xi}w\right)(t) = \frac{1}{\Gamma(1-\zeta_1)}\int_0^t \Xi^{'}(s)(\Xi(t)-\Xi(s))^{-\zeta_1}w(s)ds.$$

Definition 2.9. [11] *Let $\zeta_1>0$, $\zeta_2>0$. Then the two parameters Mittag-Leffler function is defined as*

$$\mathbb{E}_{\zeta_1,\zeta_2}(w) = \sum_{k=0}^{\infty}\frac{w^k}{\Gamma(\zeta_1 k+\zeta_2)}, \quad w\in E_c. \tag{2.9}$$

If $\zeta_2=1$, the one-parameter Mittag-Leffler function defined by

$$\mathbb{E}_{\zeta_1}(w) = \sum_{k=0}^{\infty}\frac{w^k}{\Gamma(\zeta_1 k+1)}, \quad w\in E_c, \zeta_1>0. \tag{2.10}$$

Definition 2.10. [11] *The fuzzy problem* (1.1) *is said to be finite time stable with respect to $\{0,J,\tau,\sigma,\epsilon\}$, $0<\delta_1<\epsilon$, $\epsilon\in E_c$, such that for any solution w of fuzzy problem* (1.1)*, if and if $D_0[w_0,\widehat{0}]<\sigma$ and $D_0[\chi,\widehat{0}]<\sigma$, implies a solution w of fuzzy problem* (1.1) *satisfying $D_{\mathfrak{J}}^{\gamma}[w,\widehat{0}]<\epsilon$.*
For our convenience, we define $\mathscr{N}(w)=\{w\in C_\gamma(\mathfrak{J},E_c).\ w$ satisfies (3.1)$\}$.

Lemma 2.1. [12] *Let $\zeta_1,\zeta_2,\vartheta_1>0$. Then*

(*i*) $\mathscr{I}_{0^+}^{\zeta_1\Xi}\mathscr{I}_{0^+}^{\zeta_2,\Xi}w(t) = \mathscr{I}_{0^+}^{\zeta_1,\zeta_2}w(t).$

(*ii*) $\mathscr{I}_{0^+}^{\zeta_1,\Xi}(\Xi(t)-\Xi(0))^{\vartheta_1-1} = \dfrac{\Gamma(\vartheta_1)}{\Gamma(\zeta_1+\vartheta_1)}(\Xi(t)-\Xi(0))^{\zeta_1+\vartheta_1-1.}$

Lemma 2.2. [12] *Let* $\zeta_1 > 0$, $0 \leq \gamma < 1$. *If* $w \in C_{\gamma,\Xi}[0,b]$ *and* $\mathscr{I}_{0^+}^{1-\zeta_1,\Xi} w \in C^1_{\gamma,\Xi}[0,b]$, *then*

$$\mathscr{I}_{0^+}^{\zeta_1,\Xi}\mathscr{D}_{0^+}^{\zeta_1,\zeta_2,\Xi} w(t) = w(t) - \frac{\mathscr{I}_{0^+}^{1-\zeta_1,\Xi} w(t)}{\Gamma(\zeta_1)}(\Xi(t)-\Xi(0))^{\zeta_1-1}.$$

Lemma 2.3. [12] *Let* $w \in L^1(0,b)$. *If* $\mathscr{D}_{0^+}^{\zeta_2(1-\zeta_1),\Xi} w$ *exists on* $L^1(0,b)$, *then*

$$\mathscr{D}_{0^+}^{\zeta_1,\zeta_2,\Xi}\mathscr{I}_{0^+}^{\zeta_1,\Xi} w = \mathscr{I}_{0^+}^{\zeta_2(1-\zeta_2),\Xi}\mathscr{D}_{0^+}^{\zeta_2(1-\zeta_1),\Xi} w, \quad \textit{for all} \quad t \in (0,b].$$

Theorem 2.4. [25]**(Schauder fixed point theorem)** *Let* $H \neq 0$ *be a bounded, closed, convex subset of a fuzzy Banach space in* X. *If* $T: H \to H$ *be a continuous compact operator. Then,* T *has at least one fixed point in* H.

Lemma 2.5. [11]**(Generalized Gronwall's Inequality)** *Let* $\zeta_1 > 0$ *and* $x_1(t), x_2(t)$ *be two nonnegative function locally integrable on* $[0,T]$. *Assume that* g *is nonnegative and nondecreasing, and let* $\Xi \in C^1([0,T],E_c)$ *an increasing function such that* $\Xi'(t) \neq 0$ *for all* $t \in [0,T]$. *If*

$$x_1(t) \leq x_2(t) + g(t)\int_0^t \Xi'(s)(\Xi(t)-\Xi(s))^{\zeta_1} x_1(s)ds, \quad t \in [0,T].$$

Then

$$x_1(t) \leq x_2(t) + \int_0^t \sum_{n=1}^{\infty} \frac{[g(t)\Gamma(\zeta_1)]}{\Gamma(n\zeta_1)}\Xi'(s)(\Xi(t)-\Xi(s))^{n\zeta_1} x_2(s)ds, \quad t \in [0,T].$$

If x_2 *be a nondecreasing function on* $[0,T]$. *Then*

$$x_1(t) \leq x_2(t)\mathbb{E}_{\zeta_1}\{g(t)\Gamma(\zeta_1)[\Xi(t)-\Xi(0)]^{\zeta_1}\}, \quad t \in [0,T].$$

Lemma 2.6. [12] *Let* $g : (0,b] \times E_c \to E_c$ *be a continuous fuzzy function. Then the following problem*

$$\begin{cases} \mathscr{D}_{0^+}^{\zeta_1,\zeta_2,\Xi} w(t) = g(t,w_t), \quad t \in (0,b], \\ \mathscr{I}_{0^+}^{1-\gamma,\Xi} w(0^+) = w_0, \quad \gamma = \zeta_1+\zeta_2-\zeta_1\zeta_2, \end{cases}$$

is equivalent to integral equation

$$w(t) = \frac{(\Xi(t)-\Xi(0))^{\gamma-1}}{\Gamma(\gamma)} w_0 + \frac{1}{\Gamma(\zeta_1)}\int_0^t \Xi'(s)(\Xi(t)-\Xi(s))^{\zeta_1} g(s,w_s)ds.$$

3. Existence and stability theory

In this section, we estabilish and demonstrate the existence and stabilty of (1.1). We assume the following assumptions before begining and examining the key outcomes. $(A1)$ There exists a positive constants $\mathscr{L}$ such that

$$D_0[g(t,w),\widehat{0}] \leq \mathscr{L}(\Xi(t)-\Xi(0))^{1-\gamma} D_{[-\tau,0]}[w,\widehat{0}],$$
$$\text{for all} \quad w \in C([-\tau,0],E_c), w \in E_c, t \in \mathfrak{J}.$$

with $\mathscr{L} \in \big[0, \Gamma(\zeta_1+1)\big(\frac{1}{\Xi(b)-\Xi(0)}\big)^{1+\zeta_1-\gamma}\big]$.
$(A2)$ There exists a positive constants $\mathscr{L}^*$ such that

$$\begin{aligned} D_0[g(t,w_t),g(t,w_t^*)] &\leq \mathscr{L}^* D_{[-\tau,0]}[w_t,w_t^*] \\ &= \mathscr{L}^* D_{[t-\tau,t]}[w,w^*], \quad \text{for all} \quad w,w^* \in E_c. \end{aligned}$$

Lemma 3.1. *Let $g : [0,b] \times C([-\tau,0],E_c) \to E_c$ be a continuous fuzzy function, $\chi \in C([-\tau,0],E_c)$. Then a d- monotone fuzzy function $w \in C(\mathfrak{J},E_c)$ is a solution of initial value problem* (1.1) *if and only if w satisfies the integral equation*

$$w(t) \ominus_{gH} \frac{(\Xi(t)-\Xi(0))^{\gamma-1}}{\Gamma(\gamma)} w_0 = \frac{1}{\Gamma(\zeta_1)} \int_0^t \Xi^{'}(s)(\Xi(t)-\Xi(s))^{\zeta_1-1} g(s,w_s)ds, \quad t \in \mathfrak{J} \tag{3.1}$$

and $w(t) = \chi(t)$, $t \in [-\tau,0]$, and the fuzzy function $t \mapsto \mathscr{I}_{0^+}^{1-\gamma,\Xi} g(t,w_t)$ is d-increasing on $\mathfrak{J}$.

Proof. Let us assume $w \in C(\mathfrak{J},E_c)$ be a d-monotone solution of (1.1) and let $y(t) = w(t) \ominus_{gH} (\mathscr{I}_{0^+}^{1-\gamma,\Xi})$, $t \in \mathfrak{J}$. Since w is d-monotone on $\mathfrak{J}$, it follows that $t \mapsto y(t)$ is d-increasing on $\mathfrak{J}$. It follow from (1.1) and Lemma 2.12 we have

$$\mathscr{I}_{0^+}^{\zeta_1,\Xi} \mathscr{D}_{0^+}^{\zeta_1,\zeta_2,\Xi} w(t) = w(t) \ominus_{gH} \frac{w_0}{\Gamma(\gamma)} (\Xi(t)-\Xi(0))^{1-\gamma}, \quad t \in \mathfrak{J}.$$

Since $g(t,w) \in C_{\gamma,\Xi}(\mathfrak{J},E_c)$ for any $w \in E_c$ and by using the Eqn.(1.1), it follows that

$$\begin{aligned}\mathscr{I}_{0^+}^{\zeta_1,\Xi} \mathscr{D}_{0^+}^{\zeta_1,\zeta_2,\Xi} w(t) &= \mathscr{I}_{0^+}^{\zeta_1,\Xi} g(t,w_t) \\ &= \frac{1}{\Gamma(\zeta_1)} \int_0^t \Xi^{'}(s)(\Xi(t)-\Xi(s))^{\zeta_1-1} g(s,w_s)ds, \quad t \in \mathfrak{J}.\end{aligned}$$

In addition, since $y(t)$ is d-increasing on $(0,b]$, due to $t \mapsto g_{\zeta_1,\Xi}(t,w)$ is also d-increasing on $(0,b]$. We obtain that

$$w(t) \ominus_{gH} \frac{(\Xi(t)-\Xi(0))^{\gamma-1}}{\Gamma(\gamma)} w_0 = \frac{1}{\Gamma(\zeta_1)} \int_0^t \Xi^{'}(s)(\Xi(t)-\Xi(s))^{\zeta_1-1} g(s,w_s)ds, \quad t \in \mathfrak{J}$$

For every $t \in [-\tau,0]$, we have $w(t) = \chi(t)$. This implies that (3.1) is satisfied.
Conversely, assume that $w \in C(\mathfrak{J},E_c)$ satisfies (1.1). If $t \in [0,b]$, then $w(0^+) = w_0$, and applying $\mathscr{D}_{0^+}^{\zeta_1,\zeta_2,\Xi}$ on both sides, we obtain
$\mathscr{D}_{0^+}^{\zeta_1,\zeta_2,\Xi} w(t) = g(t,w_t), \quad t \in (0,b]$.
And we can easily prove that $w(t) = \chi(t)$ for $t \in [-\tau,0]$. □

Lemma 3.2. *Let $g : [0,b] \times C([-\tau,0],E_c) \to E_c$ be a continuous fuzzy function, $\chi \in C([-\tau,0],E_c)$. Assume that (A1) is satisfied. Then for any $w \in C([-\tau,b],E_c)$ of Eqn.*(1.1)*, there exists a constant $\eta > 0$ such that $D_{[-\tau,b]}[w,\widehat{0}] \le \eta$.*

Proof. Let us assume $w \in C([-\tau,0],E_c)$. If $t \in [-\tau,0]$, then we have that $w(t) = \chi(t)$. In according to the boundedness of χ, which gives $w(t)$ is bounded.
Suppose $t \in \mathfrak{J}$, which is $w \in \mathscr{N}(y)$. Then, for $\xi \in [0,t]$, $t \in (0,b]$, it follows that , we have

$$\begin{aligned}D_{[-\tau,0]}[w_\xi,\widehat{0}] &= \sup_{\theta\in[-\tau,0]} D_0[y_\xi(\theta),\widehat{0}] \\ &= \sup_{\theta\in[-\tau,0]} D_0[y(\xi+\theta),\widehat{0}] \\ &\le \sup_{r\in[-\tau,0]} D_0[y_r,\widehat{0}] + \sup_{r\in[0,\xi]} D_0[y(r),\widehat{0}] \\ &\le D_{[-\tau,0]}[\chi,\widehat{0}] + \sup_{r\in[0,\xi]} D_0[y_r,\widehat{0}].\end{aligned} \tag{3.2}$$

Hence, for $t \in (0,b]$, by using (3.1), (3.2), $(A1)$, Definition 2 and the Beta Function $B(\cdot,\cdot)$, we have

$$
\begin{aligned}
&D_0[(\Xi(t)-\Xi(0))^{1-\gamma}w(t),\widehat{0}]\\
&\quad\le D_0\left[(\Xi(t)-\Xi(0))^{1-\gamma}\left(\frac{(\Xi(t)-\Xi(0))^{\gamma-1}}{\Gamma(\gamma)}w_0\right),\widehat{0}\right]\\
&\quad+D_0\left[(\Xi(t)-\Xi(0))^{1-\gamma}\frac{1}{\Gamma(\zeta_1)}\int_0^t\Xi^{'}(s)(\Xi(t)-\Xi(s))^{\zeta_1-1}g(s,w_s)ds,\widehat{0}\right]\\
&\quad\le\frac{1}{\Gamma(\gamma)}D_0[w_0,\widehat{0}]\\
&\quad+\frac{(\Xi(b)-\Xi(0))^{1-\gamma}}{\Gamma(\zeta_1)}\int_0^t\Xi^{'}(s)(\Xi(t)-\Xi(s))^{\zeta_1-1}\{\mathscr{L}(\Xi(s)-\Xi(0))^{1-\gamma}D_{[-\tau,0]}[w_s,\widehat{0}]\}ds\\
&\quad\le\frac{1}{\Gamma(\gamma)}D_0[w_0,\widehat{0}]\\
&\quad+\mathscr{L}(\Xi(b)-\Xi(0))^{1-\gamma}\left\{D_{[-\tau,0]}[\chi,\widehat{0}]\frac{1}{\Gamma(\zeta_1)}\int_0^t\Xi^{'}(s)(\Xi(t)-\Xi(s))^{\zeta_1-1}(\Xi(s)-\Xi(0))^{1-\gamma}ds\right.\\
&\quad\left.+\frac{1}{\Gamma(\zeta_1)}\int_0^t\Xi^{'}(s)(\Xi(t)-\Xi(s))^{\zeta_1-1}\sup_{r\in[0,\xi]}D_0[(\Xi(s)-\Xi(0))^{1-\gamma}y_s,\widehat{0}]\right\}\\
&\quad\le\frac{1}{\Gamma(\gamma)}D_0[w_0,\widehat{0}]\\
&\quad+\frac{\mathscr{L}}{\Gamma(\zeta_1)}(\Xi(b)-\Xi(0))^{\zeta_1+2-2\gamma}B(2-\gamma,\zeta_1)D_{[-\tau,0]}[\chi,\widehat{0}]\\
&\quad+\frac{\mathscr{L}}{\Gamma(\zeta_1)}(\Xi(b)-\Xi(0))^{1-\gamma}\int_0^t\Xi^{'}(s)(\Xi(t)-\Xi(s))^{\zeta_1-1}\sup_{r\in[0,\xi]}D_0[(\Xi(s)-\Xi(0))^{1-\gamma}y_s,\widehat{0}]ds.
\end{aligned}
$$

It follows from the generalized Gronwall inequality gives that,

$$N(t)\le M^{*}\mathbb{E}_{\zeta_1}(\mathscr{L}(\Xi(b)-\Xi(0))^{1-\gamma}(\Xi(t)-\Xi(0))^{\zeta_1})=\eta,$$

where

$$
\begin{aligned}
N(t)&=\sup_{r\in[0,\xi]}D_0[(\Xi(s)-\Xi(0))^{1-\gamma}y_s,\widehat{0}],\\
M^{*}&=\frac{1}{\Gamma(\gamma)}D_0[w_0,\widehat{0}]+\frac{\mathscr{L}}{\Gamma(\zeta_1)}(\Xi(b)-\Xi(0))^{\zeta_1+2-2\gamma}B(2-\gamma,\zeta_1)D_{[-\tau,0]}[\chi,\widehat{0}].
\end{aligned}
$$

This implies that, there exists a constant $\eta>0$ such that $D_{[-\tau,b]}[w,\widehat{0}]\le\eta$. □

Theorem 3.3. *Let $g:[0,b]\times C([-\tau,0],E_c)\to E_c$ be a continuous fuzzy function, $\chi\in C([-\tau,0],E_c)$. Assume that $(A1)$ is satisfied. Then the fuzzy problem (1.1) has at least one solution $w\in C([-\tau,b],E_c)\cap C_\gamma(\mathfrak{J},E_c)$.*

Proof. Let us define the operator $\Theta: C([-\tau,b],E_c)\to C([-\tau,b],E_c)\cap C_\gamma(\mathfrak{J},E_c)$ is given by

$$
(\Theta w)(t)=\begin{cases}(Tw)(t)\ominus_{gH}\frac{(\Xi(t)-\Xi(0))^{\gamma-1}}{\Gamma(\gamma)}w_0=\frac{1}{\Gamma(\zeta_1)}\int_0^t\Xi^{'}(s)(\Xi(t)-\Xi(s))^{\zeta_1-1}g(s,w_s)ds,\\ t\in\mathfrak{J},\\ w(t)=\chi(t),\quad t\in[-\tau,0],\end{cases}
$$

where $T : C_\gamma(\mathfrak{J}, E_c) \to C_\gamma(\mathfrak{J}, E_c)$, let us assume $w \in C([-\tau, 0], E_c)$. Because $w(t) = \chi(t)$, $t \in [-\tau, 0]$.

Step 1. $T(B_{\eta_1}) \subseteq B_{\eta_1}$.

Let us define a bounded, closed and convex set $B_{\eta_1} \in C_{1-\gamma}[0, b]$ as follows

$$B_{\eta_1} = \{y \in C_\gamma(\mathfrak{J}, E_c) | D^\gamma_{[0,b]}[y, \widehat{0}] \le \eta_1\}.$$

with

$$\eta_1 \ge max\Bigg\{\bigg(\frac{1}{\Gamma(\gamma)}D_0[w_0, \widehat{0}] + \frac{\mathscr{L}}{\Gamma(\zeta_1)}(\Xi(b) - \Xi(0))^{\zeta_1+2-2\gamma}B(2-\gamma, \zeta_1)D_{[-\tau,0]}[\chi, \widehat{0}]\bigg) \times \frac{\Gamma(\zeta_1+1)}{\Gamma(\zeta_1+1) - \mathscr{L}(\Xi(b) - \Xi(0))^{1+\zeta_1-\gamma}}, \eta\Bigg\}.$$

If $w \in B_{\eta_1}$. Then, for $r \in [0, t]$, $t \in (0, b]$, we get

$$\begin{aligned} D_{[-\tau,0]}[w_t, \widehat{0}] &\le \sup_{r\in[-\tau,0]} D_0[w_t(s), \widehat{0}] \\ &= \sup_{\xi\in[t-\tau,t]} D_0[w(\xi), \widehat{0}] \\ &\le D_{[-\tau,0]}[\chi, \widehat{0}] + D_{[0,b]}[w, \widehat{0}]. \end{aligned} \tag{3.3}$$

Therefore, for each $t \in (0, b]$, we get

$$\begin{aligned} &D_0[(\Xi(t) - \Xi(0))^{1-\gamma}(Tw)(t), \widehat{0}] \\ &\le \frac{1}{\Gamma(\gamma)}D_0[w_0, \widehat{0}] + \frac{\mathscr{L}}{\Gamma(\zeta_1)}(\Xi(b) - \Xi(0))^{\zeta_1+2-2\gamma}B(2-\gamma, \zeta_1)D_{[-\tau,0]}[\chi, \widehat{0}] \\ &+ \frac{\mathscr{L}}{\Gamma(\zeta_1)}(\Xi(b) - \Xi(0))^{1-\gamma}\int_0^t \Xi'(s)(\Xi(t) - \Xi(s))^{\zeta_1-1}\sup_{r\in[0,\xi]} D_0[(\Xi(s) - \Xi(0))^{1-\gamma}y_s, \widehat{0}]]ds \\ &\le \frac{1}{\Gamma(\gamma)}D_0[w_0, \widehat{0}] + \frac{\mathscr{L}}{\Gamma(\zeta_1)}(\Xi(b) - \Xi(0))^{\zeta_1+2-2\gamma}B(2-\gamma, \zeta_1) + D_{[-\tau,0]}[\chi, \widehat{0}] \\ &+ \frac{\kappa}{\Gamma(\zeta_1+1)}(\Xi(b) - \Xi(0))^{1+\zeta_1-\gamma}\sup_{t\in\mathfrak{J}} D_0[\Xi(t) - \Xi(0))^{1-\gamma}w_t, \widehat{0}]. \end{aligned}$$

This proves that $T(B_{\eta_1}) \subseteq B_{\eta_1}$.

Step 2. T is continuous on B_{η_1}.

Let $\{w_n\}^\infty_{n\ge1}$ $(n = 1, 2, ...)$ be a sequence in B_{η_1} such that $w_n \to w$ in $C([-\tau, 0], E_c)$. Then, for each $t \in \mathfrak{J}$, we have

$$\begin{aligned} &D_0[(\Xi(t) - \Xi(0))^{1-\gamma}(Tw_n)(t), (\Xi(t) - \Xi(0))^{1-\gamma}(Tw)(t)] \\ &\le D_0\bigg[(\Xi(t) - \Xi(0))^{1-\gamma}\bigg((Tw_n)(t) \ominus_{gH} \frac{(\Xi(t) - \Xi(0))^{\gamma-1}}{\Gamma(\gamma)}w_0\bigg), \\ &(\Xi(t) - \Xi(0))^{1-\gamma}\bigg((Tw)(t) \ominus_{gH} \frac{(\Xi(t) - \Xi(0))^{\gamma-1}}{\Gamma(\gamma)}w_0\bigg)\bigg] \\ &\le (\Xi(b) - \Xi(0))^{1-\gamma}\frac{1}{\Gamma(\zeta_1)}\int_0^t \Xi'(s)(\Xi(t) - \Xi(s))^{\zeta_1-1}D_0[g(s, w_{ns}), g(s, w_s)]ds \\ &\le (\Xi(b) - \Xi(0))^{1-\gamma}\frac{\mathscr{L}^*}{\Gamma(\zeta_1)}\int_0^t \Xi'(s)(\Xi(t) - \Xi(s))^{\zeta_1-1}D_{[-\tau,0]}[w_{ns}, w_s]ds \\ &\to 0 \quad as \quad n \to \infty. \end{aligned}$$

Now, $\lim_{n\to\infty} w_n = w \in B_{\eta_1}$. Then, for each $t \in (0, b]$, we get $\lim_{n\to\infty}(\Xi(t) - \Xi(0))^{1-\gamma}w_n = (\Xi(t) - \Xi(0))^{1-\gamma}w$. Furthermore, for each $t \in [-\tau, 0]$,

due to $\chi \in C([-\tau,0],E_c)$ that is, $\lim_{n\to\infty} w_n(t) = \chi(t) = g_1(w)(t) = w(t)$. Moreover, one has $\|w_n\|_{C_{1-\zeta_1}} \leq \eta_1$ and $\|w\|_{C_{1-\zeta_1}} \leq \eta_1$. Hence, for $r \in [0,t]$, $t \in (0,b]$, we get $\lim_{n\to\infty} w_{ns} = w_s, \quad D_{[-\tau,0]}[w_{ns},\widehat{0}] \leq \eta_1 + D_{[-\tau,0]}[\chi,\widehat{0}]$,
$D_{[-\tau,0]}[w_s,\widehat{0}] \leq \eta_1 + D_{[-\tau,0]}[\chi,\widehat{0}]$. This implies that, it follows from $(A1)$ $(A2)$, for $r \in (0,t)$, $t \in (0,b]$, one's get

$$\begin{aligned} D_0[g(s,w_{ns}),g(s,w_s)] &\leq D_0[g(s,w_{ns}),\widehat{0}] + D_0[g(s,w_s),\widehat{0}] \\ &\leq 2\mathscr{L}((\Xi(b)-\Xi(0))D_{[-\tau,0]}[\chi,\widehat{0}] + \kappa\eta_1). \end{aligned}$$

Taking into account the fact that T is continuous, that is, $D_0[g(s,w_{ns}),g(s,w_s)] \to 0 \quad \text{as} \quad w_n \to w$, which gives $\|w_{ns} - w_s\|_0 \to 0 \quad \text{as} \quad w_n \to w$,
where $\sup_{t\in\mathfrak{J}} D_0[(Tw_n)(t),(Tw)(t)] \leq \|Tw_n - Tw\|_0$. Thus T is continuous
Step 3. T is compact in B_{η_1}
First, we have to prove T maps bounded sets into equicontinuous sets in B_{η_1}.
For any $t_1,t_2 \in (0,b]$, $t_1 < t_2$ and $w \in B_{\eta_1}$, we get
$D_0[(\Xi(t_2)-\Xi(0))^{1-\gamma}(Tw)(t_2),(\Xi(t_1)-\Xi(0))^{1-\gamma}(Tw)(t_1)]$

$$\begin{aligned} &\leq D_0\Bigg[(\Xi(t_2)-\Xi(0))^{1-\gamma}\Bigg((Tw)(t_2) \ominus_{gH} \frac{(\Xi(t_2)-\Xi(0))^{\gamma-1}}{\Gamma(\gamma)}w_0\Bigg), \\ &(\Xi(t_1)-\Xi(0))^{1-\gamma}\Bigg((Tw)(t_1) \ominus_{gH} \frac{(\Xi(t_1)-\Xi(0))^{\gamma-1}}{\Gamma(\gamma)}w_0\Bigg)\Bigg] \\ &\leq \frac{1}{\Gamma(\zeta_1)}(\Xi(t_2)-\Xi(0))^{1-\gamma}\int_{t_1}^{t_2}\Xi'(s)(\Xi(t_2)-\Xi(s))^{\zeta_1-1}D_0[g(s,w_s),\widehat{0}]ds \\ &+ \frac{1}{\Gamma(\zeta_1)}\int_0^{t_1}\Xi'(s)[(\Xi(t_2)-\Xi(0))^{1-\gamma}(\Xi(t_2)-\Xi(s))^{\zeta_1-1} - (\Xi(t_1) \\ &- \Xi(0))^{1-\gamma}(\Xi(t_1)-\Xi(s))^{\zeta_1-1}]D_0[g(s,w_s),\widehat{0}]ds \\ &\to 0 \quad as \quad t_2 \to t_2. \end{aligned}$$

The right hand sides of the above equation tends to zero independently of $w \in B_{\eta_1}$ as $t_2 \to t_1$, which means that $D_0[(Tw)(t_2),(Tw)(t_1)] \to 0$. Thus, it follows from the Arzela-Ascoli theorem gives that the operator T is completely continuos. Consequently, by using the Schauder's fixed point theorem gives that the operator T has at least one fixed point. Hence Eqn.(1.1) has at least one solution on $\mathfrak{J}$. This completes the proof. □

Theorem 3.4. *Assume that $g : [0,b] \times C([-\tau,0],Ec) \to E_c$ be a continuous fuzzy function, $\chi \in C([-\tau,0],E_c)$. Assume that (A1)-(A2) is satisfied, then the Eqn.(1.1) is finite-time stable with respect to $\{0,[-\tau,b],\tau,\sigma,\epsilon\}$, $0 < \sigma < \epsilon$, $\sigma,\epsilon \in \mathbb{R}^c$.*
If $M_1^*\mathbb{E}_{\zeta_1}(\mathscr{L}(\Xi(b)-\Xi(0))^{1-\gamma}(\Xi(b)-\Xi(0))^{\zeta_1}) < 1$, $t \in \mathfrak{J}$
where

$$M_1^* = \frac{1}{\Gamma(\gamma)} + \frac{\mathscr{L}}{\Gamma(\zeta_1)}(\Xi(b)-\Xi(0))^{\zeta_1+2-2\gamma}B(2-\gamma,\zeta_1).$$

Proof. According to the similar proof (**??**) and by Definition 2.10, we have

$$
\begin{aligned}
&D_0[(\Xi(t)-\Xi(0))^{1-\gamma}w(t),\widehat{0}] \\
&\leq D_0\left[(\Xi(t)-\Xi(0))^{1-\gamma}\left(\frac{(\Xi(t)-\Xi(0))^{\gamma-1}}{\Gamma(\gamma)}w_0\right),\widehat{0}\right] \\
&+D_0\left[(\Xi(t)-\Xi(0))^{1-\gamma}\frac{1}{\Gamma(\zeta_1)}\int_0^t \Xi^{'}(s)(\Xi(t)-\Xi(s))^{\zeta_1-1}g(s,w_s)ds,\widehat{0}\right] \\
&\leq \frac{1}{\Gamma(\gamma)}D_0[w_0,\widehat{0}]+\frac{\mathscr{L}}{\Gamma(\zeta_1)}(\Xi(b)-\Xi(0))^{\zeta_1+2-2\gamma}B(2-\gamma,\zeta_1)D_{[-\tau,0]}[\chi,\widehat{0}] \\
&+\frac{\mathscr{L}}{\Gamma(\zeta_1)}(\Xi(b)-\Xi(0))^{1-\gamma}\int_0^t \Xi^{'}(s)(\Xi(t)-\Xi(s))^{\zeta_1-1}\sup_{r\in[0,\xi]}D_0[(\Xi(s)-\Xi(0))^{1-\gamma}y(s),\widehat{0}]ds, \\
&\leq \frac{1}{\Gamma(\gamma)}\sigma+\frac{\mathscr{L}}{\Gamma(\gamma)}(\Xi(b)-\Xi(0))^{\zeta_1+2-2\gamma}B(2-\gamma,\zeta_1)\sigma+\frac{\mathscr{L}}{\Gamma(\zeta_1)}(\Xi(b)-\Xi(0))^{1-\gamma} \\
&\times\int_0^t \Xi^{'}(s)(\Xi(t)-\Xi(s))^{\zeta_1-1}\sup_{r\in[0,\xi]}D_0[(\Xi(s)-\Xi(0))^{1-\gamma}y(s),\widehat{0}]ds
\end{aligned}
$$

Now, we put

$$
\begin{aligned}
N(t)&=\sup_{r\in[0,t]}D_0[(\Xi(s)-\Xi(0))^{1-\gamma}y(s),\widehat{0}], \\
M_1^*&=\frac{1}{\Gamma(\gamma)}+\frac{\mathscr{L}}{\Gamma(\zeta_1)}(\Xi(b)-\Xi(0))^{\zeta_1+2-2\zeta_1}B(2-\gamma,\zeta_1).
\end{aligned}
$$

It follows from the generalized Gronwall inequality gives that, we have

$$N(t)=D^{\gamma}_{[0,b]}[w,\widehat{0}]\leq\sigma M_1^*\mathbb{E}_{\zeta_1}(\mathscr{L}(\Xi(b)-\Xi(0))^{1-\gamma}(\Xi(t)-\Xi(0))^{\zeta_1})<\sigma<\epsilon.$$

Therefore, Eqn.(1.1) is finite-time stable. This completes the proof. □

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