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Production lot sizing with quality screening, rework and shortages backordered

Kalite kontrol, tamir ve sonradan karşılama ile üretim parti büyüklüklerinin belirlenmesi

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Abstract

In the classic EOQ and EPQ models, the assumption that all products are of good quality is not always valid. In some cases, the production system may produce a certain amount of defective products due to human errors and/or wear of machinery and equipment. In the literature, defective products obtained with quality screening are categorized as scrap, reworkable, low quality, etc. Generally, low-quality products are sold at a reduced price in lots at the end of the selection procedure, while scrap products are eliminated from the inventory at a certain cost. Reworkable products are also reworked at the end of the quality control period and turned into good products. In our study, screening activities are carried out during the production period in order to acquire the perfect items to run into the demand. For the remaining products, screening activity begins at the end of the production period. Allowing backorder in production systems both provides efficient production schedules and reduces operating costs. In this study, an EPQ model was constructed in which all defective products are taken at coincidental and reworked, and shortages are allowed. Sensitivity analyzes are also provided to evaluate the effects of changes in design variables on the optimal solution and to obtain feasible conditions.

Keywords: EPQ, Shortages, Rework, Backorder, Quality screening, Production lot sizing, Symbolic computation.

Öz

Klasik ESM ve EÜM modellerinde, tüm ürünlerin iyi kalitede olduğu varsayımı her zaman geçerli değildir. Bazı durumlarda, üretim sistemi, insan hataları ve/veya makine ve ekipmanların aşınması nedeniyle belirli miktarda kusurlu ürün üretebilir. Literatürde kaliteli kontrol ile elde edilen hatalı ürünler hurda, yeniden işlenebilir, düşük kalite vb. olarak sınıflandırılır. Genel olarak düşük kaliteli ürünler, eleme işlemi sonunda partiler halinde indirimli fiyattan satılırken, hurda ürünler ise belirli bir maliyetle envanterden çıkartılır. Ayrıca kalite kontrol süreci sonunda tamir edilebilir ürünler yeniden çalışılarak kusursuz ürünlere dönüştürülür. Çalışmamızda talebi karşılayacak kusursuz ürünleri elde etmek için üretim döneminde kalite kontrol çalışmaları yapılmaktadır. Kalan ürünler için ise üretim döneminin sonunda kalite kontrol faaliyeti başlar. Üretim sistemlerinde ön siparişe izin vermek, hem verimli üretim programları sağlar hem de işletme maliyetlerini düşürür. Bu çalışmada, tüm kusurlu ürünlerin rastgele bir oranla alınıp yeniden çalışıldığı, stoksuzluğa izin verilen bir EÜM modeli geliştirilmiştir. Model parametrelerindeki değişikliklerin optimal çözüm üzerindeki etkilerini incelemek ve uygun çözüm koşullarını elde etmek için duyarlılık analizleri de verilmektedir.

Anahtar kelimeler: EÜM, Stoksuzluk, tamir, Sonradan karşılama, Kalite kontrol, Üretim parti büyüklüğü, Sembolik hesaplama.

1 Introduction

Logistics systems started in the hunter-gatherer era, before human beings settled down in the Neolithic period. The rapid transformation paradigms that occurred in the social life of human beings with the industrial revolution paved the way for the production and marketing processes to evolve to a different point, and the complex relations in production and consumption relations made it necessary to manage these processes with scientific methods. Following the industrial revolution, scientific studies on logistics processes have found more place in human life with the First and Second World Wars. The period following the Second World War passed into a different phase with the influence of computers in industrial processes until the 90s. Especially since the 90s, with the developments in computer and Internet technologies, logistics applications in the industrial field have clearly revealed their effective presence. Inventory control models constitute one of the most critical areas in logistics applications in the industrial field.

Inventory control models concerning inventory management started with Harris and Taft. The economic order quantity (EOQ) model was introduced by F.W. Harris in 1913 following the industrial revolution [1]. This model ensures optimal order quantity by minimizing inventory holding and ordering costs. The second development in this area was the development of the economic production quantity (EPQ) model by E.W. Taft in 1918 [2]. The EPQ model is the basic model that enables a production company to find the optimal production amount with production costs, holding costs and production preparation costs.

The classical EOQ and EPQ models are based on the assumption that all manufactured products are robust. This assumption may not always be valid. EPQ models taking into account production lot size containing defective products and EOQ models considering order lot size have been developed. Salameh and Jaber (2000) extends the classical EOQ model by accounting for defective product rate that fits a certain probability distribution [3]. Hayek and Salameh (2001) built an EPQ model in which the defective product rate is a random

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variable and allows for shortages [4]. These two studies can be described as pioneering locomotive studies in this field. After these aforementioned studies, models in which defective products are categorized as scrap, imperfect quality and repairable have started to be produced intensively.

Chiu (2003) has developed an EPQ model in which a certain proportion of defective products are scraped and removed from the inventory at a certain cost, while the rest are reworked and turned into good quality products by assuming that the defective product rate is a random variable conforming to a uniform probability distribution and that shortages are allowed [5]. Chiu and Chiu (2003) introduced an EPQ model for the situation where imperfect goods are scrapped and the remainder are reworked and included in the inventory of good quality products, where shortages are not allowed [6]. In their study, Chan et al. (2003) classified defective items into three categories as scrap, reworkable and imperfect. Their rates are obtained from the normal distribution curve. They constructed an EPQ model assuming that scrap items are take out from the depot at a certain cost, imperfect items are sold in lots at a discounted price, reworkable items are reworked and added to the good quality product inventory [7]. Chiu and Gong (2004) built up an EPQ model for the situation where the rate of defective products is a coincidental variable, all of them are reworked and a certain proportion is scrapped due to the imperfect rework process, others are included in the good quality product inventory, and shortages are allowed [8]. Chiu et al (2004) classified defective products into scrap and reworkable. Because of the imperfect rework process, they had constructed an EPQ model for situations where some of the reworked products are included in the inventory as scrap and others as good quality products and shortages are not allowed [9]. Jamal et al. (2004) have developed a variety of models to solve the optimum lot size in a one-phase system where rework is carried out under two operation policies to obtain optimum overall system costs [10].

Chiu et al (2006) classified defective items into two categories as scrap and reworkable. After the well-ordered manufacturing, the rework operation starts and the reworked ones are included in the inventory as good quality products. They developed an EPQ model that allows shortages and investigated the effects of using rework process on the optimal solution [11]. Chiu and Chiu (2006) obtained an EPQ model for coincidental rate of defective produced goods, imperfect rework process and shortages [11]. Chiu et al (2007) studied the optimization of a production system that is subject to level of service and backlogging constraints and the reworking of defective and scrap products that are randomly produced [12]. Chiu (2007) designed a model in which all products are screened and the defective ones are classified as d reworkable and scrap. Then a model is designed to obtain the optimal replenishment policy for inferior goods under rework and backordering assumptions [13].

The system considered by Ojha et al (2007) which is imperfect and produces defective items at a constant rate, assumed that the produced items possibly be transported if the full quantities are quality approved. Therefore, defective items must be reworked and the quality of full lot should be completed until the end of the cycle [14]. Eroğlu and Özdemir (2007) [15] and Wee (2007) [16] improved the model of Salameh and Jaber (2000) [3] to allow shortages.

Models dealing with machine failures in the production system also have an important place in the literature. Chiu et al (2007) discussed a model in which defective items are divided into two categories as scrap and reworkable items, and that does not allow shortages. When the machine fails, production ends and the machine is immediately repaired. The number of machine failures is assumed to fit the Poisson distribution. After the machine is repaired, the rework process is performed and then the cycle ends when the stock is exhausted [17]. Chiu (2007) examines the production uptime problem with random machine malfunctions and rework of produced defective items within the scope of abort/ resume policy. Under this policy, once the interruption is cleared and the machine is restored, the interrupted lot production will resume immediately. Also, production inventory cost functions for systems with and without machine malfunctions are shown by Chiu [18]. The studies on machine breakdowns until today can be listed as [19]-[33].

Biswas and Sarker (2008) discussed a production system in which final goods are obtained with randomly appearing defective and scrap items. They entered a production process in which defective products, scrap and final goods were obtained at irregular intervals. Since the system occasionally malfunction, a number of scrap is generated during the production and rework processes [34].

Hejazi et al (2008) divided the products into 4 categories as good, imperfect quality, reworkable and scrap. They had developed an EPQ model that does not allow shortages and imperfect quality items are sold at reduced prices, where reworked items are added to the inventory as good items [35]. In an EPQ model with an imperfect rework process, Chiu et al (2008) develop an effective rule to speed up the process of choosing if an item is scrap or rework and then calculated the expected total cost of production inventories and the optimal size of the lot [36]. Cardanes-Barron (2009) expanded the model developed by Jamal et al. (2004) to a model that allows shortages [37]. Krishnamoorthi and Panayappan (2012) proposed an EPQ model considering defective items which are observed on the production quality control process, cannot be observed during the quality control process and are delivered to the customer [38]. Sivashankari and Panayappan (2014) had developed two EPQ models that allow shortages based on whether defective items are reworked or not [39]. Sarkar et al (2014) extended an inventory model to allow random defective rates and had developed EPQ models for cases where the defective rate which is a random variable that fits three different distributions as uniform, triangular and beta [40]. Haidar et al (2016) had developed two EPQ models that do not allow shortages. In model 1, imperfect quality goods are sold at a reduced price at the end of the screening operation, and in model 2, all of the imperfect products are reworked [41].

In their study Nobil et al (2020) classified the products as high-quality, low-quality, reworkable and scrap. The demand rate of high quality goods is fixed, and the demand rate of low quality goods is considered as a decreasing mathematical relation of its selling price. Under their aforementioned assumptions, they developed an EPQ model that allows shortages [42]. In our study, an EPQ model was developed in which the production system produces imperfect goods to a certain extent and all imperfect goods are reworked. Mostly in the literature, the quality control rate is considered higher than the production rate and thus quality control and production activities are carried out simultaneously. However, the screening activity can be difficult and inefficient during the production period as highlighted in the related literature. Therefore, in our developed model, screening activities are carried out during the production period in order to obtain the good items to meet the demand. For the remaining products, screening activity begins at the end of the production period that is a structure similar to the screening activity in Haidar et al (2016) study [41]. Additionally, in our study, shortages are permitted and fully backordered. The outline of the rest of the paper is as follows. In section 2, the assumptions, notation, mathematical formulation are given. Then analysis of the model using expected total cost per unit time and expected total profit function unit time are considered. And also, the feasibility conditions of the solution are given. In section 3, a numerical example is given to show the validity and applicability of the developed model. Also, the sensitivity analysis to examine the effects of changes in model-parameters on the optimal solution and the necessary conditions for the model to produce feasible solutions are considered in Section 3 as a sub-section.

2 Proposed EPQ model

In the proposed EPQ model, the production system produces defective items due to human errors, equipment and machine failures. Since the demand will be met from good items, the products should go through the screening process and defective and good items must be separated from each other. On the other hand, with the assumption that simultaneous production and screening activities may be difficult and inefficient, it is recommended to start the screening process after production. Therefore, it is assumed that the unit screening cost during the production period is greater than that in the screening period. In order to meet the demand from good items during the production period, the good items production rate must be greater than the demand rate. Also symbolic computation software SageMath was used in the analysis of the model and numerical analysis processes. With this point of view, the assumptions of the model have been obtained in the next subsection.

2.1 Notation

The following notation will be used to develop the mathematical model.

α : Production rate per unit time,
α_1 : Rework rate of defective items per unit time,
β : Demand rate of good quality product per unit time,
c_p : Unit production cost,
c_r : Unit rework cost,
d_1 : Screening cost per item during production,
d_2 : Screening cost per item after production stops,
h : Holding cost per unit per unit time,
h_1 : Holding cost of defective items being reworked per unit per unit time,
P : Random proportion of defective items, with probability density function $f(P)$,
s : Unit selling price of good quality items,
K : Fixed production setup cost,
x_1 : Screening rate per unit time during production,
x_2 : Screening rate per unit time after production stops,
y : Total number of items produced during a production cycle,
t_1 : Backordered build up time length,
t_2+t_3 : Production time length,
t_4 : Screening time length,
t_5 : Rework time length,
t_6 : Inventory consumption time length,
T : Production cycle length, $T=\sum_{i=1}^{6} t_i$.

2.2 The assumption of the model

The assumptions of the model are given as follows:

Shortages are allowed and fully backordered

The demand during the production is met from good items only.

The production rate of good items is greater than the demand rate, $(1-P)\alpha>\beta$

The unit screening cost during the production period is greater than that in the screening period, $d_1>d_2$.

The screening rate during the production is $x_1=\frac{\beta}{1-P}$

The holding cost of defective items being reworked is greater than that of the good items, $h_1>h$

2.3 Mathematical formulation and analysis of the model

Let's consider a production system where a single item of product is produced. The system produces at α rate and its P ratio is assumed to be defective production. The defective product ratio, P, is considered to be a random variable that fits a certain probability distribution. In order to meet the demand from good items during the production period, it is assumed that screening is performed at $x_1=\frac{\beta}{1-P}$ rate (instead of full screening). This assumption was made by Haidar et al [41], stating the difficulty of performing full screening during production. After the production is finished, the screening process of the remaining products begins. Since the $\frac{\beta}{1-P}(t_2+t_3)$ amount of the produced y products are screened in the production period, the remaining $y-\frac{\beta}{1-P}(t_2+t_3)$ amount will be screened in the t_4 period with x_2 rate.

After the screening process is over, the rework process begins, in which defective items are turned into good items. The graphs of the inventories of good and defective items are given in Figure 1 and Figure 2, respectively.

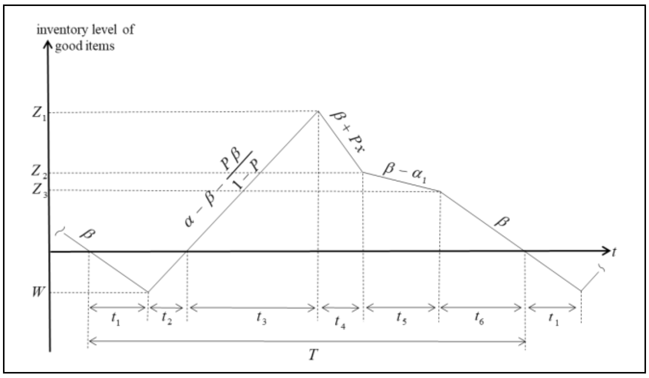

Figure 1. Inventory level of good items.

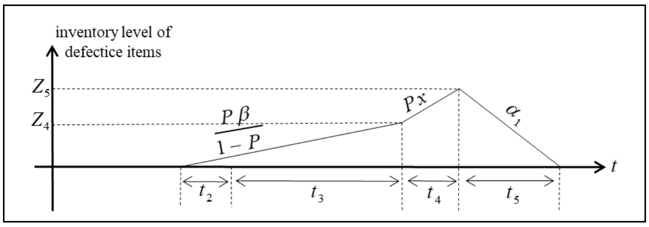

Figure 2. Inventory level of defective items.

During (t_2+t_3) period, at least β products should be examined in order to meet the demand from good items. Therefore, during, $(t_2 + t_3)$ period, screening rate will be $x_1 = \frac{\beta}{1-p}$ and defective rate will be $\frac{\beta p}{1-p}$

In the t_4 period, on the other hand amount of products not inspected in the $(t_2 + t_3)$ period will be examined with the screening rate of x_2.

After the screening period, that is, during t_5, defective items are reworked with α_1 rate and become good items. Since the produced y products are totally good items after reworking the defects, the cycle length T and rework time t_5 can be expressed as follows:

$$T = y/\beta \tag{1}$$

$$t_5 = \frac{py}{\alpha_1} \tag{2}$$

As y products are produced with α rate at $(t_2 + t_3)$ period, one can write:

$$t_2 + t_3 = y/\alpha \tag{3}$$

And as $y - \frac{\beta(t_2+t_3)}{1-P}$ products are screened with x_2 rate, the screening time length t_4 can be written as:

$$t_4 = \left(\frac{1}{x_2}\right)\left(1 - \frac{\beta}{\alpha(1-p)}\right)y \tag{4}$$

Considering Figure 1 and Figure 2, the inventory level at the end of the production period (z_1) is as follows:

$$z_1 = \left(1 - \frac{\beta}{\alpha(1-p)}\right)y - w \tag{5}$$

The inventory level at the end of the screening period (z_2) is:

$$z_2 = (1 - p - \beta/x_2)\left(1 - \frac{\beta}{\alpha(1-p)}\right)y - w \tag{6}$$

The inventory level at the end of the rework period (z_3) is:

$$z_3 = \left[(1 - \beta/\alpha_1)p + (1 - p - \beta/x_2)\left(1 - \frac{\beta}{\alpha(1-p)}\right)\right]y - w \tag{7}$$

The inventory consumption time length (t_6) is:

$$t_6 = \left[(1 - \beta/\alpha_1)p + (1 - p - \beta/x_2)\left(1 - \frac{\beta}{\alpha(1-p)}\right)\right]y/\beta - w/\beta \tag{8}$$

Defective product quantities; at the end of the production period (z_4) and at the end of the screening period (z_5) are:

$$z_4 = \left(\frac{p\beta}{\alpha(1-p)}\right)y \tag{9}$$

$$z_5 = py \tag{10}$$

backorder build uptime (t_1) is

$$t_1 = w/\beta \tag{11}$$

The t_2 period in order to eliminate the backorder is:

$$t_2 = \frac{w}{\alpha - \frac{\beta}{1-p}} \tag{12}$$

And the production period t_3 in which the inventory is positive:

$$t_3 = \frac{y}{\alpha} - \frac{w}{\alpha - \frac{\beta}{1-p}} \tag{13}$$

Total cost per cycle , $TC(y, w)$, can be given as

$$TC(y,w) = c_p y + c_r P y + d_1\left[\frac{\beta(t_2+t_3)}{1-P}\right] + d_2\left[y - \frac{\beta(t_2+t_3)}{1-P}\right] + K + h\left[\frac{z_1 t_3}{2} + \frac{(z_1+z_2)t_4}{2} + \frac{(z_2+z_3)t_5}{2} + \frac{z_3 t_6}{2} + \frac{z_4(t_2+t_3)}{2} + \frac{(z_4+z_5)t_4}{2}\right] + h_1\left(\frac{z_5 t_5}{2}\right) + h_b\left(\frac{t_1+t_2}{2}\right)w \tag{14}$$

$$\begin{aligned} TC(y,w) = {} & \left[c_p + c_r P + d_1\left(\frac{\beta}{\alpha(1-P)}\right) + d_2\left(1 - \frac{\beta}{\alpha(1-P)}\right)\right]y + K \\ & + \left\langle \frac{h}{2}\left\{\frac{1}{\beta} + \frac{1-\frac{\beta}{\alpha}}{\alpha} - \frac{2}{\alpha}\left(\frac{1}{1-P}\right) + \frac{1}{\alpha}\left[2\left(\frac{P}{1-P}\right) + \frac{\beta}{\alpha}\left(\frac{1}{(1-P)^2}\right)\right] - 2\left(\frac{\beta}{\alpha^2}\right)\left(\frac{P}{(1-P)^2}\right) - \left(\frac{1}{\alpha_1}\right)P^2 + \left(\frac{\beta}{\alpha^2}\right)\left(\frac{P}{1-P}\right)^2\right\} + \frac{h_1}{2}\left(\frac{P^2}{\alpha_1}\right)\right\rangle y^2 \\ & - \left(\frac{h}{\beta}\right)wy \\ & + \left(\frac{h+h_b}{2}\right)\left\{\frac{1}{\beta} + \frac{1}{\alpha - \frac{\beta}{(1-P)}}\right\}w^2 \end{aligned} \tag{15}$$

expected total cost per cycle, $ETC(y, w)$:

$$\begin{aligned} ETC(y,w) = {} & \left[c_p + c_r E[P] + d_1\left(\frac{\beta}{\alpha}E\left[\frac{1}{1-P}\right]\right) + d_2\left(1 - \frac{\beta}{\alpha}E\left[\frac{1}{1-P}\right]\right)\right]y + K \\ & + \left\langle \frac{h}{2}\left\{\frac{1}{\beta} + \frac{1-\frac{\beta}{\alpha}}{\alpha} - \frac{2}{\alpha}E\left[\frac{1}{1-P}\right] + \frac{1}{\alpha}\left[2E\left[\frac{P}{1-P}\right] + \frac{\beta}{\alpha}E\left[\frac{1}{(1-P)^2}\right]\right] - 2\left(\frac{\beta}{\alpha^2}\right)E\left[\frac{P}{(1-P)^2}\right] - \left(\frac{1}{\alpha_1}\right)E[P^2] + \left(\frac{\beta}{\alpha^2}\right)E\left[\left(\frac{P}{1-P}\right)^2\right]\right\} + \frac{h_1}{2\alpha_1}E[P^2]\right\rangle y^2 \\ & - \left(\frac{h}{\beta}\right)wy \\ & + \left(\frac{h+h_b}{2}\right)\left\{\frac{1}{\beta} + E\left[\frac{1}{\alpha - \frac{\beta}{(1-P)}}\right]\right\}w^2 \end{aligned} \tag{16}$$

Expected cycle length, $E[T]$ is

$$E[T] = y/\beta \tag{17}$$

And expected total cost per unit time, $ETCU(y,w)$, can be written as

$$ETCU(y,w) = \left[c_p + c_r E[P] + d_1\left(\frac{\beta}{\alpha}E\left[\frac{1}{1-P}\right]\right) + d_2\left(1 - \frac{\beta}{\alpha}E\left[\frac{1}{1-P}\right]\right)\right]\beta + \frac{K\beta}{y} + \left\langle \frac{h}{2}\left\{\frac{1}{\beta} + \frac{1-\frac{\beta}{\alpha}}{\alpha} - \frac{2}{\alpha}E\left[\frac{1}{1-P}\right] + \frac{1}{\alpha}\left[2E\left[\frac{P}{1-P}\right] + \frac{\beta}{\alpha}E\left[\frac{1}{(1-P)^2}\right]\right] - 2\left(\frac{\beta}{\alpha^2}\right)E\left[\frac{P}{(1-P)^2}\right] - \left(\frac{1}{\alpha_1}\right)E[P^2] + \left(\frac{\beta}{\alpha^2}\right)E\left[\left(\frac{P}{1-P}\right)^2\right]\right\} + \frac{h_1}{2\alpha_1}E[P^2]\right\rangle \beta y - hw + \left(\frac{h+h_b}{2}\right)\left\{\frac{1}{\beta} + E\left[\frac{1}{\alpha - \frac{\beta}{(1-P)}}\right]\right\}\frac{\beta w^2}{y} \tag{18}$$

Similarly, by defining

$$A_1 = \left[c_p + c_r E[P] + d_2 + (d_1 - d_2)\left(\frac{\beta}{\alpha}\right)E\left[\frac{1}{1-P}\right]\right]\beta \tag{19}$$

$$A_2 = \left\langle \frac{h}{2}\left\{\frac{1}{\beta} + \frac{1-\frac{\beta}{\alpha}}{\alpha} - \frac{2}{\alpha}E\left[\frac{1}{1-P}\right] + \frac{1}{\alpha}\left[2E\left[\frac{P}{1-P}\right] + \frac{\beta}{\alpha}E\left[\frac{1}{(1-P)^2}\right]\right] - 2\left(\frac{\beta}{\alpha^2}\right)E\left[\frac{P}{(1-P)^2}\right] - \left(\frac{1}{\alpha_1}\right)E[P^2] + \left(\frac{\beta}{\alpha^2}\right)E\left[\left(\frac{P}{1-P}\right)^2\right]\right\} + \frac{h_1}{2\alpha_1}E[P^2]\right\rangle \beta \tag{20}$$

$$A_3 = \left(\frac{h+h_b}{2}\right)\left\{\frac{1}{\beta} + E\left[\frac{1}{\alpha - \frac{\beta}{(1-P)}}\right]\right\}\beta \tag{21}$$

$ETCU(y,w)$, can be written more compactly as follows:

$$ETCU(y,w) = A_1 + \frac{K\beta}{y} + A_2 y - hw + A_3\frac{w^2}{y} \tag{22}$$

The expected total profit function unit time $ETPU(y,w)$, using A_1, A_2, A_3 can be written as follows:

$$ETPU(y,w) = s\beta - ETCU(y,w) \tag{23}$$

$$ETPU(y,w) = s\beta - A_1 - \frac{K\beta}{y} - A_2 y + hw - A_3\frac{w^2}{y} \tag{24}$$

Since $ETPU(y,w)$ is concave, the optimum amount of y produced and the optimum amount of deficiency w are obtained by differentiating $ETPU(y,w)$ by y and then by w and setting the partial derivatives to zero. The partial differential equations of $ETPU(y,w)$ are

$$\frac{\partial ETPU(y,w)}{\partial y} = \frac{K\beta}{y^2} - A_2 + A_3\frac{w^2}{y^2} = 0 \tag{25}$$

$$\frac{\partial ETPU(y,w)}{\partial w} = h - 2A_3\frac{w}{y} = 0 \tag{26}$$

Where

$$y = \sqrt{\frac{4A_3 K\beta}{4A_2A_3 - h^2}} \tag{27}$$

$$w = \sqrt{\frac{h^2 K\beta}{A_3(4A_2A_3 - h^2)}} \tag{28}$$

The Hessian matrix, H, can be written as:

$$H = \begin{bmatrix} \frac{\partial^2 ETPU(y,w)}{\partial y^2} & \frac{\partial^2 ETPU(y,w)}{\partial y \partial w} \\ \frac{\partial^2 ETPU(y,w)}{\partial w \partial y} & \frac{\partial^2 ETPU(y,w)}{\partial w^2} \end{bmatrix} \tag{29}$$

If $[y \quad w]H[y \quad w]^T < 0$, $y, w \neq 0$ then $ETPU(y,w)$ function is strictly concave.

From (29), as $[y \quad w]H[y \quad w]^T = \frac{-2K\beta}{y} < 0$, $ETPU(y,w)$ function is therefore strictly concave.

1.1 Feasibility conditions of the solution

In order to have a feasible solution, the conditions $y, w, z_1, z_2, z_3 > 0$ must be satisfied. Therefore, for $y, w > 0$, since $A_3 > 0$, it must be $4A_2A_3 - h^2 > 0$. Thus, condition-1 is obtained as follows:

Condition-1:

$$4A_2A_3 > h^2 \tag{30}$$

On the other hand, for $z_1, z_2, z_3 > 0$, the following conditions must be satisfied:

Condition-2:

$$\frac{h}{2A_3} + \frac{\beta}{\alpha}E\left[\frac{1}{1-P}\right] < 1 \tag{31}$$

Condition-3:

$$E[P] + \frac{\beta}{x} < 1 \tag{32}$$

Condition-4:

$$\frac{\beta}{\alpha_1} + \frac{h}{2A_3} - (1 - \beta/x)\left(1 - \frac{\beta}{\alpha}E\left[\frac{1}{1-P}\right]\right) + E[P] - \frac{\beta}{\alpha}E\left[\frac{P}{1-P}\right] < 1 \tag{33}$$

A feasible solution is obtained in cases where these 4 conditions are simultaneously satisfied.

2 Numerical example

2.1 Parameters of the numerical example

In this section, a numerical example will be given to show the validity and applicability of the developed model. Let's consider that a firm produces a single item of goods. Let's also assume that the demand for the goods is 1200 pieces/day, the production capacity is 2000 pieces/day, the rework capacity is 1600 pieces/day and the screening capacity is 5000 pieces/day. On the other hand, let's consider the setup cost as 1500 $, production cost as 104 $/unit, rework cost as 8 $/unit, inventory holding cost as 20 $/unit/unit time, shortages cost as

24 \$/unit/unit time, the inventory holding cost of defective items during reworking as 22 \$/unit/unit time, screening costs during production and screening as 0.6 \$/unit and 0.5 \$/unit, respectively and selling price of good items as 200 \$/unit. Assuming that the defect rate is a uniformly distributed random variable, its probability density function is taken as follows:

$$f(p) = \begin{cases} \frac{1}{g-a}, a \leq p \leq g \\ 0, otherwise \end{cases} \quad (34)$$

where $a = 0$ and $g = 0.1$. For the numerical example, the parameters of the model are taken as:

$$\alpha = 2000, \beta = 1200, \alpha_1 = 1600, x_2 = 5000, K = 1500, s = 200,$$

$$c_P = 104, c_r = 8, h = 20, h_1 = 22, h_b = 24, d_1 = 0.6, d_2 = 0.5$$

2.4 Feasibility of the numerical example

From the formula $E[g(p)] = \int g(p)f(p)dp$, the following expected values are obtained:

$$E\left[\frac{1}{1-p}\right] = 1.053605$$

$$E\left[\frac{p}{1-p}\right] = 0.053605$$

$$E\left[\frac{p}{(1-p)^2}\right] = 0.057506$$

$$E\left[\left(\frac{p}{1-p}\right)^2\right] = 0.003901$$

$$E\left[\frac{1}{(1-p)^2}\right] = 1.111111$$

$$E[p] = 0.05$$

$$E[p^2] = 0.003333$$

$$E\left[\frac{1}{\alpha - \frac{\beta}{1-p}}\right] = 0.001363$$

As a result of the solution of the model, it is seen from equation (50) that the production amount is $y = 888.94$ units, from equations (45) and (44) the monthly average profit and cost are $ETPU(y,w) = 109{,}994.37\$$ and $ETCU(y,w) = 130{,}005.63\$$, respectively. Other values obtained from the solution are as follows.

$$w = 153.3, z_1 = 173.7, z_2 = 78.9$$

$$z_3 = 90.5, z_4 = 28.1, z_5 = 44.4$$

$$t_1 = 0.1278, t_2 = 0.2090, t_3 = 0.2355$$

$$t_4 = 0.0654, t_5 = 0.0278, t_6 = 0.0754$$

From the conditions in (30)-(33):

condition-1 : $400 < 928.31$,

condition-2: $0.8046 < 1$,

condition-3: $0.29 < 1$,

condition-4: $0.6607 < 1$,

are obtained. Since all the conditions are satisfied, the optimum solution obtained is feasible. The graphical representation of the numerical solution is given in Figure 3.

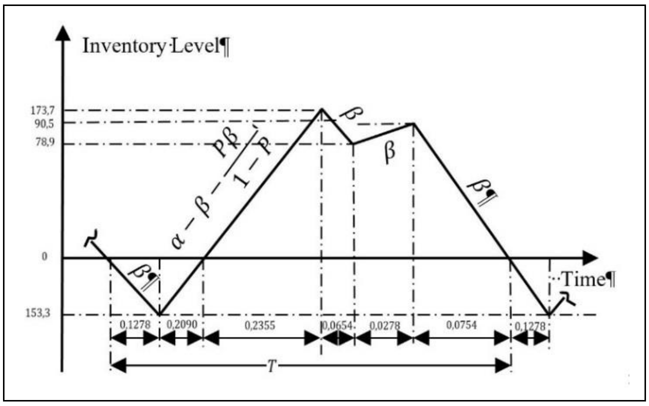

Figure 3. Graphical representation of the numerical solution.

2.5 Sensitivity analysis of the numerical example

On the other hand, when the sensitivity of the model according to the defect rate is examined, it is observed that the expected monthly profit decreases when the defective product rate increases, and this is an expected result. At the same time, as the defect rate increases, both the optimum production quantities (y) and the maximum shortages quantities (w) decrease. Solution values for this analysis are given in Table 1.

In Table 2, it is observed that the average defective rate ($E[p]$) did not change, but the expected monthly profit decreased and the expected cost increased as a result of the increase in the range of the defect rate ($g - a$). Thus, it can be concluded that in addition to the efforts of companies to reduce the defect rate, it is also important for their profitability to reduce the variability of the defect rate. The changes of increasing the production capacity (α) on the optimum solution are given in Table 3. Unless demand changes, increasing capacity does not have a significant effect on profitability. Even if the effect is small, the profit decreases while the capacity increases. To examine the effects of demand (β) on the optimum solution, it is necessary to take a look at the solution results in Table 4.

Let $\alpha = 12{,}000$, $\alpha_1 = 9{,}600$, $x = 15{,}000$ in order not to encounter the infeasible solution as much as possible. Naturally, with the increase in the amount of demand, the production amount (y), the monthly expected total profit ($ETPU(y,w)$) and the cost ($ETCU(y,w)$) are also expected to increase. The analyses given in Table 4 confirm with the aforementioned explanations. The effects of defective rate, production capacity and demand on the optimum solution are given in Figure 4, Figure 5 and Figure 6.

Table 1. The effects of expected defective rate on optimum solution.

$E[p]$	g	a	$ETPU(y,w)$	y	w	$ETCU(y,w)$
0.0500	0.1000	0	109,994.37	888.94	153.31	130,005.63
0.0700	0.1400	0	109,757.99	879.67	147.68	130,242.01
0.1000	0.2000	0	109,390.65	863.51	137.82	130,609.35
0.1200	0.2400	0	109,134.92	850.91	130.05	130,865.08
0.1500	0.3000	0	108,723.22	827.36	115.25	131,276.78
0.1999	0.3998	0	107,567.84	716.96	38.59	132,432.16
0.2000	0.4000	0	*infeasible*	*infeasible*	*infeasible*	*infeasible*

Table 2. Effects of defective rate variation interval on optimum solution.

$E[p]$	g	a	$ETPU(y,w)$	y	w	$ETCU(y,w)$
0.15	0.30	0.00	108,723.22	827.36	115.25	131,276.78
0.15	0.25	0.05	108,777.52	837.72	121.84	131,222.48
0.15	0.20	0.10	108,805.77	843.20	125.29	131,194.23
0.15	0.17	0.13	108,812.72	844.56	126.13	131,187.28

Table 3. The effects of production capacity on optimum solution.

α	$ETPU(y,w)$	y	w	$ETCU(y,w)$
2000	109,994	888.9	153.3	130,006
2200	109,759	838.8	166.9	130,241
2400	109,572	802.7	177.3	130,428
3200	109,091	722.6	202.7	130,909
9000	108,265	616.6	242.6	131,735
11000	108,188	608.3	246.1	131,812
16500	108,074	596.3	251.3	131,926
17000	108,067	595.7	251.6	131,933
24000	108,004	589.2	254.4	131,996
25000	107,998	588.6	254.7	132,002

Table 4. Effects of demand on optimum solution.

β	$ETPU(y,w)$	y	w	$ETCU(y,w)$
1200	108160	605.4	247.5	131840
1500	136073	686.3	272.7	163927
1800	164071	762.6	294.2	195929
2100	192134	835.9	312.8	227866
2500	229636	930.6	333.9	270364
3000	276622	1040.7	355.2	323378
3500	323709	1162.3	371.8	376291
4500	418126	1399.6	392.6	481874
6500	607753	1944.8	390.5	692247
7000	655307	2017.8	380.6	744693
7500	infeasible $z_2 < 0$	infeasible $z_2 < 0$	infeasible $z_2 < 0$	infeasible $z_2 < 0$

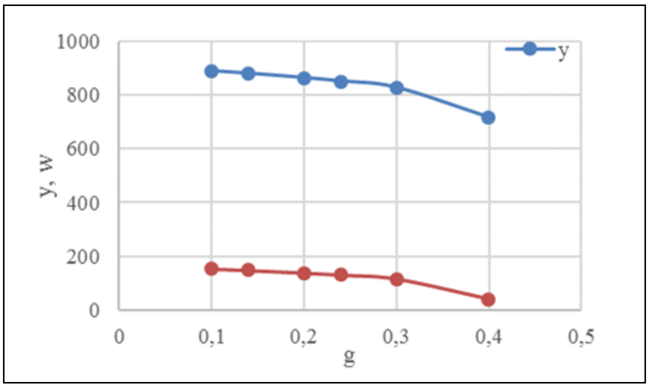

Figure 4. The effects of defective rate on optimum solution.

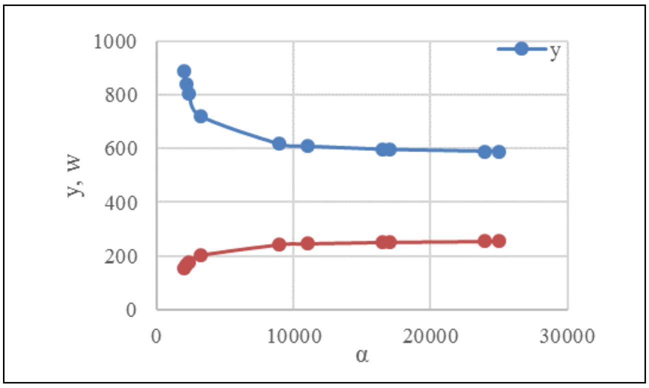

Figure 5. The effects of production capacity on optimum solution.

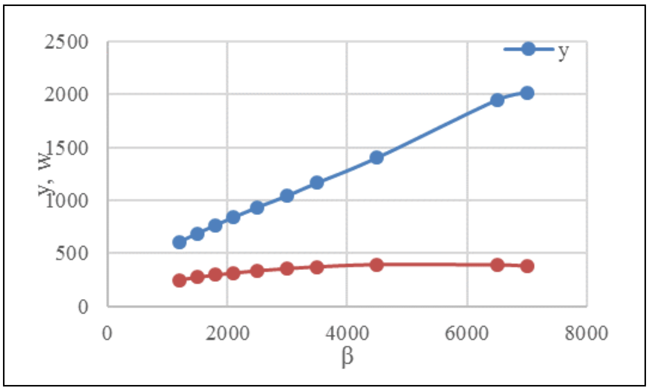

Figure 6. The effects of demand on optimum solution.

3 Conclusions

In this study, a production inventory system that produces defective products to a certain extent and in which all defective products become good items by rework is taken into consideration. Thus, an EPQ model has been developed for a process in which shortages is allowed and the production, quality control and rework periods follow each other.

The operation of the model and its feasibility conditions have been mathematically proven. At the same time, the operation and feasibility conditions of the model were calculated on the numerical example. Also, using sensitivity analysis, the transition points of the model from solution to non-solution were shown under the given parameters.

As a further study, the model can be extended with parameters as carbon emission, taxes, etc. that take into account the environmental and social responsibility, which has been focused on recently. Moreover, the model can be extended by taking into account the partial backorder, machine breakdowns, production and rework setup times.

4 Author contribution statements

Both Nigar KARAGÜL and Abdullah EROĞLU developed the theory, contributed to the design and implementation of the research, to the analysis of the results and to the final version of the manuscript. Nigar Karagül performed the computations. Abdullah Eroğlu supervised the project.

5 Ethics committee approval and conflict of interest statement

There is no conflict of interest with any person/institution in the article prepared.

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