International Journal of Thermodynamics (IJoT)
ISSN 1301-9724 / e-ISSN 2146-1511
https://dergipark.org.tr/en/pub/ijot

Vol. 26 (No. 1), pp. 026-035, 2023
doi: 10.5541/ijot.1169691
Published online: March 1, 2023

Research Article

The Bosons of the Conventional Superconductors

*U. Köbler

Research Center Jülich, Institute PGI, D-52425, Jülich, Germany
E-mail: u.koebler@fz-juelich.de

Received 1 September 2022, Revised 15 December 2022, Accepted 15 December 2022

Abstract

For the conventional superconductors it will be shown that not only the superconducting energy gap, $E_{gap}(T=0)$, and the critical field, $B_c(T=0)$, but also the London penetration depth, $\lambda_L(T=0)$, scale in a reasonable approximation with the superconducting transition temperature, T_{SC}, as $\sim T_{SC}$, $\sim T_{SC}^2$ and $\sim T^{-1/2}$, respectively. From these scaling relations the conclusion obtained earlier, using a completely different method, is confirmed that the London penetration depth corresponds to the diameter of the Cooper-pairs. As a consequence, only one layer of Cooper pairs is sufficient to shield an external magnetic field completely. The large diamagnetism of the superconductors is caused by the large orbital area of the Cooper-pairs. From the fact that, in the zero-field ground state, the temperature dependence of the superconducting heat capacity is given above and below T_{SC} by power functions of absolute temperature it follows that the only critical point is T=0. The superconducting transitions of the element superconductors, therefore, are all within the critical range at T=0. As a consequence, above and below T_{SC} there is short-range order only. As we know from Renormalization Group (RG) theory, in the critical range the dynamics is the dynamics of a boson field, exclusively. Evidently, the Cooper-pairs have to be considered as the short-range ordered units created by this boson field. It is reasonable to assume that the relevant bosons in the superconducting state are identical with the bosons giving rise to the universal linear-in-T electronic heat capacity above T_{SC}. Plausibility arguments will be given that these bosons must be electric quadrupole radiation generated by the non-spherical charge distributions in the soft zones between the metal atoms. The radiation field emitted by an electric quadrupole can be assumed to be essentially curled or circular. In the ordered state below T_{SC}, the bosons are condensed in resonating spherical modes which encapsulate the two Cooper-pair electrons and shield their charge perfectly.

Keywords: *Cooper-pairs; ordered boson fields; stimulated emission.*

1. Introduction

The postulation of Cooper pairs [1] marks a historical breakthrough in our understanding of the phenomenon of superconductivity. However, the detailed nature of the coupling mechanism between the two Cooper-pair electrons is still unclear [2-8]. Further experimental information on the properties of the Cooper-pairs, such as the temperature dependence of their size and their density, therefore, is of vital importance. The present study aims to contribute in a rather phenomenological way to the solution of these problems. A final confirmation of the here advanced ideas must come from field-theoretical studies.

It is evident that at the superconducting transition temperature, T_{SC}, Cooper-pairs get formed and are responsible for the two prominent superconducting properties: a vanishing electrical resistivity and the emergence of a huge diamagnetism (Meissner-effect) [1]. Formation of Cooper-pairs at T_{SC} reminds on the formation of domains at the magnetic ordering temperature. In fact, Cooper-pairs and domains have to be considered as the characteristic ordered units with linear dimensions of much larger than the inter-atomic distance. As a consequence, domains and Cooper-pairs cannot result from atomistic short-range interactions. In order to illustrate the similarities and differences between the magnetic domains and the Cooper-pairs, it is useful to discuss first the better understood magnetic domains.

Quite generally, at virtually all order-disorder phase transitions a boson field orders [9,10]. The visible atomic or magnetic order results from a coupling of the atoms or spins to the ordered boson field. Essential for a long-range and coherent order of a boson field is that the emission characteristics of the individual boson source is axial and that the bosons get generated by stimulated emission. These conditions hold for the bosons that order at the magnetic ordering transition [9]. It could be shown that these bosons are magnetic dipole radiation generated by the precessing spins [11]. We have called these bosons, in honor of J. Goldstone, Goldstone-bosons [12,13]. Due to stimulated emission the bosons get collimated along those crystallographic directions with a high density of the boson sources and, eventually, condense in a single quantum state. In the ordered state of the Goldstone-boson field, all bosons are in a one-dimensional, perfectly coherent and long-range ordered state. This is realized in each magnetic domain. The reduction from a spatially isotropic propagation of the bosons to a few propagation directions is an example of broken symmetry. In fact, stimulated emission seems to be an important origin of the phenomenon of broken symmetry [12].

*Corresponding Author

The ordered boson field has a well-defined, self-organized, spatially limited contour. In the magnetic case this are the well-known domains. The domains result from a self-constriction mechanism of the ordered boson field and define the limits of a region with homogenously ordered spins. They are self-contained units. The ordered spins are, so to say, enclosed in the domain. Moreover, the domains have the functionality of a resonator. Within each domain, bosons and magnons are standing plane waves. The dynamics in the individual domain, therefore, is perfectly one-dimensional. The one-dimensionally ordered structure is a consequence of the axial radiation characteristics of the individual boson source, i.e., the precessing spin. As a result, for many magnets, the observed magnon dispersions are as for the linear spin chain, independent of the assumed locally anisotropic exchange interactions that are not relevant in the critical range above the ordering temperature and for all lower temperatures as well [14,15]. A three-dimensional dynamic symmetry results by a coupling of the one-dimensional boson fields of the domains along x-, y-, and z-axis. The observed critical exponents are defined by the dimensionality of the global boson field [15].

The superconducting transition is particular in that it is a transition into a state with a short range ordered boson field. The ordered units generated by these bosons are the Cooper-pairs. In contrast to the magnetic domains that are weakly fixed to the crystal lattice, the Cooper-pairs can move without any resistance across the metal [16]. The fact that there is no correlation between the positions of the Cooper-pairs conforms to the short-range order in the superconducting state. While the surface of the magnetic domains consists of planes, the shape of the Cooper-pairs must be rather spherical. A resistivity-free propagation mode of the Cooper-pairs is surprising in view of the two-fold electric charge of the Cooper-pair. Evidently, the boson resonator surrounding the two Cooper-pair electrons shields the charge of the two electrons perfectly against all other charges of the metal. There are good reasons to assume that the condensed bosons in the superconducting state are identical with the bosons giving rise to the universal linear-in-T electronic heat capacity above T_{SC}. We will call the bosons of the <u>c</u>ontinuous <u>m</u>etallic <u>s</u>olid CMS-bosons [17]. At T_{SC}, these bosons change the type of short-range order and assume a definite shape below T_{SC}. It is suggestive to identify the CMS-bosons with electric quadrupole radiation generated by the anisotropic charge distributions in the rather soft zones between the metal atoms. One experimental observation supporting this idea is the strong dependence of T_{SC} upon application of an external pressure [18]. It is evident that application of pressure leads to deformations of the mechanically soft zones between the metal atoms and therefore to changes of the quadrupole moments. This has a direct effect on the generation process of the CMS-bosons, and, as a consequence, on T_{SC}. Note that the ordering transition of a boson field occurs for a sufficiently high density of identical bosons [9].

The Cooper-pairs behave as a dense gas of neutral particles [16]. Possibly, the mobility of the Cooper-pairs is by tunnel effect. In recently performed new analyses of published superconducting heat capacity data of the conventional superconductors, low-temperature crossover events were identified that could be interpreted as Bose-Einstein (B-E) condensation of the Cooper-pairs [16]. From the observed B-E condensation temperature, T_{BE}, it is possible to obtain the density, n, of the Cooper-pairs at T_{BE} according to T_{BE}~$n^{2/3}$ [19]. The observed T_{BE} temperatures turned out to scale, to a good approximation, with the superconducting transition temperature, T_{SC}, as $T_{BE}=0.135\cdot T_{SC}$ [16]. As a consequence, the density of the Cooper-pairs at T_{BE} scales with the superconducting transition temperature, T_{SC}, as $n(T\sim0)\sim T_{SC}^{3/2}$ [16]. Since T_{BE} is much lower than T_{SC}, the density of the Cooper-pairs at T_{BE} can be taken as representative of the density at T=0. Assuming that at all temperatures the Cooper-pairs form a dense-packed gas of bosons with a spin of S=0, their diameter at T~0, $\lambda(T\sim0)$ is given by $\lambda(T\sim0)\sim n^{-1/3}\sim T_{SC}^{-1/2}$.

Interestingly, the same scaling relation as for the diameter of the individual Cooper-pair, $\lambda(T\sim0)\sim T_{SC}^{-1/2}$, holds for the London penetration depth, $\lambda_L(T\sim0)\sim T_{SC}^{-1/2}$ (see Figure 2 below). The diameter of the Cooper-pair, therefore, corresponds to the London penetration depth. In other words, only one layer of Cooper-pairs, next to the inner surface of the superconductor, is sufficient to shield an applied magnetic field completely. As a consequence, the diamagnetic moment of the individual Cooper-pair, i.e., its cross-section area, is given by the square of the London penetration depth. As the experimental data show, the London penetration depth, λ_L, is divergent at T_{SC} and decreases strongly with decreasing temperature towards a finite value for $T\rightarrow0$ (see Figures 5-7 below) [20]. We can assume that the proportionality $\lambda_L\sim\lambda$ holds for all temperatures. As a consequence, the diameter of the individual Cooper-pair decreases with decreasing temperature, in proportionality to the London penetration depth. This allows one to obtain the temperature dependence of the Cooper-pair diameter from measurements of the temperature dependence of the London penetration depth. The decreasing diameter of the Cooper-pairs is indicative of an increasing binding energy between the two Cooper-pair electrons. This is certainly a dynamic, i.e., temperature-dependent effect and has to be ascribed to a constricting force, inherent to the condensed boson shield that surrounds the two Cooper-pair electrons. In fact, there is a reasonable proportionality between the Cooper pair coupling energy, given by the gap energy $E_{gap}(T)$, and the reciprocal London penetration depth $\lambda_L^{-1}(T)$ (compare Figure 4 and Figure 5 below). In other words, the larger the coupling energy, $E_{gap}(T)$, is, the lower is the diameter of the Cooper-pair. The condensed boson shell surrounding the two Cooper-pair electrons has the functionality of a cage that exerts the necessary force, needed to counteract the electrostatic repulsion between the two electronic charges. Due to an increasing constricting force of the boson cage with decreasing temperature, the size of the Cooper pairs decreases with decreasing temperature, in parallel to the London penetration depth [16]. This "electrostriction" has some similarity with the spontaneous magnetostriction in the ordered magnets [21]. In both cases the constricting forces are a dynamic property of the ordered boson field. However, magnetostriction acts on the collective of a nearly constant configuration of dense packed domains and lets the lattice parameter decrease with decreasing temperature [21]. The corresponding electrostriction acts on each of the increasing number of Cooper-pairs and has little effect on the temperature dependence of the lattice parameter [22].

The decreasing size of the Cooper-pairs with decreasing temperature gets compensated by a corresponding increase of their density such that the volume of the superconductor is always completely filled with Cooper-pairs, similar to the volume of the ordered magnets that is completely filled with domains. A complete filling seems to be specific to a homogeneous phase. The increasing number of electrons, needed for the increasing number of Cooper-pairs, can be assumed to be delivered by the conduction band. The superconducting system thereby gains an increasing mass which is certainly of importance on the superconducting dynamics, i.e., on the temperature dependence of the superconducting heat capacity [10,16]. At the same time, the conduction band gets depleted and the lattice parameter increases [22]. As a consequence, the superconductors resemble a two-phase system with the Cooper-pairs as the condensed phase and the conduction electrons as the vapor phase. The Cooper-pairs correspond, so to say, to the droplets in a vapor-liquid mixture.

Even in the limit $T \rightarrow 0$ where the London penetration depth and the diameter of the Cooper-pairs have a minimum, the orbital diamagnetism of the Cooper-pairs is sufficiently large such that, in the superconductors of the first kind, only one layer of Cooper-pairs at the inner surface of the superconductor shields a magnetic field completely. For the superconductors of the second kind, the low-temperature diameter of the Cooper-pairs, and therefore the orbital diamagnetism seems not to be sufficient to shield an applied magnetic field completely. The applied magnetic field then penetrates the superconductor as vortices.

Very peculiar is that the zero-field heat capacities of the conventional superconductors exhibit no critical behavior at T_{SC} but the Cooper-pair binding energy, $E_{gap}(T)$, and the London penetration depth, $\lambda_L(T)$, show critical behavior at $T=T_{SC}$ (see Figures 4-7 below), as it is familiar for the spontaneous magnetization [21,23] and for the magnon gap in the ordered magnets [21,24]. The typical critical power functions of the argument $|T_{SC}-T|$ are absent in the zero-field heat capacity of the conventional superconductors. Note that $E_{gap}(T)$ and $\lambda_L(T)$ are quantities that are not specific to the dynamics of the ground state of the unperturbed superconductor. Observation of these quantities requires special excitation conditions. As the finite critical range at T_{SC} and at T=0 shows, the temperature dependence of $E_{gap}(T)$ and $\lambda_L(T)$ is controlled by a long-range ordered boson-field that, apparently, has a higher dispersion energy than the bosons that are responsible for the dynamics of the superconducting ground state. As a consequence, there seem to exist two boson types in the superconductor. Only one of them can be relevant. The excited state bosons are, evidently, not relevant for the dynamics of the superconducting ground state and appear to be completely absent in zero-field measurements. For the ground state bosons T=0 is the only critical point. These bosons are in a short-range ordered state. The complicated temperature dependence of the superconducting zero-field heat capacity is another indication of a complicated excitation spectrum of the superconducting elements [10,16]. Note that there can be an interaction between thermally not occupied excited states and the thermally occupied ground state. This interaction can modify the dynamics of the ground state.

A possible explanation of the existence of two boson types could be that the radiation field emitted by an electric quadrupole is rather complicated. It is possible that this radiation field includes a linear and a curled component. In the superconducting state, the linear component is long-range ordered but not relevant for the dynamics of the ground-state. The heat capacity of this boson field is responsible for the temperature dependence of $E_{gap}(T)$ and $\lambda_L(T)$, in particular for the critical behavior of the two quantities at T_{SC} in addition to T=0. In the ordered state, the curled component gives the Cooper-pairs their spherical shape. Due to the exclusion principle of relevance, the binding mechanism of the two Cooper-pair electrons seems to be decoupled from the dynamics of the Cooper-pairs in the zero-field ground state. The Cooper-pairs are rather stable objects. No critical behavior at T_{SC} is observed not only in the zero-field heat capacity but also for the critical field, B_c [25].

In the first part of this communication, we discuss the relation between the zero-temperature values of the Cooper-pair gap energy, $E_{gap}(T=0)$, of the London penetration depth, $\lambda_L(T=0)$, and of the critical field, $B_c(T=0)$ of the superconducting elements [26]. As is well-known, the Cooper-pair gap energy, $E_{gap}(T=0)$ and the critical field $B_c(T=0)$ scale to a good approximation with the superconducting transition temperature, T_{SC}, as $E_{gap}(T=0) \sim T_{SC}$ [1] and $B_c(T=0) \sim T_{SC}^2$ [25]. For the London penetration-depth the scaling relation will be shown to be $\lambda_L(T=0) \sim T_{SC}^{-1/2}$ [16]. From these scaling relations it follows conclusively that the diamagnetic moment of the Cooper pair, $\mu(T=0)$, is proportional to the square of the London penetration depth, i.e., $\mu(T=0) \sim \lambda_L^2(T=0) \sim T_{SC}^{-1}$. In other words, the Cooper-pairs are closed objects with a diameter that corresponds to the London penetration depth. The strong superconducting diamagnetism results from the large orbital area of the Cooper-pair wave function. The two Cooper-pair electrons are evidently in a spherical symmetric s-state. The antiparallel coupling of the spins of the two Cooper-pair electrons seems to be by the rather weak dipole-dipole interaction. The Cooper-pairs, therefore, receive a net magnetic moment in a rather low applied magnetic field and superconductivity breaks down at the moderate critical field, B_c [25].

In the second part of this work, representative data of the temperature dependence of the Cooper-pair gap energy, $E_{gap}(T)$, and of the reciprocal London penetration depth, $\lambda_L(T)^{-1}$, are analyzed. Typical of the long-range order of the boson field that controls the dynamics of the two quantities is that the complete temperature dependence of $E_{gap}(T)$ and of $\lambda_L(T)^{-1}$ is given by the two critical power functions at T=0 and at $T=T_{SC}$. This is as for the spontaneous magnetization of the ordered magnets [21,23]. As the identical critical exponents of $E_{gap}(T)$ and of $\lambda_L(T)^{-1}$ show, the two quantities are proportional to each other. In conformity with the increasing binding energy between the two Cooper-pair electrons as a function of a decreasing temperature, given by $E_{gap}(T)$, the diameter of the Cooper-pair orbital, i.e., $\lambda_L(T)$ shrinks.

2. Properties at T=0

As done by the BCS-theory [1], we make use of the empirical fact that the superconducting elements have similar electronic properties and differ essentially by their transition temperatures, T_{SC}, only. In this way it could plausibly be proven that the gap energy at T=0, $E_{gap}(T=0)$ is proportional to T_{SC} [1] and that the critical field $B_c(T=0)$ is proportional to T_{SC}^2 [25]. For the London penetration depth, it turns out that $\lambda_L(T=0)$ is proportional to $T_{SC}^{-1/2}$ (Figure 2) [16]. As a consequence, for a high T_{SC}, the Cooper-pairs are strongly bound and, as a consequence, are small objects with a small diameter given by λ_L.

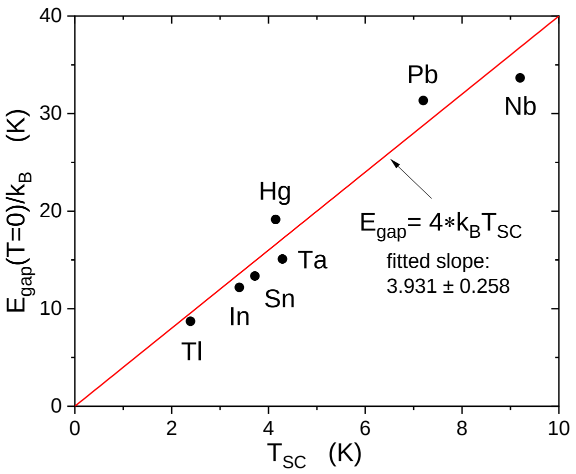

Figure 1. Experimental Cooper-pair gap energies at T~0, $E_{gap}(T=0)$, *converted to temperatures for some superconducting elements as a function of the transition temperature,* T_{SC} [26]. *A linear fit of these data results reasonably into a slope of four, which deviates only slightly from the prediction of the BCS-theory of ~3.528 (see text)* [20].

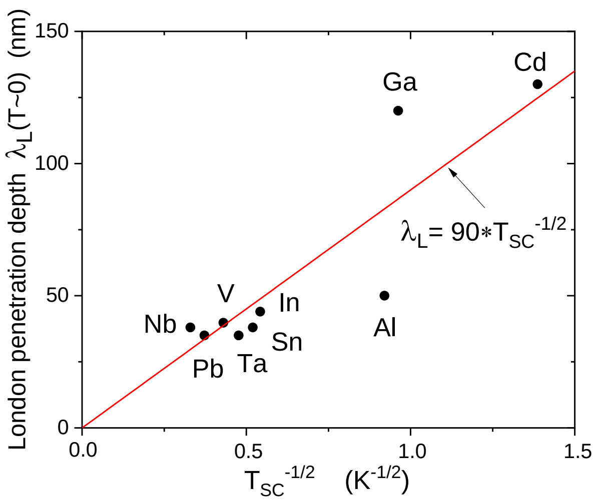

Figure 2. Experimental data of the London penetration depth at T~0, $\lambda_L(T=0)$, *of some superconducting elements versus the reciprocal root of the superconducting transition temperature* [26].

Figure 1 reproduces the famous proportionality between the gap energy at T=0 and T_{SC}. In Figure 1 the gap energies are converted to temperatures [26]. It can be seen, that $E_{gap}(T=0)/k_B$ is to a good approximation proportional to T_{SC}, as predicted by the BCS-theory [1]. A linear fit of the data in Figure 1 results in a proportionality constant of 3.931±0.258 which is somewhat larger than ~3.528 predicted by the BCS-theory [1,20]. It seems justified to consider the fit result for the proportionally constant as consistent with four, although 3.5 cannot be excluded.

Considering the gap energy as the thermodynamic stability limit of the Cooper-pairs and that T_{SC} defines the temperature scale, one can, formally, consider $E_{gap}(T=0)$ as a multiple of $1/2 \cdot k_BT_{SC}$. Note, however, that this proportionality does not mean a thermal equilibrium. In other words, the individual Cooper-pair has between 7 and 8 degrees of freedom. Since there are certainly three translational degrees of freedom, the Cooper-pair has between 4 and 5 additional energy degrees of freedom. This could mean that the Cooper-pairs are not perfectly spherical in shape and/or have a number of internal degrees of freedom such as breathing or pump modes. According to the generally low superconducting transition temperatures it is clear that the two electrons of the Cooper pair are not rigidly bound to each other. In fact, as the temperature dependence of $E_{gap}(T)$ shows, the coupling of the two Cooper-pair electrons is a dynamic process.

Another characteristic quantity of the superconductors is the low-temperature minimum of the London penetration depth, $\lambda_L(T=0)$, that gives the distance from the surface of the sample over which an external magnetic field can penetrate into the superconductor. Note that the penetration depth is divergent at T_{SC} and assumes a finite minimum for $T\rightarrow0$ (see Figures 5-7 below) [20]. For T=0, the volume of the superconductor (of the first kind) is field-free, except for a thin layer at the surface with a thickness of $\lambda_L(T=0)$. It is evident that this phenomenon, known as Meissner-Ochsenfeld effect [27], is a direct consequence of the strong diamagnetism of the Cooper-pairs owing to their large orbital area. The strong diamagnetism is consistent with the view that the two electrons of the Cooper-pair can move on closed loops. They circulate the stronger, the larger the applied magnetic field is. The diamagnetic moment can be expected to be proportional to the applied magnetic field. Note that in the normal-conducting state there are no Cooper-pairs. Most conventional superconductors are paramagnetic in the normal state. Figure 2 shows experimental data of the London penetration depth at T~0 as a function of the square root of the reciprocal transition temperature, $T_{SC}^{-1/2}$ [26]. From this data representation it follows that, the lower the transition temperature is, the larger is the penetration depth for a magnetic field. Eventually, for $T_{SC}\rightarrow0$, the penetration depth, $\lambda_L(T=0)$, diverges and the superconductor does no longer shield the magnetic field. This proves consistently that the strong diamagnetism is restricted to the superconducting state. It is evident that for a low T_{SC}, the two electrons of the Cooper-pair are weakly coupled only and the size of the Cooper-pair orbital is correspondingly large. The London penetration depth then is correspondingly large as well. In other words, the T=0 values of the London penetration depth and of the size of the Cooper pairs increase with decreasing T_{SC}.

Another characteristic quantity of the conventional superconductors is the critical field at T=0, $B_c(T=0)$. As is well-known, $B_c(T=0)$ scales with the square of the transition temperature (Figure 3) [25]. The relation $B_c(T=0)\sim T_{SC}^2$ is satisfactorily confirmed by the data representation of Figure 3 [26].

The three quantities $E_{gap}(T=0)$, $\lambda_L(T=0)$ and $B_c(T=0)$ are certainly not independent of each other. In order to find out a correlation between them we make use of the formal energy equation

$$\mu(T=0)\cdot B_c(T=0)=E_{gap}(T=0) \qquad (1)$$

with $\mu(T=0)$ as diamagnetic moment of the Cooper-pair orbital at T=0 for an applied magnetic field of $B_c(T=0)$. Inserting into equation (1) the two relations:

$$B_c(T=0)\sim T_{SC}^2 \quad \text{(Figure3) and}$$
$$E_{gap}(T=0)\sim T_{SC} \quad \text{(Figure 1)} \qquad (2)$$

it results that $\mu(T=0) \sim T_{SC}^{-1}$. Considering that the London penetration depth is $\lambda_L(T=0) \sim T_{SC}^{-1/2}$ (Figure 2) it follows that

$$\mu(T=0) \sim \lambda_L(T=0)^2 \quad (3)$$

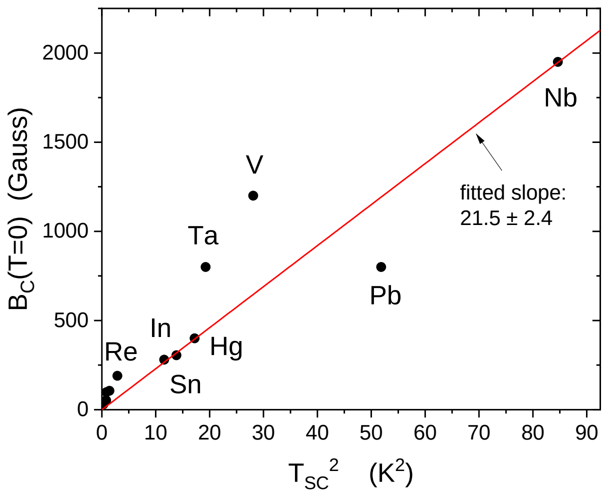

Figure 3. The critical field at T~0, $B_c(T=0)$, *as a function of the superconducting transition temperature squared for a selection of superconducting elements* [25,26].

As a consequence, the diamagnetic moment μ of the Cooper-pair is proportional to the square of the London penetration depth. In other words, λ_L^2 gives the cross-section area of the Cooper-pair, and the diameter of the Cooper-pair is proportional to the London penetration depth, λ_L. This result provides a plausible microscopic explanation of the London penetration depth λ_L and supports the real-space, or particle picture of the Cooper-pair. As a consequence, only one layer of diamagnetic Cooper-pairs next to the inner surface of the superconductor is sufficient to shield the external magnetic field completely. The London penetration depth deceases as a function of a decreasing temperature because the Cooper-pair orbital area decreases as a function of a decreasing temperature. For the type I superconductors the minimum of the orbital area of the Cooper-pairs for $T \rightarrow 0$ and, as a consequence, the associated small diamagnetic moment is still sufficient to shield the applied magnetic field completely. On the other hand, a complete shielding of a magnetic field for all temperatures by only one layer of Cooper-pairs requires that the decreasing size of the Cooper-pairs, with decreasing temperature, gets compensated by a corresponding increase of their density such that the volume of the sample is always nearly completely filled with Cooper-pairs [16]. For the type II superconductors, either the orbital area or the density of the Cooper-pairs seems not to be sufficiently large to shield the magnetic field completely. The magnetic field then penetrates the superconductor as vortices.

The result expressed by the proportionality (3) agrees with a recent experimental study of the Bose-Einstein (B-E) condensation temperatures, T_{BE}, of the Cooper-pairs of the superconducting elements [16]. Note that Cooper-pairs are bosons with an integer spin of S=0 [2]. In spite of their two-fold charge, the Cooper-pairs can move completely freely across the metallic matrix, which is a condition for an electrical resistivity of zero of the superconducting current. This shows that the charges of the two Cooper-pair electrons get completely shielded by the surrounding CMS-boson cage. As an empirical fact, the thermodynamics of the Cooper-pair gas can be described by the same algorithm as it applies to the dilute alkali-metal atom gases [19]. From the observation of the B-E condensation temperature, T_{BE}, it is possible to evaluate the density of the Cooper-pairs, n, at T_{BE}. For the uniform Bose gas, confined to a three-dimensional box, the dependence of T_{BE} on the density of the gas particles, n, is given by

$$k_B \cdot T_{BE} \approx 3.31(\hbar^2 n^{2/3}/m) \quad (4)$$

with $\hbar = h/2\pi$ as Planck constant and $m=2m_e$ as the mass of the Cooper-pair (m_e is the mass of the electron) [19]. Because of the low mass of the electron, the B-E condensation temperatures of the Cooper pairs are five to six orders of magnitude higher than for the alkali-atom condensates. However, in contrast to the dilute alkali-atom condensates the density of the Cooper-pairs is temperature dependent. As we have already argued, the density of the Cooper-pairs increases as a function of a decreasing temperature according to their decreasing size such that for all temperatures the volume of the superconductor is nearly completely filled with Cooper-pairs. In other words, the observed condensation temperature, T_{BE}, corresponds to the Cooper-pair density at T_{BE}. Since the T_{BE} values turned out to be proportional to T_{SC} as $T_{BE} \sim 0.135 \cdot T_{SC}$ [16] the density of the Cooper-pairs at T_{BE} can be taken as representative for the density at T=0.

The B-E condensation of the Cooper-pairs gives rise to a crossover event in the heat capacity of the superconductor [10,16]. Inserting the experimental scaling relation $T_{BE}=0.135 \cdot T_{SC}$ into formula (4), the density of the Cooper pairs at T_{BE}, i.e., at T~0, follows as

$$n(T=0)=0.88 \cdot 10^{15} \cdot T_{SC}^{3/2} \quad (5)$$

with n in units of cm^{-3} [16].

Another, completely independent estimate of the Cooper-pair density at T=0 is possible from the London penetration depth data in Figure 2, assuming that the Cooper pairs form a dense gas of particles with no significant distance between them. Under this condition, the distance between the Cooper-pairs corresponds to their diameter $\lambda_L(T \sim 0)$. The density of the Cooper pairs therefore is given by $n \sim \lambda_L^{-3}$. Using $\lambda_L(T=0) \sim T_{SC}^{-1/2}$ from Figure 2 it follows that $n(T=0) \sim T_{SC}^{3/2}$. As a consequence, the same relation $n(T=0) \sim T_{SC}^{3/2}$ results from two completely different experimental methods. Inserting $\lambda_L = 90 \cdot T_{SC}^{-1/2}$ (in nm) from Figure 2 into $n \sim \lambda_L^{-3}$ it follows that

$$n(T=0)=1.37 \cdot 10^{15} \cdot T_{SC}^{3/2} \quad (6)$$

with n in units of cm^{-3}. The pre-factor in equation (6) is larger by a factor of ~1.5 compared to the pre-factor in equation (5). This indicates that the assumption of no space at all between the Cooper-pairs is not perfectly correct and that the density of the Cooper-pairs is over-estimated, assuming that their distance corresponds to their diameter. Nevertheless, it seems to be a reasonable approximation that for all temperatures the available space in the superconductor is nearly completely filled with Cooper-pairs.

Using the fitted slopes in Figure 1 and Figure 3, an estimate of the diamagnetic moment of the Cooper pairs at T=0 for an applied magnetic field of $B=B_c(T=0)$ can be obtained. According to formula (1) the diamagnetic moment of the Cooper pairs results as $\mu(T=0)=-2.8\cdot 10^3\mu_B/T_{SC}$, with μ_B as Bohr magneton. This surprisingly large diamagnetic moment is consistent with the view that for an applied field of $B_c(T=0)$ the two electrons of the Cooper-pair circulate with a high frequency on a closed loop. However, in view of the enormous diamagnetic moment obtained in this way, we cannot exclude that formula (1) is correct except for an unknown proportionality constant.

We should recall that in 1935 when F. and H. London proposed a theoretical explanation of the Meissner-Ochsenfeld effect [27,28,29], Cooper-pairs were unknown [1-8]. Using the free electron model of the metals, the strongly increasing diamagnetism of the superconductors with decreasing temperature had to be explained by a strong increase of the electron density, n_e, with decreasing temperature. In other words, no specific assumption on the superconducting electronic state was made. There is, however, neither a physical reason, nor some experimental evidence for such a strong temperature dependence of the electron density. It is clear that because of the large orbital diamagnetism of the Cooper-pairs, the shielding of an external magnetic field is much more efficient by Cooper pairs than by a gas of free electrons. According to the London theory, the relation between the density of the free electron gas, n_e, and the London penetration depth λ_L is given by [28,29]:

$$\lambda_L^2=m_e/\mu_0 e^2 n_e \qquad (7)$$

In formula (7), m_e is the mass of the electron, μ_0 is the vacuum permeability and e is the charge of the electron. Inserting $\lambda_L(T=0)=90\cdot T_{SC}^{-1/2}$ (in nm) according to Figure 2 into formula (7), results for the electron density at T=0:

$$n_e(T=0)=3.48\cdot 10^{21}\cdot T_{SC} \qquad (8)$$

with n_e in cm^{-3}. Note that in contrast to the Cooper-pair density at T=0 that is proportional to $T_{SC}^{3/2}$, according to formula (6), the electron density at T=0 of the London theory is proportional to T_{SC}, according to formula (8). As a conclusion, the historical London theory explains the low penetration depth of a magnetic field for $T\rightarrow 0$ by an electron density of the assumed free-electron gas that is larger by a factor of $\sim 10^6$ compared to a Cooper-pair density that results into the same shielding effect (formula (5)).

In 1935, the importance of bosons for the dynamics of solids was unknown. This new chapter of solid-state physics began only in 1974 when the Renormalization-Group theory appeared [30]. Although RG-theory has restricted to the magnetic degrees of freedom, it became more and more clear that bosons are essential for the dynamics of all other degrees of freedom as well. In particular ordered boson fields are responsible for the generation of Cooper-pairs and magnetic domains.

3. Temperature Dependence of E_{gap} and λ_L

As we have already mentioned, in contrast to the critical field and the zero-field heat capacity which exhibit critical behavior at T=0 only, the Cooper-pair gap energy, $E_{gap}(T)$, and the reciprocal London penetration depth, $\lambda_L(T)^{-1}$, exhibit critical behavior additionally at $T=T_{SC}$, as it is known for the spontaneous magnetization and for the magnon gap of the ordered magnets [23,24,31]. Since we know that the temperature dependence of the two magnetic quantities is controlled by the heat capacity of the long-range ordered Goldstone-boson field (magnetic dipole radiation), it can be concluded that the temperature dependence of $E_{gap}(T)$ and of $\lambda_L(T)^{-1}$ is controlled also by the heat capacity of a long-range ordered boson field. This boson field is evidently different from the boson field that is responsible for the temperature dependence of the zero-field heat capacity. For these low-energy bosons, T=0 is the only critical point, i.e., these bosons do not order into a long-range ordered state. The bosons that control the temperature dependence of $E_{gap}(T)$ and $\lambda_L(T)^{-1}$ become apparent only under the special excitation conditions necessary for the observation of the two quantities, and, evidently, have high dispersion energies. This shows that the excitation spectra of the superconductors are very complicated [10]. Relevance of excited-state bosons requires a thermal population of the dispersion relation of these bosons. Population of the dispersion relation of excited state-bosons is, however, not a continuous process, according to the Boltzmann-factor, but occurs in the discrete manner of a crossover event. At this crossover, the excited state bosons suddenly become relevant and T_{SC} appears as a second critical temperature. Below this crossover temperature, the excited state bosons seem to be completely absent and $T=T_{SC}$ is not a critical point. Due to the symmetry selection principle of relevance, only one boson type can be relevant [30]. On the other hand, thermal population of the dispersion relation of the excited state bosons that control the temperature dependence of $E_{gap}(T)$ and $\lambda_L(T)^{-1}$ is certainly never given considering that the $E_{gap}(T=0)/k_B$ values are about four times larger than T_{SC} (Figure 1).

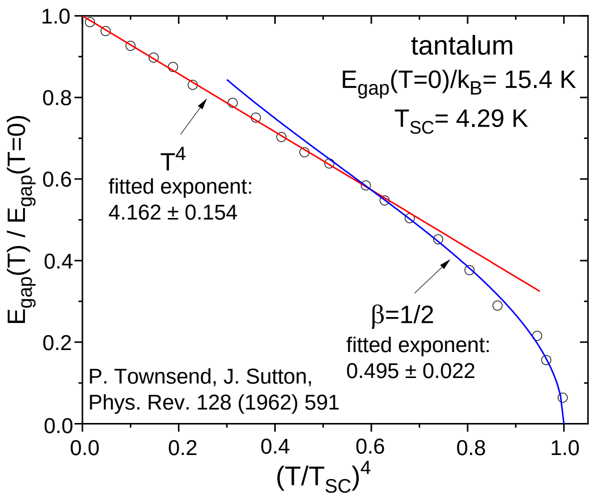

Figure 4. Normalized Cooper-pair gap energy of tantalum as a function of the reduced temperature to a power of four [32]. *As for all here investigated superconducting elements, thermal decrease of the gap energy with respect to saturation at* T=0 *is given by a* T^4 *power function, followed by a crossover to the critical power function of the argument* $(T_{SC}-T)$ *with mean field exponent of* β=1/2 [21].

As far as $E_{gap}(T)$ data are available, the same type of universal temperature dependence as in Figure 4 [32] is observed for all superconducting elements [21]. Universality holds in the vicinity of the two critical points T=0 and $T=T_{SC}$. At the critical point T=0 the critical power function is a power function of absolute temperature and exhibits the critical exponent of ε=4. The critical power function at $T=T_{SC}$ is a power function of the argument $(T_{SC}-T)$ and exhibits mean field exponent of β=1/2. Formally, this critical exponent agrees with the BCS-theory [1,20]. However, in contrast to the atomistic BCS-theory, the observed, boson-defined critical power function holds over a finite distance from the critical point [31]. The finite width of the critical range provides clear evidence of boson dynamics. The critical range at T_{SC} is limited by the crossover to the critical power function at T=0. Crossover events are analytical changes of the temperature function and are clearly beyond the atomistic models.

As for the spontaneous magnetization of the ordered magnets [21,23], the two critical power functions at $T=T_{SC}$ and at T=0 intersect and give a complete description of the temperature dependence of $E_{gap}(T)$. For niobium, the same universal exponents as in the temperature dependence of $E_{gap}(T)$ occur in the temperature dependence of the reciprocal London penetration depth, λ_L^{-1} (Figure 5) [33]. This one can reasonably expect since the temperature dependence of the two quantities is controlled by the same boson type. Observation of the same critical exponents does, however, not mean that the two quantities are perfectly proportional to each other. In fact, as we have seen, for $T\rightarrow 0$ $E_{gap}(T=0)\sim T_{SC}$ (Figure 1) but $\lambda_L^{-1}(T=0)\sim T_{SC}^{1/2}$ (Figure 2). Moreover, the pre-factors of the two universal power functions at T=0 and at $T=T_{SC}$, i.e., the critical amplitudes, can be different for the two quantities. The similar temperature dependence of E_{gap} and λ_L^{-1} proves that the larger the gap energy is, i.e., the stronger the two Cooper-pair electrons are coupled, the lower is the diameter of the Cooper-pair orbital and the London penetration depth. Figure 5 shows, as an example, the normalized reciprocal London penetration depth of niobium as a function of the reduced temperature [33]. In other words, λ_L diverges at T_{SC} with a critical exponent of $\beta=1/2$.

For chemically and structurally more complicated superconducting compounds, the critical exponent at T_{SC} seems to remain $\beta=1/2$. This exponent is typical of an isotropic behavior as it can be expected for cubic, or weakly non-cubic materials. In structurally strongly anisotropic systems, the symmetry, i.e., the dimensionality of the excited state boson field can be lower at the critical point T=0. The exponent ε of the T^{ε} function then can assume a rational value different from $\varepsilon=4$. In this case, a symmetry crossover coincides with the common crossover between the two critical power functions at T=0 and at $T=T_{SC}$. The observed exponent of $\varepsilon\neq 4$ is characteristic of the specific low-temperature symmetry of the excited state boson field. As we know from magnetism, a crossover to a lower symmetry class at T=0 compared to T_{SC} can be caused by a sufficiently strong spontaneous lattice distortion as a function of a decreasing temperature [21]. The material then cannot be classified by only one symmetry class alone. In other words, each critical point, either T=0 or $T=T_c$ can have its own dynamic symmetry. Although spontaneous lattice distortions increase continuously with decreasing temperature, the boson-defined dynamics reacts in the discrete manner of a crossover event when the distortion has increased beyond the threshold to become relevant. Finite distortions that remain below this threshold are not relevant and have no effect on the boson-controlled thermodynamic observables, i.e., on the critical exponents. As an example of a low symmetry class at T=0, Figure 6 shows the normalized reciprocal London penetration depth of the two-dimensional organic salt κ-(BEDT-TTF)$_2$Cu(NCS)$_2$ as a function of the reduced temperature [34]. The meaning of the exponent of $\varepsilon=5/2$ is difficult to specify as long as this exponent is not reproduced by many other similar materials.

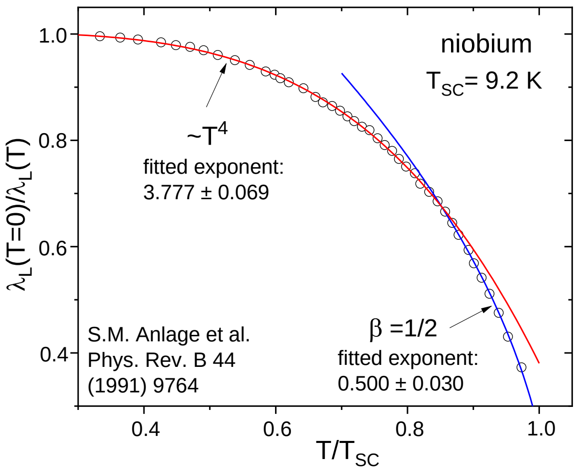

Figure 5. Normalized reciprocal London penetration depth of niobium as a function of the reduced temperature [33]. *The same critical exponents as for* $E_{gap}(T)$ *in Figure 4 are observed at T=0 and at* $T=T_{SC}$.

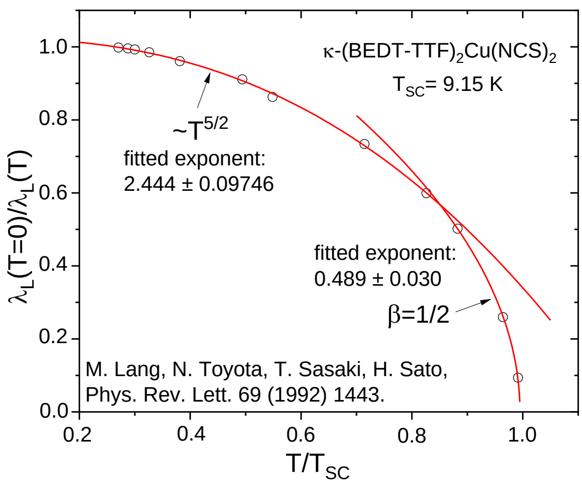

Figure 6. Normalized reciprocal London penetration depth of the strongly anisotropic organic salt κ-(BEDT-TTF)$_2$Cu(NCS)$_2$ *as a function of the reduced temperature* [34]. *The exponent of* $\varepsilon=5/2$ *of the critical power function at* T=0, T^{ε}, *is indicative of a low-symmetry class.*

For hexagonal MgB_2, investigated in [35], a critical exponent of $\varepsilon=2$ can be identified in the reciprocal London penetration depth (Figure 7). According to the unusual intersection of the two critical power functions in Figure 7, compared to Figure 5 and Figure 6, the symmetry of the excited state boson field must be considerably lower at T=0 compared to the symmetry at $T=T_{SC}$. Quite generally, when the symmetry at the critical point T=0 is much lower than the symmetry at $T=T_{SC}$, the crossover between the critical power function at $T=T_{SC}$ and at T=0, can assume a rather anomalous appearance. It is evident that more systematic investigations are necessary for an understanding of the different observed exponents ε.

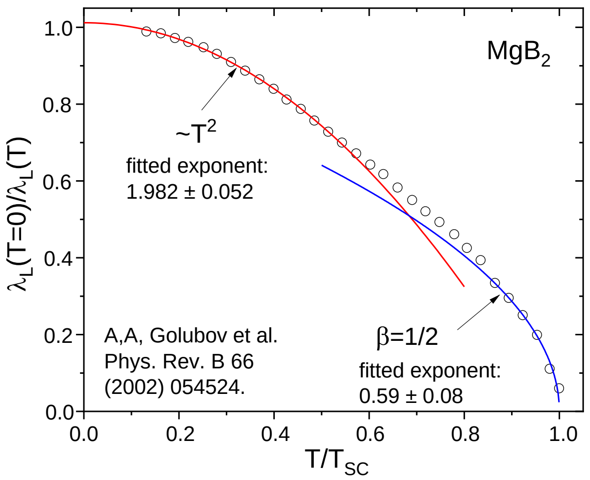

Figure 7. Normalized reciprocal London penetration depth of a MgB_2 *thin film sample as a function of the reduced temperature* [35].

4. Concluding Remarks

It seems to be now clear that at phase-transitions into a long-range and coherently ordered state, a boson field orders [9]. The visible atomic or magnetic order results from a coupling of atoms or spins to the ordered boson field. As we have mentioned, the superconducting transition is exceptional, in that the relevant bosons enter a short-range ordered state only. Nevertheless, comparison with the better understood long-range ordered boson field in the magnetic systems, is most revealing [9]. Characteristic of a transition into a long-range ordered boson field is the finite width of the critical range and the observed rational critical exponents [36,37,38]. The observed perfect collinear order of the spins results from a surprisingly strong coupling of the spins to the perfectly ordered boson field. The locally anisotropic near-neighbor interactions would never result into a perfect coherent, long-range order. If the anisotropic local exchange interactions would be the relevant excitations, a spin-glass like order would result. This is realized in Ising magnets only [11]. Note that in Ising magnets, Goldstone-bosons get not generated because Ising spins do not precess. The dynamics therefore is atomistic, i.e., due to the local exchange interactions. In other words, the order realized by ordered boson fields is the highest possible one and provides an entropy argument for the dominance of the bosons. The long-range ordered objects generated by the ordered Goldstone-boson field are the domains. Domains are typical of the ordered magnets [39] and of the ordered ferroelectrics [40]. The mosaic blocks, occurring in practically all single crystals, have also to be viewed as domains, generated by the bosons that order at the melting transition [38]. These bosons are, however, completely unexplored. The bosons that order at the magnetic ordering temperature, i.e., the Goldstone-bosons, are magnetic dipole radiation, generated by the precessing spins [11]. The bosons of the ferroelectric materials are evidently electric dipole radiation [40]. Although the individual domain is a stable unit, the domain configurations are not very stable and can easily be manipulated by suitable external means. As is well known, upon ferromagnetic saturation the whole sample gets transformed into the mono-domain state. The dynamic symmetry then is one-dimensional [41]. In other words, a dimensionality crossover occurs upon ferromagnetic saturation [41]. Since the linear dimensions of the domains are much larger than the inter-atomic distance, it is evident that the domains do not result from atomistic near-neighbor interactions. The domain is, so to say, a universal, self-contained geometrical unit. The size and shape of the individual domain, therefore, must result from the ballistically propagating bosons. The finite dimension of each domain is indicative of a self-constriction mechanism of the ordered boson field. Self-constriction has to be considered as a dynamic particularity of the ordered boson fields and seems to be one origin of the spontaneous magnetostriction [21]. The domains are resonators, self-organized by the ordered boson field. In each domain, bosons and magnons are standing one-dimensional waves. As a consequence, in many magnets the magnon dispersions are as for the linear spin chain, irrespective of the locally anisotropic exchange interactions that are not relevant in the sense of the RG-theory [15,23]. Condition for a one-dimensional long-range order of the boson field is that the emission characteristics of the individual boson source is axial and that the dominant generation process of the bosons is by stimulated emission. This holds for the magnetic and for the electric dipole radiation. Due to stimulated emission, the number of bosons propagating along those crystallographic directions with a high density of the boson sources gets enhanced [42,43]. This can be viewed as a self-collimation mechanism of the bosons. Eventually, for a sufficiently sharp collimation, the critical boson density for the spontaneous onset of stimulated will be reached and the boson field orders perfectly one-dimensional [9,42,43]. Now all bosons are condensed in the same quantum state. This is realized in each domain. The ordered boson field resembles the beam of a LASER. A three-dimensional global boson field results by a coupling between the one-dimensional boson fields of the domains along x-, y- and z-axis. In other words, the dimensionality of the global boson field is given by the number of inequivalently oriented domains. The observed critical exponents are defined by the global boson field.

The superconducting transition is particular in that the relevant bosons undergo a transition into a short-range ordered state. In contrast to the long-range ordered magnets, there are no domains observed in the superconducting state. Moreover, the typical critical power functions of the argument $|T_{SC}-T|$, as they occur at the magnetic ordering transition, are absent in the zero-field heat capacity of the superconductors [10,13]. Instead, universal power functions of absolute temperature are observed above and below T_{SC} [10,16]. As a consequence, the superconducting transitions are all within the critical range of the critical point T=0. As we know from RG-theory [30], the dynamics in the critical range is exclusively due to bosons. Characteristic of a critical range is short-range order. This means, at T_{SC} the type of short-range order of the relevant bosons changes. Evidently, the short-range ordered units of the superconducting state are the Cooper-pairs. The radiation field emitted by the sources of the relevant bosons, therefore, cannot be axial. It is quite clear, that the bosons that order at T_{SC} are the same as the bosons above T_{SC} [18]. We have called the bosons of the continuous metallic solid that give rise to the universal linear-in-T heat capacity above T_{SC}, CMS-bosons [17]. It is suggestive to identify these bosons with electric quadrupole radiation generated by the inhomogeneous charge distributions in the soft zones between the metal atoms. Although the radiation field generated by an electric quadrupole is theoretically unexplained, it evidently contains a component that is essentially circular or spherical in shape. Due to stimulated emission many identical quanta of the spherical waves can superimpose whereby a sharp spherical shell is generated that encapsulates the two Cooper-pair electrons. This is as for the magnetic domains that encapsulate a region with coherently ordered spins. The shape and volume of the magnetic domains (Cooper-pairs) is defined by the ordered Goldstone-boson field (ordered CMS-boson field). The short-range ordered Cooper-pairs correspond, so to say, to the domains in the long-range ordered magnets. The CMS-boson shell, surrounding the two Cooper-pair electrons, shields the charge of the two Cooper-pair electrons perfectly and allows for a free mobility of the Cooper-pairs. This is a condition for a resistivity of zero of the superconducting current. In contrast to the magnetic domains that are fixed to each other and to the crystal lattice, the spherical Cooper-pairs can move freely across the metal. They behave as a dense gas of neutral particles [16]. Consistent with the short-range order in the superconducting state is that there is no correlation between the positions of the Cooper-pairs. In contrast to the ordered magnets, for which the dynamic dimensionality is defined by the number of inequivalently oriented domains, the conventional (cubic) superconductors are isotropic system. The CMS-boson shell exerts the necessary constricting force to counteract the repulsion between the charges of the two Cooper-pair electrons. This constriction is an inherent dynamic, i.e., temperature dependent property of the ordered boson field, and is similar to the spontaneous magnetostriction in the ordered magnets [21]. Since the constriction gets stronger with decreasing temperature, the diameter of the Cooper-pairs shrinks with decreasing temperature. The corresponding binding energy between the two Cooper-pair electrons is given by the gap energy $E_{gap}(T)$ that increases with decreasing temperature. $E_{gap}(T)$ has much similarity with the magnon gap in the ordered magnets that is a measure of the stability of the spin order due to the interaction with the ordered Goldstone-boson field (magnetic dipole radiation) [44]. The decreasing diameter of the Cooper-pairs with increasing $E_{gap}(T)$ agrees with the temperature dependence of the London penetration depth for an applied magnetic field. This allows one to obtain the temperature dependence of the diameter of the Cooper-pairs from measurements of the temperature dependence of the London penetration depth. The strong superconducting diamagnetism results from the large orbital cross section of the Cooper-pairs. Since the London penetration depth agrees with the diameter of the individual Cooper-pair, it results that only one layer of Cooper-pairs next to the inner surface of the superconductor is sufficient to shield an applied magnetic field completely. The fact that, for all temperatures, a magnetic field is expelled out of the superconductor shows that the decreasing size of the Cooper-pairs gets compensated by a corresponding increase of their density such that the volume of the superconductor is always nearly completely filled with Cooper pairs [16]. This is as for the volume of the ordered magnets that is completely filled with domains. It can be assumed that the increasing number of electrons needed for the increasing number of Cooper-pairs is delivered by the conduction band. The superconductor, therefore, resembles a two-phase system with the Cooper-pairs as the condensed phase and the conduction-band electrons as the vapor phase. With the increasing number of Cooper-pairs with decreasing temperature the superconducting systems receives an increasing mass which is certainly of influence on the dynamics, i.e., on the temperature dependence of the heat capacity of the zero-field ground state.

Concluding it has to be remarked that the just sketched scenarios need more detailed investigations, in particular field-theoretical studies, for a final approval. Many of the statements and ideas advanced in this rather phenomenological work are heuristic guides only, intended to stimulate further research activities. It cannot be excluded that with the continuing progress of our understanding of the dynamics in solids, one or the other of the here presented ideas will need a considerable correction. Nevertheless, the dominant role of bosons for the dynamics of solids is now a firmly established experimental fact. Essential for this dominance seems to be that the bosons get generated by stimulated emission. Due to stimulated emission the bosons can order, i.e., they condense all in the same or a few quantum states, whereby extremely high, local electromagnetic fields get generated. These high fields affect the microscopic near-neighbor interactions and determine the dynamics in the ordered state, instead of the non-relevant near-neighbor interactions.

Nomenclature

λ_L, *London penetration depth (nm)*
E_{gap}, *Cooper-pair energy gap (meV)*
T_{SC}, *superconducting transition temperature (K)*
λ, *diameter of the Cooper-pair*
T_{BE}, *Bose-Einstein condensation temperature*
μ_B, *Bohr-magneton (Vsm)*
μ, *diamagnetic moment of the Cooper-pair (Vsm)*
B_c, *critical magnetic field (Gauss)*
n, *spatial density of the Cooper-pairs* (cm^{-3})
n_e, *conduction-band electron density* (cm^{-3})
m_e, *mass of the electron (Kgr)*
β, *critical exponent at* T_{SC}
ε, *critical exponent at* $T=0$
e, *charge of the electron (C)*
μ_0, *vacuum permeability (Vs/Am)*

References:

[1] J. Bardeen, L.N. Cooper, J.R. Schrieffer, “Theory of Superconductivity”, *Phys. Rev*. vol. 108, pp. 1175-1204, 1957.

[2] A.M. Kadin, “Spatial Structure of the Cooper Pair” *Supercond. And Novel Magnetism*, vol. 20, pp. 285-282, 2007.

[3] N. Andrenacci, M. Capezzali, H. Beck, “Internal structure of fluctuating Cooper pairs”, *Eur. Phys. J. B*, vol. 53, pp. 417-432, 2006.

[4] W.V. Pogosov, “Applicability of Bardeen-Cooper-Schrieffer theory to small-sized superconductors: Role of Cooper-pair binding energy”, *Solid State Commun*. vol. 207, pp. 1-4, 2015.

[5] T. Örd, K. Rägo, A. Vargunin, G. Litak, “Strong temperature effect on the size of the Cooper-pairs in a two-band superconductor” *Eur. Phys. J* . B, 91:2, pp. 1-6, 2018.

[6] N. Ahmad, S.H. Naqib, “Estimation of Cooper pair density and its relation to the critical current density in

Y(Ca)BCO high-T_c cuprate superconductors" *Results in Physics*, vol. 17, pp.103054, 1-6, 2020.

[7] F.D. Neto, M.A. Neto, O.D. Rodriguez Salmon, "Cooper-pair size and binding energy for unconventional superconducting systems", *Physica C:Supercond. and its Appl.* 549 pp. 159-163, 2018.

[8] T. Böhm et al., "Microscopic origin of Cooper pairing in the iron-based superconductor $Ba_{1-x}K_xFe_2As_2$", *Quantum Materials,* 3:48, pp. 1-6, 2018.

[9] U. Köbler, "On the precise value of the magnetic ordering temperature", *J. Magn. Magn. Mater.* 546, pp. 168839, 1-6, 2022.

[10] U. Köbler, "New Explanation of the Excitation Spectra of Conventional Superconductors *J. of Low Temp. Phys*. https://doi.org/10.1007/s10909-022-02886-7.

[11] A. Hoser, U. Köbler, "Boson Fields in Ordered Magnets", *Acta Phys. Pol.* A, vol. 127, pp. 350-352, 2015.

[12] J. Goldstone, A. Salam, S. Weinberg, "Broken Symmetries", *Phys. Rev.* vol 127, pp. 965-970, 1962.

[13] U. Köbler, "Universality in the temperature dependence of the heat capacity of magnetic solids", *Int. J. of Thermo*. vol. 23, pp.147-174, 2020.

[14] A. Hoser, U. Köbler, "Linear spin chains in paramagnetic and in ordered bulk magnets", *Physica* B, vol. 551, pp.83-88, 2018.

[15] U. Köbler, "Bosonic and magnonic magnon dispersions", *J. Magn. Magn. Mater*. vol. 502, pp. 166533, 1-18, 2020.

[16] U. Köbler, "Bose-Einstein Condensation of Cooper-Pairs in the Conventional Superconductors" *Int. J. of Thermo*. 24, pp. 238-246, 2021.

[17] U. Köbler, "On the Thermal Conductivity of Metals and of Insulators", *Int. J. of Thermo*., vol. 20, pp. 210-218, 2017.

[18] C. Probst, J. Wittig, "Superconductivity of bcc Barium under Pressure", *Phys. Rev. Lett.* Vol. 39, pp.1161-1163, 1977.

[19] C.J. Pethick, H. Smith: "Bose-Einstein condensation in dilute gases", Cambridge Univ. Press, 2008.

[20] B. Mühlschlegel, "Die thermodynamischen Funktionen des Supraleiters", *Z. Physik,* vol. 155, pp. 313-327, 1959.

[21] U. Köbler, A. Hoser: *Experimental Studies of Boson Fields in Solids,* World Scientific, Singapore, 2018.

[22] G.K. White, "Thermal Expansion at Low Temperatures IV. Normal and Superconducting Lead", *Phil. Mag*. Vol. 7, pp. 271-278, 1962.

[23] U. Köbler, "One-Dimensional Boson Fields in the Critical Range of EuS and EuO", *Acta Phys. Pol*. A, vol. 128, pp. 398-407, 2015.

[24] O.W. Dietrich, "Critical magnetic fluctuations in MnF_2", *J. Phys. C: Solid State Phys.* vol. 2, pp. 2022-2036, 1969.

[25] B.B. Goodman, E. Mendoza, "The Critical Magnetic Field of Aluminium, Gallium and Zinc", *Philosophical Magazine* vol. 42, pp. 594-602, 1951.

[26] R. Kleiner, W. Buckel: *Superconductivity: an introduction*, Wiley-VCH, Weinheim, 2016.

[27] W. Meissner, R. Ochsenfeld, "Ein neuer Effekt bei Eintritt der Supraleitfähigkeit" *Naturwissenschaften,* vol. 21, pp. 787-788, 1933.

[28] F. London, H. London, "Supraleitung und Diamagnetismus", *Physica,* vol. 2, pp. 341-354, 1935.

[29] M. v. Laue, F. u. H. London, "Zur Theorie der Supraleitung", *Z. Physik,* vol. 96, pp. 359-364, 1935.

[30] K.G. Wilson, J. Kogut, "The renormalization group and the ε expansion", *Phys. Rep*. vol. 12C, pp. 75-199, 1974.

[31] U. Köbler, "Crossover phenomena in the critical range near magnetic ordering transition", *J. Magn. Magn. Mater.* Vol. 453, pp. 17-29, 2018.

[32] P. Townsend, J. Sutton, "Investigation by Electron Tunneling of the Superconducting Energy Gaps in Nb, Ta, Sn, and Pb", *Phys. Rev*. vol. 128, pp. 591-595, 1962.

[33] S.M. Anlage, B.W. Langley, G. Deutscher, J. Halbritter, M.R. Beasley, "Measurements of the temperature dependence of the magnetic penetration depth in $Yba_2Cu_3O_{7-\delta}$ superconducting thin films", *Phys. Rev*. B, vol. 44, pp. 9764-9767, 1991.

[34] M. Lang, N. Toyota, T. Sasaki, H. Sato, "Magnetic Penetration Depth of κ-$(BEDT\text{-}TTF)_2Cu(NCS)_2$: Strong Evidence for Conventional Cooper Pairing" *Phys. Rev. Lett*. vol. 69, pp. 1443-1446, 1992.

[35] A.A. Golubov, A. Brinkman, O.V. Dolgov, J. Kortus, O. Jepsen, "Multiband model for penetration depth in MgB_2", *Phys. Rev*. B vol. 66, pp. 054524, 1-5, 2002.

[36] R.K. Pathria: *Statistical Mechanics*, 2nd ed., Butterworth-Heinemann, Oxford, 1996.

[37] P. Heller, "Experimental investigations of critical phenomena", *Rep. Prog. Phys.* vol. 30, pp. 731-826, 1967.

[38] U. Köbler, V. Bodryakov, "On the melting process of solids", *Int. J. of Thermo*. vol. 18, pp. 200-204, 2015.

[39] A. Hubert, R. Schäfer: *Magnetic Domains,* Springer, Berlin, 2000.

[40] E. Fatuzzo, W.J. Merz: *Ferroelectricity*, North-Holland, Amsterdam, 1967.

[41] U. Köbler, A. Hoser, C. Thomas: "Dimensionality crossover upon magnetic saturation of Fe, Ni and Co", *J. Magn. Magn. Mater*. vol. 321, pp. 1202-1208, 2009.

[42] U. Köbler, "Sound waves and phonons in crystalline solids", *J. Chem. Thermo.*, to be published.

[43] U. Köbler, "Magnetic ordering by boson fields", *Eur. Phys. J. B.*, to be published.

[44] U. Köbler, "Thermal decay of magnons in MnF_2", *J. Magn. Magn. Mater*. vol. 551, pp. 169129, 1-9, 2022.