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The Investigation of Some Tensor Conditions for α-Kenmotsu Pseudo-Metric Structures

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Abstract

This paper aims to study some semi-symmetric and curvature tensor conditions on α-Kenmotsu pseudo-metric manifolds. Some conditions of semi-symmetric, locally symmetric, and the Ricci semi-symmetric are considered on such manifolds. Also, the relationships between the M-projective curvature tensor and conformal curvature tensor, concircularly curvature tensor, and conharmonic curvature tensor are investigated. Finally, an example of α-Kenmotsu pseudo-metric structure is given.

Bazı Tensör Koşullarının α-Kenmotsu Pseudo-Metrik Yapılar İçin İncelenmesi

Anahtar kelimeler
α-Kenmotsu manifold; Pseudo metriği; Konharmonik eğrilik tensörü; M-projektif eğrilik tensörü

Öz

Bu makale α-Kenmotsu pseudo-metrik manifoldlar üzerinde bazı yarı simetrik ve eğrilik tensör şartlarını çalışmayı amaçlamaktadır. Bazı yarı-simetrik, lokal simetrik ve Ricci yarı-simetrik şartlar bu tür manifoldlar için göz önüne alınmıştır. Ayrıca, M-projektif eğrilik tensörü ile konformal eğrilik tensörü, konsirküler eğrilik tensörü ve konharmonik eğrilik tensörü arasındaki ilişkiler araştırılmıştır. Makale açıklayıcı bir α-Kenmotsu pseudo-metrik manifold örneği ile sonlandırılmıştır.

1. Introduction

Almost contact manifolds are critical to the manifold theory. A $(2n+1)$-dimensional differentiable manifold M of class C^∞ is said to have an almost contact structure if the structural group of its tangent bundle reduces to $U(n)\times 1$ (Gray 1959); equivalently an almost contact structure is given by a triple (φ,ξ,η) satisfying the following conditions

$$\eta(U)\xi - U = \varphi^2 U, \qquad \eta(\xi) = 1. \tag{1}$$

In 1960, Sasaki defined a metric g such that

$$g(U,V) - \eta(U)\eta(V) = g(\varphi U, \varphi V) \tag{2}$$

$$\eta(U) = g(U,\xi) \tag{3}$$

on the structure (φ,ξ,η,g). Thus Sasaki introduced the concept of almost contact metric structure to the literature. Then, Sasaki and Hatakeyama gave the normality condition on almost contact manifolds (Sasaki and Hatakeyama 1962). After these studies, Goldberg and Yano defined cosymplectic manifolds, a class of almost contact metric manifolds. (Goldberg and Yano 1969). The most extensive and detailed research in this field of study began with Blair (Blair 1976). After that many studies were handled by Olszak on almost cosymplectic manifolds (Olszak 1981, 1989).

The manifold, which forms the basic structure of our study, was first mentioned by Kenmotsu (Kenmotsu 1972). These manifolds are known as Kenmotsu which are certain subclasses of almost contact metric structures, and it is defined as follows:

$$\nabla_U \xi = U - \eta(U)\xi, \tag{4}$$

$$(\nabla_U \varphi)V = g(\varphi(U), V)\xi - \eta(V)\varphi(U) \tag{5}$$

for any $U, V \in \chi(M)$ (Kenmotsu 1972). Then the structure (φ, ξ, η, g) is normal. Since it is not a Killing, the ξ is not Sasakian.

For contact metric manifolds, the first study with the help of pseudo-metric was made in 1969 by Takahashi on Sasakian manifolds (Takahashi 1969). After this study, contact manifolds and Sasakian pseudo-Riemannian manifolds defined with the help of pseudo-Riemann metrics attracted the attention of some researchers (Duggal 1990, Calvaruso and Perrone 2010). Calvaruso and Perrone put forward the most systematic study on this subject. The similarities and differences between the Riemann and pseudo-Riemann metrics were discussed. Contact pseudo-metric structures whose sectional curvature is constant, contact pseudo-metric structures with local symmetry, and homogeneous contact Lorentzian metric structures of dimension 3 were investigated and classified (Calvaruso and Perrone 2010). Also, Oneil's book is the most significant source of motivation for all authors studying in this field (O'neil 1983).

A classification of almost α-Kenmotsu pseudo-metric manifolds has not been undertaken yet. The first study, one of the detailed studies in the literature, was investigated by Wang and Liu (Wang ve Liu 2016). Moreover, some authors have also focused on this topic (Öztürk 2020, Öztürk 2021, Naik *et al.* 2020). In particular, Öztürk studied the η-parallelity conditions on almost α-cosymplectic pseudo-metric manifolds (Öztürk 2021).

This paper aims to study some semi-symmetric and curvature tensor conditions on α-Kenmotsu pseudo-metric manifolds such that a smooth function defined $d\alpha \wedge \eta = 0$ on M. Semi-symmetry, local symmetry, Ricci semi-symmetry, and the curvature tensors of conformal, concircularly, conharmonic, and M-projective are investigated. Many results are obtained related to such tensor conditions. Many results have been obtained related to such tensor conditions. An illustrative example is provided at the end of the paper.

2. Preliminaries

A $(2n+1)$-dimensional differentiable manifold M with a triple (φ, ξ, η), such that φ is a tensor field type of (1,1), ξ is a vector field, and η is a 1-form on M defined by (1).

Admitting a Riemannian metric g, given by (2) and (3), then M is called almost contact metric structure (φ, ξ, η, g). Also, the fundamental 2-form Φ of M is defined by

$$\Phi(U, V) = g(\varphi U, V). \tag{6}$$

If the Nijenhuis tensor vanishes, then M is called normal. Here it is noted that the Nijenhuis tensor defined by

$$N_\varphi(U, V) = [\varphi U, \varphi V] - \varphi[\varphi U, V] - \varphi[U, \varphi V] + \varphi^2[U, V] + 2d\eta(U, V)\xi, \tag{7}$$

and Kenmotsu manifold is normal.

The conformal curvature tensor defined as

$$C(U, V)Z = R(U, V)Z - \left(\frac{1}{2n-1}\right)[S(V, Z)U - S(U, Z)V + g(V, Z)QU - g(U, Z)QV] + \left(\frac{r}{2n(2n-1)}\right)[g(V, Z)U - g(U, Z)V] \tag{8}$$

for any $U, V, Z \in \chi(M)$ (Yano and Kon 1984).

Besides, the other essential curvature tensor fields, which are called concircular $\bar{C}$, conharmonic H, and M-projective M^* in Riemannian geometry, are presented as follows, respectively:

$$\bar{C}(U, V)Z = R(U, V)Z - \left(\frac{r}{2n(2n+1)}\right)(g(V, Z)U - g(U, Z)V), \tag{9}$$

$$H(U, V)W = R(U, V)W$$

$$-\left(\frac{1}{2n-1}\right)[S(V,W)U - S(U,W)V$$
$$+g(V,W)QU - g(U,W)QV], \tag{10}$$

$$M^*(U,V)W = R(U,V)W$$
$$-\left(\frac{1}{4n}\right)[S(V,W)U - S(U,W)V$$
$$+g(V,W)QU - g(U,W)QV] \tag{11}$$

(Yano and Kon 1984).

If the structure of Kenmotsu satisfies the condition of Nomizu ($R \cdot R = 0$), then it has constant negative curvature. In addition, if the Kenmotsu manifold holds the conformal flatness condition, then the manifold has constant negative curvature -1 for dimension greater than 3.

The definition of semi-symmetry for the Riemannian manifold is defined by

$$R(U,V) \cdot R = 0, \tag{12}$$

for any $U, V \in \chi(M)$. Here $R(U,V)$ acts as a derivation on R (Nomizu 1968). A semi-symmetric space is the same space as the curvature tensor of a symmetric space at a point $p \in M$. Of course, this may change according to the point of $p \in M$. So, it is obvious that local symmetry implies semi-symmetry. Nevertheless, the converse of this proposition is not necessarily correct (Calvaruso and Perrone 2002). These spaces were studied in the sense of a complete intrinsic classification by Szabó (Szabó 1982).

Moreover, the other semi-symmetric curvature conditions, $R \cdot C = 0$ and $R \cdot \bar{C} = 0$, are called conformally semi-symmetric and concircularly semi-symmetric, respectively (Bagewadi and Venkatesha 2007).

3. α-Kenmotsu Pseudo-Metric Manifolds

Assume that (M, φ, ξ, η) is an almost contact manifold of dimension $(2n+1)$. Now, let us include the pseudo-Riemannian metric in (2). In other words, a compatible pseudo Riemannian metric g satisfies

$$g(U,V) - \varepsilon\eta(U)\eta(V) = g(\varphi U, \varphi V) \tag{13}$$

such that $\varepsilon = \pm 1$ (Calvaruso and Perrone 2010). Then $(M, \varphi, \xi, \eta, g)$ is called an almost contact pseudo-metric manifold. Also, we have

$$g(\varphi U, V) = -g(U, \varphi V),$$
$$\eta(U) = \varepsilon g(U, \xi),\ g(\xi, \xi) = \varepsilon \tag{14}$$

(Wang and Liu 2016). In this case, the ξ is either space-like or time-like and can not be light-like $(g(\xi,\xi) = 0)$.

If $(M, \varphi, \xi, \eta, g)$ satisfies the conditions $d\eta = 0$ and $d\Phi = 2\alpha(\eta \wedge \Phi)$, then M is called an almost α-Kenmotsu pseudo-metric manifold for $\alpha \neq 0$, $\alpha \in IR$. We note that if (7) vanishes, then M is said to be an α-Kenmotsu pseudo-metric manifold.

Proposition 3.1 An almost α-Kenmotsu pseudo-metric manifold M of dimension $(2n+1)$ is an α-Kenmotsu pseudo-metric manifold if and only if

$$(\nabla_U \varphi)V = \alpha\varepsilon g(\varphi U, V)\xi$$
$$-\alpha\eta(V)\varphi U \tag{15}$$

for any $U, V \in \chi(M)$ (Öztürk 2020).

Proposition 3.2 An α-Kenmotsu pseudo-metric manifold M of dimension $(2n+1)$ holds

$$\nabla_U \xi = -\alpha\varphi^2 U \tag{16}$$

$$(\nabla_U \eta)V = \alpha\varepsilon g(U,V)$$
$$-\alpha\eta(U)\eta(V), \tag{17}$$

for any $U, V \in \chi(M)$. Here α is a smooth function such that $d\alpha \wedge \eta = 0$ (Öztürk 2020).

Proposition 3.3 An α-Kenmotsu pseudo-metric manifold M of dimension $(2n+1)$ satisfies the following curvature properties

$$R(U,V)\xi = [\alpha^2 + \xi(\alpha)][\eta(U)V - \eta(V)U] \tag{18}$$

$$R(U,\xi)V = -[\alpha^2 + \xi(\alpha)]$$
$$[-\varepsilon g(U,V)\xi + \eta(V)U] \tag{19}$$

$$R(U,\xi)\xi - \varphi R(\varphi U, \xi)\xi =$$

$$2[\alpha^2+\xi(\alpha)][-U+\eta(U)\xi] \quad (20)$$

$$g(R(U,V)W,\xi)=-\varepsilon[\alpha^2+\xi(\alpha)]$$
$$[-g(U,W)\eta(V)+g(V,W)\eta(U)] \quad (21)$$

$$S(\xi,U)=-2n\eta(U)[\alpha^2+\xi(\alpha)] \quad (22)$$

$$S(\xi,\xi)=-2n\varepsilon[\alpha^2+\xi(\alpha)] \quad (23)$$

$$Q\xi=-2n\varepsilon[\alpha^2+\xi(\alpha)]\xi \quad (24)$$

$$S(\varphi U,\varphi V)=[\alpha^2+\xi(\alpha)]$$
$$\varepsilon\{-2n[-\varepsilon\eta(V)\eta(U)+g(V,U)]+S(V,U)\}. \quad (25)$$

Here $\nabla_\xi\alpha=\xi(\alpha)$, $\varepsilon=g(\xi,\xi)$ and α is a smooth function defined $d\alpha\wedge\eta=0$, for any $U,V,W\in\chi(M)$ (Öztürk 2020).

Corollary 3.1 If α is parallel along the characteristic vector field ξ, then Proposition 3.3 becomes

$$R(U,V)\xi=\alpha^2[\eta(U)V-\eta(V)U] \quad (26)$$

$$R(U,\xi)V=-\alpha^2[-\varepsilon g(U,V)\xi+\eta(V)U] \quad (27)$$

$$R(U,\xi)\xi-\varphi R(\varphi U,\xi)\xi=2\alpha^2[-U+\eta(U)\xi] \quad (28)$$

$$g(R(U,V)W,\xi)=$$
$$-\varepsilon\alpha^2[-g(W,U)\eta(V)+g(W,V)\eta(U)] \quad (29)$$

$$S(\xi,U)=-2n\eta(U)\alpha^2 \quad (30)$$

$$S(\xi,\xi)=-2n\varepsilon\alpha^2 \quad (31)$$

$$Q\xi=-2n\varepsilon\alpha^2\xi \quad (32)$$

$$S(\varphi U,\varphi V)=$$
$$\alpha^2(\varepsilon S(U,V)-2n[-\varepsilon\eta(V)\eta(U)+g(V,U)]) \quad (33)$$

for any $U,V\in\chi(M)$ (Öztürk 2020).

Moreover, an α-Kenmotsu pseudo-metric manifold M is called Einstein if its Ricci tensor S holds

$$S(U,V)=\mu g(U,V) \quad (34)$$

where μ is constant and it is called η-Einstein if its Ricci tensor S holds

$$S=ag+b(\eta\otimes\eta) \quad (35)$$

for any $U,V\in\chi(M)$, where $a,b: M\to R$ are the functions on M (Yano and Kon 1984).

4. Certain Semi-Symmetric Tensor Fields

We study some semi-symmetric conditions and obtain some results in this section.

Theorem 4.1 Let M be a $(2n+1)$-dimensional α-Kenmotsu pseudo-metric manifold and α is parallel along the characteristic vector field ξ. If M holds the Ricci semi-symmetric condition, then it is Einstein with $S=-2n\alpha^2\varepsilon g$ and $r=-2n(2n+1)\alpha^2$.

Proof. By the hypothesis, the tensor product of $R\cdot S$ is defined as

$$(R(V,Z)\cdot S)(U,X)=S(R(V,Z)U,X)-$$
$$S(U,R(V,Z)X). \quad (36)$$

For $R\cdot S=0$, we have

$$S(R(V,Z)U,X)+S(U,R(V,Z)X)=)0. \quad (37)$$

Taking $V=\xi$ in (37) and using (18), we get

$$0=S(\alpha^2\eta(U)Z-\alpha^2\eta(Z)U+\xi(\alpha)\eta(U)Z-$$
$$\xi(\alpha)\eta(Z)U,X)-2n[\alpha^2+\xi(\alpha)]\eta(R(U,Z)X) \quad (38)$$

Again taking $Z=\xi$ and from $\xi(\alpha)=0$ and (22), it follows that

$$0=-\alpha^2S(X,U)-2n\alpha^4\eta(X)\eta(U)$$
$$+2n\alpha^4\eta(U)\eta(X)-2n\varepsilon\alpha^4g(X,U). \quad (39)$$

Hence, (39) becomes

$$S(U,X)=-2n\alpha^2\varepsilon g(U,X). \quad (40)$$

Let $\{E_1,E_2,\ldots,E_{2n},\xi\}$ be a local pseudo φ-basis of M for $i=1,2,\ldots,2n+1$. Putting $U=X=E_i$ in (40) and then taking contraction over the index i, we obtain

$$r = -2\varepsilon n\alpha^2 \sum_{i=1}^{2n+1} g(E_i, E_i).$$

Thus the proof is completed.

Theorem 4.2 If an α-Kenmotsu pseudo-metric manifold M of dimension $(2n+1)$ is semi-symmetric and α is parallel along the characteristic vector field ξ, then it is Einstein.

Proof. In view of (12), the tensor product $R \cdot R$ is defined as follows:

$$(R(X,Y) \cdot R)(Z,U)V =$$
$$R(X,Y)R(Z,U)V - R(R(X,Y)Z,U)V$$
$$-R(Z,R(X,Y)U)V - R(Z,U)R(X,Y)V. \quad (41)$$

By the hypothesis, we have

$$R(X,Y)R(Z,U)V - R(R(X,Y)Z,U)V$$
$$-R(Z,R(X,Y)U)V - R(Z,U)R(X,Y)V = 0. \quad (42)$$

By the help of (18) and (19), we consider four tensor product expressions separately on the left side of (42) for $U = \xi$. Then we obtain

$$R(X,Y)R(Z,\xi)V$$
$$= [\alpha^2 + \xi(\alpha)](\eta(V)R(Y,X)Z$$
$$-\varepsilon g(V,Z)R(Y,X)\xi) \quad (43)$$

$$-R(R(X,Y)Z,\xi)V$$
$$= [\alpha^2 + \xi(\alpha)](-\eta(V)R(Y,X)Z$$
$$+\varepsilon g(R(Y,X)Z,V)\xi) \quad (44)$$

$$-R(Z,R(X,Y)U)V$$
$$= [\alpha^2 + \xi(\alpha)](\eta(X)R(Y,Z)V$$
$$-\eta(Y)R(X,Z)V) \quad (45)$$

$$-R(Z,\xi)R(X,Y)V = [\alpha^2 + \xi(\alpha)]$$
$$(g(R(X,Y)V)Z,\xi) - \varepsilon g(R(X,Y)V,Z)\xi). \quad (46)$$

Then summing (43)-(46) side by side and using $X = \xi$ ve $\xi(\alpha) = 0$ we deduce

$$R(Z,Y)V = \varepsilon\alpha^2 g(Z,V)Y - \varepsilon\alpha^2 g(Y,V)Z.$$

Lastly, taking contraction with respect to $Z = E_i$ for the local pseudo φ-basis in the above equation, it yields

$$S(Y,V) = -2n\varepsilon\alpha^2 g(Y,V).$$

This ends the proof.

Theorem 4.3 Let M be a $(2n+1)$-dimensional α-Kenmotsu pseudo-metric manifold and α is parallel along the characteristic vector field ξ. If M satisfies the locally symmetric condition, then it is Einstein.

Proof. Due to the hypothesis, we can write

$$0 = \nabla_U(R(V,W)Z) - R(\nabla_U V,W)Z$$
$$-R(V,\nabla_U W)Z - R(V,W)\nabla_U Z. \quad (47)$$

Let us remember that the locally symmetric conditon holds

$$(\nabla_U R)(V,W)Z = 0 \quad (48)$$

for $U,V,W,Z \in \chi(M)$.

Putting $Z = \xi$, (47) gives

$$0 = \nabla_U(R(V,W)\xi) - R(\nabla_U V,W)\xi$$
$$-R(V,\nabla_U W)\xi - R(V,W)\nabla_U \xi. \quad (49)$$

Taking into account of (16), (18) and $\xi(\alpha) = 0$,

(49) reduces to

$$R(V,W)U =$$
$$-\alpha^2\varepsilon[-g(V,U)W + g(W,U)V]. \quad (50)$$

Then taking contraction with respect to $V = E_i$ for the local pseudo φ-basis, it follows that

$$S = -2n\alpha^2\varepsilon g$$

for any $U,W \in \chi(M)$. So the last equation implies that M is Einstein and $r = -2n(2n+1)\alpha^2$.

5. Certain Curvature Tensor Fields

This section is devoted to investigating specific curvature tensors. We study the relationships between the M-projective curvature tensor and conformal, concircular, and conharmonic curvature tensors.

Theorem 5.1 If an α-Kenmotsu pseudo-metric manifold M of dimension $(2n+1)$ is M-projectively flat and α is parallel along the characteristic vector field ξ, then it is Einstein.

Proof. Follows from $M^*=0$ and (11), we have

$$R(U,V)\xi = \left(\frac{1}{4n}\right)[S(\xi,V)U - S(\xi,U)V + g(\xi,V)QU - g(\xi,U)QV]. \quad (51)$$

Making use of (18) and (22), (51) becomes

$$[\alpha^2+\xi(\alpha)][\eta(U)V-\eta(V)U] = \left(\frac{1}{4n}\right)[-2n(\alpha^2+\xi(\alpha))(\eta(V)U-\eta(U)V) + \varepsilon\eta(V)QU - \varepsilon\eta(U)QV]. \quad (52)$$

Then simplifying (52) for $\xi(\alpha)=0$, we get

$$QU = -2n\alpha^2\varepsilon U. \quad (53)$$

Putting $U=\xi$ in (53), we have

$$Q\xi = -2n\alpha^2\varepsilon\xi. \quad (54)$$

Also, taking account of (52) for $V=\xi$ and (54), it follows that

$$S(V,U) = g(QV,U) = -2n\alpha^2\varepsilon g(U,V). \quad (55)$$

This completes the proof.

Corollary 5.1 Suppose that M is an α-Kenmotsu pseudo-metric manifold of dimension $(2n+1)$ and α is parallel along the characteristic vector field ξ. If it is M-projectively flat, then its scalar curvature is $r=-2n(2n+1)\alpha^2$.

Proof. Using (55) and then taking contraction with respect to $U=V=E_i$ in (55) for the local pseudo φ-basis, the proof is obvious.

Theorem 5.2 Let M be a $(2n+1)$-dimensional α-Kenmotsu pseudo-metric manifold and α is parallel along the characteristic vector field ξ. If M-projectively curvature and concircular curvature tensors are linearly independent, then M is Einstein.

Proof. Considering (9) and (11), by the help of the hypothesis, assume that

$$e\bar{C}(U,V)X = M^*(U,V)X \quad (56)$$

where e is a reel constant and $e\neq 0$.

Taking into account of (9), (11) and (56), it follows that

$$R(U,V)X = \left(\frac{1}{4n}\right)\frac{1}{1-e}[-S(U,X)V + S(V,X)U - QVg(U,X) + QUg(V,X)] - \left(\frac{er}{2n(2n+1)}\right)\frac{1}{1-e}[-g(U,X)V + g(V,X)U]. \quad (57)$$

Now, taking contraction with respect to $U=E_i$ in (57) for the local pseudo φ-basis, we get

$$S(V,X) = \left(\frac{1}{4n}\right)\frac{1}{1-e}[(2n-1)S(V,X) - (2n+1)\,2n[\alpha^2+\xi(\alpha)]\varepsilon g(V,X)] - \left(\frac{er}{2n(2n+1)}\right)\frac{1}{1-e}[2ng(V,X)]. \quad (58)$$

Simplifying (58), thanks to $\xi(\alpha)=0$, we obtain

$$\left(-\frac{2n-1}{4n}+1-e\right)S(X,V) + \left(\frac{er}{2n+1}-\frac{r}{4n}\right)g(X,V) = 0. \quad (59)$$

Then putting $2n+1=t$ and $4nk=s$ in (59), the last equation reduces to

$$S(V,X) = \left(\frac{rt-rs}{t(t-s)}\right)g(V,X). \quad (60)$$

Hence, it is clear that

$$S = \left(\frac{r}{2n+1}\right)g$$

for $V,X\in\chi(M)$. Therefore, it completes the proof.

Theorem 5.3 Let M be a $(2n+1)$-dimensional α-Kenmotsu pseudo-metric manifold and α is parallel along the characteristic vector field ξ. If M-projectively curvature and conharmonic curvature tensors are linearly independent, then M is Einstein.

Proof. According to the hypothesis, using (10) and (11), let us denote

$$dH(U,V)X = M^*(U,V)X \quad (61)$$

where d is a reel constant and $d\neq 0$.

With the help of (10), (11) and (61), it yields

$$\left(\frac{1}{4n}-\frac{d}{2n-1}\right)\frac{1}{d-1}[g(X,V)QU-g(X,U)QV \quad (62)$$

$$+S(V,X)U-S(U,X)V]=R(V,U)X.$$

After the necessary arrangements, taking contraction with respect to $U=E_i$ in (62) for the local pseudo φ-basis, it gives

$$S(V,X)=\left(-\frac{cr}{-1+d+c(2n-1)}\right)g(V,X) \quad (63)$$

such that $c=\left(\frac{1}{4n}-\frac{d}{2n-1}\right)$. So (63) can be written as

$$S(V,X)=\left(\frac{r(2n-1-4nd)}{4n^2-1}\right)g(V,X) \quad (64)$$

which is the desired result.

Theorem 5.4 Let M be a $(2n+1)$-dimensional α-Kenmotsu pseudo-metric manifold and α is parallel along the characteristic vector field ξ. If M-projectively curvature and conformal curvature tensors are linearly independent, then M is an Einstein manifold.

Proof. In view of (8) and (11), suppose that

$$\lambda C(U,V)X=M^*(U,V)X \quad (65)$$

where λ is a reel constant and $\lambda\neq 0$.

Making use of (8), (11) and (65), we have

$$0=R(U,V)X+\left(\frac{1}{4n}-\frac{\lambda}{2n-1}\right)\frac{1}{\lambda-1}[-S(U,X)V+$$

$$S(V,X)U-g(U,X)QV+g(V,X)QU]$$

$$+\left(\frac{\lambda r}{2n(2n-1)}\right)\frac{1}{\lambda-1}[-g(U,X)V+g(V,X)U]. \quad (66)$$

The set of $\{E_1,E_2,\ldots,E_{2n},\xi\}$ is a local pseudo φ-basis of M for $i=1,2,\ldots,2n+1$. Putting $\xi(\alpha)=0$ and $Y=E_i$ in (66) and then taking contraction over the index i, (66) reduces to

$$S(V,X)=-\left(\frac{ar+2nb}{\lambda+a(2n-1)-1}\right)g(V,X) \quad (67)$$

such that $a=\left(\frac{1}{4n}-\frac{\lambda}{2n-1}\right)$ and $b=\left(\frac{\lambda r}{2n(2n-1)}\right)$.

Thus the proof is completed.

Example 5.1 Denoting the standart coordinates of $R^3(u,v,w)$ and considering the set of M is given by

$$M=\{(u,v,w)\in R^3, \qquad w\neq 0\}.$$

Then we choose the local pseudo φ-basis as follows:

$$E_1=e^{w^4}\left(\frac{\partial}{\partial u}\right)$$

$$E_2=e^{w^4}\left(\frac{\partial}{\partial v}\right)$$

$$E_3=\left(\frac{\partial}{\partial w}\right).$$

The pseudo-Riemannian metric tensor product is defined as

$$g=\left(e^{-2w^4}\right)(\varepsilon_1 du\otimes du+\varepsilon_2 dv\otimes dv)$$

$$+\varepsilon(dw\otimes dw)$$

Here $\varepsilon_i=g(E_i,E_i)$ for $i=1,2,3$.

Suppose that η is an 1-form given by

$$\eta(U)=\varepsilon g(U,E_3)$$

for any $U\in\chi(M)$ and φ is a (1,1)-type tensor defined by

$$\varphi(E_1)=E_2, \varphi(E_2)=-E_1, \varphi(E_3)=0.$$

Then using linearity of g and φ, we have (13) and

$$\varphi^2U=-U+\eta(U)E_3, \quad g(E_3,E_3)=\varepsilon,$$

Furthermore, the Levi-Civita connection ∇ gives

$$[E_1,E_3]=-4\varepsilon w^3E_1, \quad [E_2,E_3]=-4\varepsilon w^3E_2,$$

$$[E_1,E_2]=0.$$

Thus the structure (φ,ξ,η,g) has almost contact metric one and it yields

$$\Phi\left(\frac{\partial}{\partial u},\frac{\partial}{\partial v}\right)=-\Phi\left(\frac{\partial}{\partial v},\frac{\partial}{\partial u}\right)=-\varepsilon_i e^{-2w^4}$$

and

$$\Phi=-\varepsilon_i e^{-2w^4}(du\wedge dv).$$

Here it is noted that $\Phi(E_1,E_2)=-1$ and otherwise $\Phi(E_i,E_j)=0$ for $i\leq j$. So we get

$$d\Phi=8w^3\varepsilon_i e^{-2w^4}(du\wedge dv\wedge dw).$$

Since $\eta=dw$, we have

$$d\Phi=-8\varepsilon_i w^3(\eta\wedge\Phi).$$

Here α is defined by $\alpha(z)=-4\varepsilon_i w^3$. Also, since the Nijenhuis torsion tensor of φ is identically zero, then M is an α-Kenmotsu manifold.

6. Discussion and Conclusion

This study deals with the conformal, conharmanic, concircular, and M-projective curvature tensors, which are essential in Riemannian geometry. There are many physical applications of such curvature tensors.

In general relativity, Weyl tensor curvatures provide space-time when the Ricci tensor vanishes. The origin of the Ricci tensor consists of the energy-momentum of the local matter distribution. If the matter distribution vanishes, then the Ricci tensor will have vanished. Since the Weyl conformal curvature participates in curvature to the Riemannian curvature tensor, spacetime does not have to be flat in this case.

The conharmonic curvature tensor symbolizes the deviation of the manifold M from conharmonic flatness. This situation holds the whole symmetric properties of R. Pokhariyal and Mishra introduced the M-projective curvature tensor as in (11). This tensor indicates the deviation of the manifold from $M-$projective flatness. Also, the concircular geometry is concerned with concircular transformations. The concircular curvature tensor $\bar{C}$ symbolizes the deviation of the manifold M from the constant curvature.

Following this work, our aim in future studies is to investigate certain symmetric and curvature tensor conditions on almost α-Kenmotsu pseudo-metric manifolds using some special Einstein structures, D-homothetic deformation, nullity distribution, and certain Ricci solitons. In addition, it is among our priorities to examine their physical properties.

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