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Almost Kaehlerian and Hermitian Structures on Four Dimensional Indecomposable Lie Algebras

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Abstract

It is known that from a given almost Hermitian structure on a simply connected Lie group, one can obtain left-invariant almost Hermitian structure on its Lie algebra. In this work, we consider Mubarakzyanov's classification of four-dimensional real Lie algebras and evaluate the existence of almost Hermitian structures on four dimensional decomposable real Lie algebras. In particular, we focus on almost Kaehlerian and Hermitian structures on these Lie algebras.

1. Introduction

An almost Hermitian manifold is an even dimensional Riemannian manifold (M,g) together with an almost complex structure J, $(J^2=-Id)$ such that

$$g(JX,JY)=g(X,Y) \tag{1.1}$$

for all $X,Y\in\mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the set of smooth vector fields on M. The fundamental 2-form (or Kähler form) of an almost Hermitian manifold (M,g,J) is defined by

$$F(X,Y)=g(JX,Y), \tag{1.2}$$

for all $X,Y\in\mathfrak{X}(M)$. Also, the Nijenhuis tensor of M will be denoted by S, that is,

$$S(X,Y)=[X,Y]+J[JX,Y]+J[X,JY]-[JX,JY], \tag{1.3}$$

for $X,Y\in\mathfrak{X}(M)$. The covariant derivative ∇F of the Kähler form F, given with

$$(\nabla_X F)(Y,Z)=g((\nabla_X J)(Y),Z) \tag{1.4}$$

is a covariant tensor of degree 3 having the following symmetry properties [1]:

$$(\nabla_X F)(Y,Z)=-(\nabla_X F)(Z,Y)=-(\nabla_X F)(JY,JZ). \tag{1.5}$$

The space of those tensors possessing the same symmetries is a finite dimensional vector space, $\mathscr{W}$. Then $\mathscr{W}$ can be expressed as

$$\mathscr{W}=\{\alpha\in\otimes_3^0 T_pM \,|\, \alpha(X,Y,Z)=-\alpha(X,Z,Y)=-\alpha(X,JY,JZ)\} \tag{1.6}$$

for all $X,Y,Z\in\mathfrak{X}(M)$. In [1], almost Hermitian manifolds were classified depending on the space, $\mathscr{W}$, the covariant derivative of the fundamental 2-form belongs to. After writing the space $\mathscr{W}$ of tensors having the same properties as the covariant derivative of F, using the representation of the unitary group $U(n)$ on $\mathscr{W}$; $\mathscr{W}$ was written as a direct sum of four $U(n)$-irreducible subspaces. Thus there are 16 invariant subspaces of $\mathscr{W}$, each corresponding to a different class of almost Hermitian manifolds, as given in the following table:

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$\mathscr{K}$	$\nabla F = 0$
$\mathscr{W}_1 = \mathscr{N}\mathscr{K}$	$\nabla_X(F)(X,Y) = 0$ (or $3\nabla F = dF$)
$\mathscr{W}_2 = \mathscr{A}\mathscr{K}$	$dF = 0$
$\mathscr{W}_3 = \mathscr{S}\mathscr{K} \cap \mathscr{K}$	$\delta F = S = 0$ (or $\nabla_X(F)(Y,Z) - \nabla_{JX}(F)(JY,Z) = \delta F = 0$)
$\mathscr{W}_4$	$\nabla_X(F)(Y,Z) = \frac{-1}{2(n-1)}\{<X,Y> \delta F(Z) - <X,Z> \delta F(Y)$ $- <X,JY> \delta F(JZ) + <X,JZ> \delta F(JY)\}$
$\mathscr{W}_1 \oplus \mathscr{W}_2 = \mathscr{Q}\mathscr{K}$	$\nabla_X(F)(Y,Z) + \nabla_{JX}(F)(JY,Z) = 0$
$\mathscr{W}_3 \oplus \mathscr{W}_4 = \mathscr{H}$	$S = 0$ (or$\nabla_X(F)(Y,Z) - \nabla_{JX}(F)(JY,Z) = 0$)
$\mathscr{W}_1 \oplus \mathscr{W}_3$	$\nabla_X(F)(X,Y) - \nabla_{JX}(F)(JX,Y) = \delta F = 0$
$\mathscr{W}_2 \oplus \mathscr{W}_4$	$\mathfrak{S}\{\nabla_X(F)(Y,Z) - \frac{1}{n-1}F(X,Y)\delta F(JZ)\} = 0$
$\mathscr{W}_1 \oplus \mathscr{W}_4$	$\nabla_X(F)(X,Y) = \frac{-1}{2(n-1)}\{\|\|X\|\|^2 \delta F(Y) - <X,Y> \delta F(X)$ $- <JX,Z> \delta F(JX)\}$
$\mathscr{W}_2 \oplus \mathscr{W}_3$	$\mathfrak{S}\{\nabla_X(F)(Y,Z) - \nabla_{JX}(F)(JY,Z)\} = \delta F = 0$
$\mathscr{W}_1 \oplus \mathscr{W}_2 \oplus \mathscr{W}_3 = \mathscr{S}\mathscr{K}$	$\delta F = 0$
$\mathscr{W}_1 \oplus \mathscr{W}_2 \oplus \mathscr{W}_4$	$\nabla_X(F)(Y,Z) + \nabla_{JX}(F)(JY,Z) = \frac{-1}{n-1}\{<X,Y> \delta F(Z)$ $- <X,Z> \delta F(Y) - <X,JY> \delta F(JZ) + <X,JZ> \delta F(JY)\}$
$\mathscr{W}_1 \oplus \mathscr{W}_3 \oplus \mathscr{W}_4 = \mathscr{G}_1$	$\nabla_X(F)(X,Y) - \nabla_{JX}(F)(JX,Y) = 0$
$\mathscr{W}_2 \oplus \mathscr{W}_3 \oplus \mathscr{W}_4 = \mathscr{G}_2$	$\mathfrak{S}\{\nabla_X(F)(Y,Z) - \nabla_{JX}(F)(JY,Z)\} = 0$
$\mathscr{W}$	No condition

Table 1: Defining relations for classes of almost Hermitian manifolds [1]

For example, the class $\mathscr{K}$, in which the covariant derivative of F is zero, is the class of Kähler manifolds. $\mathscr{W}_1$ corresponds to the class of nearly Kähler manifolds, $\mathscr{W}_2$ to the class of almost Kähler manifolds, etc. [1]. For the case dimension 4, the classification is induced to four subclasses as given in the following table:

$\mathscr{K}$	$\nabla F = 0$
$\mathscr{W}_2 = \mathscr{A}\mathscr{K}$	$dF = 0$
$\mathscr{W}_4 = \mathscr{H}$	$S = 0$
$\mathscr{W}$	No condition

Table 2: Almost Hermitian Manifolds of dimension 4 [1]

Here, the exterior derivative dF is defined as:

$$dF(X,Y,Z) = \mathfrak{S}(\nabla_X F)(Y,Z), \tag{1.7}$$

where $X,Y,Z \in \mathfrak{X}(M)$. One can see [2, 3] for more details. In the literature, there are many studies such as [4–6] that consider (almost)(para)contact structures on certain Lie algebras. In this work, by following a similar path to these studies, we will consider the classification as given in Table 2 since we focus the four dimensional almost Hermitian manifolds.

2. Four Dimensional Indecomposable Real Lie Algebras

Let G be a connected Lie group and $\mathfrak{g}$ be its Lie algebra. The almost Hermitian structures on G that we consider are invariant in the sense that the tensors g,J,F are left invariant tensors. By restricting the structures element to the left-invariant vector fields, we can directly obtain an almost Hermitian structure on the Lie algebra $\mathfrak{g}$, that will be denoted (g,J) again for convenience. In [7] and [8], four dimensional real Lie algebras are classified and with respect to this classification the decomposable Lie algebras are defined as follows with non zero commutators where $\{e_1,e_2,e_3,e_4\}$ is an orthonormal basis:

$$\begin{aligned}
\mathfrak{g}_{2,1} \oplus 2\mathfrak{g}_1 &: [e_1,e_2] = e_1, \\
2\mathfrak{g}_{2,1} &: [e_1,e_2] = e_1, [e_3,e_4] = e_3, \\
\mathfrak{g}_{3,1} \oplus \mathfrak{g}_1 &: [e_2,e_3] = e_1, \\
\mathfrak{g}_{3,2} \oplus \mathfrak{g}_1 &: [e_1,e_3] = e_1,\ [e_2,e_3] = e_1 + e_2, \\
\mathfrak{g}_{3,3} \oplus \mathfrak{g}_1 &: [e_1,e_3] = e_1,\ [e_2,e_3] = e_2, \\
\mathfrak{g}_{3,4} \oplus \mathfrak{g}_1 &: [e_1,e_3] = e_1,\ [e_2,e_3] = \alpha e_2, \quad (-1 \leq \alpha < 1,\ \alpha \neq 0) \\
\mathfrak{g}_{3,5} \oplus \mathfrak{g}_1 &: [e_1,e_3] = \beta e_1 - e_2,\ [e_2,e_3] = e_1 + \beta e_2, \quad (\beta \geq 0)
\end{aligned}$$

If (g,J) is an almost Hermitian structure on a 4-dimensional Lie algebra $\mathfrak{g}$, then by the conditions $J^2 = -1$ and (1.1), the structure J has the form:

$$J(e_1) = ae_2 + be_3 + ce_4, \quad J(e_2) = -ae_1 + de_3 + ee_4, \quad J(e_3) = -be_1 - de_2 + fe_4, \quad J(e_4) = -ce_1 - ee_2 - fe_3, \tag{2.1}$$

where $a,b,c,d,e,f \in \mathbb{R}$ satisfy:

$$\begin{aligned}&a^2+b^2+c^2=1,\ a^2+d^2+e^2=1,\ b^2+d^2+f^2=1,\ c^2+e^2+f^2=1,\\&bd+ce=0,\ ad-cf=0,\quad ae+bf=0,\quad ab+ef=0,\quad df-ac=0,\quad bc+de=0.\end{aligned} \tag{2.2}$$

Also the derivative of the fundamental 2-form F, dF becomes:

$$dF(e_i,e_j,e_k)=-F([e_i,e_j],e_k)-F([e_j,e_k],e_i)-F([e_k,e_i],e_j). \tag{2.3}$$

Now, we study the existence of almost Kaehlerian and Hermitian structures on this algebras:
The algebra $\mathfrak{g}_{2,1}\oplus 2\mathfrak{g}_1$:
Let $F=\sum_{i,j}a_{ij}e^{ij}$ be a $(0,2)$-tensor with $dF=0$ in $\mathfrak{g}_{2,1}\oplus 2\mathfrak{g}_1$. By direct calculation, it can be seen that

$$dF(e_1,e_2,e_3)=dF(e_1,e_2,e_4)=0.$$

So, F has the form

$$F=a_{12}e^{12}+a_{23}e^{23}+a_{24}e^{24}+a_{34}e^{34}.$$

On the other other, by considering the equations (2.2), one can see that the structure $J(e_1)=e_2,\ J(e_2)=-e_1,\ J(e_3)=e_4,\ J(e_4)=-e_3$ has the 2- form F, as $F=e^{12}+e^{34}$ $(a_{12}=a_{34}=1,\ a_{23}=a_{24}=0)$, for which $dF=0$. Thus, **there exists an almost Kaehlerian structure in the class $\mathfrak{g}_{2,1}\oplus 2\mathfrak{g}_1$.**
Now, we evaluate the existence of Hermitian structures on the algebra $\mathfrak{g}_{2,1}\oplus 2\mathfrak{g}_1$. Let the tensor S given with (1.3) vanishes. Then considering the equations $S(e_i,e_j)=0$, it can be seen that the almost Hermitian structures with $J(e_1)=ae_2$, $J(e_2)=-ae_1, J(e_3)=fe_4$, $J(e_4)=-fe_3$, with $a^2=f^2=1$ have Nijenhus tensors, that vanishes. Thus, **there exists Hermitian structures in the class $\mathfrak{g}_{2,1}\oplus 2\mathfrak{g}_1$.**

The algebra $2\mathfrak{g}_{2,1}$:

Let $F=\sum_{i,j}a_{ij}e^{ij}$ be a $(0,2)$-tensor with $dF=0$ in the algebra $2\mathfrak{g}_{2,1}$. Then, by (2.3), the equations

$$dF(e_1,e_2,e_3)=dF(e_1,e_2,e_4)=dF(e_2,e_3,e_4)=0$$

imply $a_{13}=a_{14}=a_{23}=0$, repsectively. Thus F becomes

$$F=a_{12}e^{12}+a_{24}e^{24}+a_{34}e^{34}.$$

It can be seen that the structure $J(e_1)=e_2$, $J(e_2)=-e_1$, $J(e_3)=e_4$, $J(e_4)=-e_3$ has the fundamental 2-form $F=e^{12}+e^{34}$ $(a_{12}=a_{34}=1,\ a_{24}=0$, for which $dF=0$. Thus, **there exists an almost Kaehlerian structure in the class $2\mathfrak{g}_{2,1}$.**
Assume the tensor S given with (1.3) vanishes. By the equations $S(e_1,e_2)=S(e_1,e_3)=S(e_1,e_4)=0$, we get $a^2=f^2=1$, $b=c=d=e=0$ in (2.2). Hence, we get the structures $J(e_1)=ae_2$, $J(e_2)=-ae_1$, $J(e_3)=fe_4$, $J(e_4)=-fe_3$, for which the Nijenhuis tensors vanish. Hence, **there exists Hermitian structures in the algebra $2\mathfrak{g}_{2,1}$.**

The algebra $\mathfrak{g}_{3,1}\oplus\mathfrak{g}_1$:

For the $(0,2)$tensor $F=\sum_{i,j}a_{ij}e^{ij}$ in the algebra $\mathfrak{g}_{3,1}\oplus\mathfrak{g}_1$, the equation $dF(e_2,e_3,e_4)=0$ implies $a_{14}=0$. Thus F has the form:

$$F=a_{12}e^{12}+a_{13}e^{13}+a_{23}e^{23}+a_{24}e^{24}+a_{34}e^{34}.$$

On the other hand, by considering the definitions of J and F, we see that the almost Hermitian structure given with $J(e_1)=e_3$, $J(e_2)=e_4$, $J(e_3)=-e_1$, $J(e_4)=-e_2$ has the fundamental 2-form

$$F=e^{13}+e^{24}\ (a_{13}=a_{24}=1,\ a_{12}=a_{23}=a_{34}=0).$$

Since $dF=0$, **there exists an almost Kaehlerian structure in the algebra $\mathfrak{g}_{3,1}\oplus\mathfrak{g}_1$.**
Let the tensor S given with (1.3) vanishes in the algebra $\mathfrak{g}_{3,1}\oplus\mathfrak{g}_1$. Then from the equations $S(e_1,e_2)=S(e_1,e_3)=S(e_2,e_3)=0$, and (2.2), we get $c^2=d^2=1$, $a=b=e=f=0$. Thus the almost Hermitian structures given with $J(e_1)=ce_4$, $J(e_2)=de_3$, $J(e_3)=-de_2$, $J(e_4)=-ce_1$, have Nijenhuis tensors S, with $S=0$. So, **there exists Hermitian structures in the algebra $\mathfrak{g}_{3,1}\oplus\mathfrak{g}_1$.**

The algebra $\mathfrak{g}_{3,2}\oplus\mathfrak{g}_1$:

Let $F=\sum_{i,j}a_{ij}e^{ij}$ be a $(0,2)$ tensor in the algebra $\mathfrak{g}_{3,2}\oplus\mathfrak{g}_1$. From the equations

$$dF(e_1,e_2,e_3)=dF(e1,e3,e4)=dF(e_2,e_3,e_4)=0,$$

we get $a_{12}=a_{14}=a_{24}=0$, respectively. Thus, the tensor F has the form

$$F=a_{13}e^{13}+a_{23}e^{23}+a_{34}e^{34}. \tag{2.4}$$

Also, by considering the definitions of J and the fundamental two form F, it can be seen that there exist no almost Hermitian structure with the fundamental 2-form given with (2.4). Thus, **there is no almost Kaehlerian structure on the algebra $\mathfrak{g}_{3,2}\oplus\mathfrak{g}_1$.**

Assume S is a tensor on the algebra $\mathfrak{g}_{3,2} \oplus \mathfrak{g}_1$, given with (1.3) and $S = 0$. By considering $S(e_1,e_2) = S(e_2,e_3) = 0$, $S(e_1,e_3) = 0$ and the equations (2.2), we get $a^2 = 0$ and $a^2 = 1$. By this contradiction, **there is no Hermitian structure on the algebra** $\mathfrak{g}_{3,2} \oplus \mathfrak{g}_1$.

The algebra $\mathfrak{g}_{3,3} \oplus \mathfrak{g}_1$:

For the tensor $F = \sum_{i,j} a_{ij} e^{ij} = 0$ in $\mathfrak{g}_{3,3} \oplus \mathfrak{g}_1$, we get $a_{12} = a_{14} = a_{24} = 0$, from the equations

$$dF(e_1,e_2,e_3) = dF(e_1,e_3,e_4) = dF(e_2,e_3,e_4) = 0,$$

respectively. So, F has the form

$$F = a_{13}e^{13} + a_{23}e^{23} + a_{34}e^{34}. \tag{2.5}$$

However, it can be seen that there is no almost Hermitian structure with fundamental 2-form of the form (2.5). Thus, **there is no almost Kaehlerian structure in** $\mathfrak{g}_{3,3} \oplus \mathfrak{g}_1$.

Let S be a tensor in $\mathfrak{g}_{3,3} \oplus \mathfrak{g}_1$ defined with (1.3) and $S = 0$. From $S(e_2,e_3) = 0$, we get $a^2 = 1$. So, by considering the equations (2.2), we get $b = c = d = e = 0$, that implies $f^2 = 1$. Thus, the structure with

$$J(e_1) = ae_2, \quad J(e_2) = -ae_1, \quad J(e_3) = fe_4, \quad J(e_4) = -fe_3,$$

with $a^2 = f^2 = 1$, has Nijenhuis tensor S, that vanishes. Thus, **there exist Hermitian structures on $\mathfrak{g}_{3,3} \oplus \mathfrak{g}_1$.**

The algebra $\mathfrak{g}_{3,4} \oplus \mathfrak{g}_1$:
For the tensor $F = \sum_{i,j} a_{ij} e^{ij} = 0$ in $\mathfrak{g}_{3,4} \oplus \mathfrak{g}_1$, we get $a_{12} = a_{24} = 0$, from the equations $dF(e_1,e_2,e_3) = dF(e_2,e_3,e_4) = 0$, respectively. Thus, F has the form,

$$F = a_{13}e^{13} + a_{14}e^{14} + a_{23}e^{23} + a_{34}e^{34}. \tag{2.6}$$

So, by considering the defining conditions of an almost Hermitian structure, it can be seen that the structure given with $J(e_1) = e_4$, $J(e_2) = e_3$, $J(e_3) = -e_2$, $J(e_4) = -e_1$ has fundamental 2-form

$$F = e^{14} + e^{23}, \quad (a_{14} = a_{23} = 1, \quad a_{13} = a_{34} = 0).$$

Thus, **there exists almost Kaehlerian structure in $\mathfrak{g}_{3,4} \oplus \mathfrak{g}_1$.**
On the other hand, $\mathfrak{g}_{3,4} \oplus \mathfrak{g}_1$ agrees with a Hermitian structure. Indeed, it can be seen that the structure $J(e_1) = ae_2$, $J(e_2) = -ae_1$, $J(e_3) = fe_4$, $J(e_4) = -fe_3$ with $a^2 = f^2 = 1$ have Nijenhuis tensor S, with $S = 0$. Thus, **there exist Hermitian structures in** $\mathfrak{g}_{3,4} \oplus \mathfrak{g}_1$.

The algebra $\mathfrak{g}_{3,5} \oplus \mathfrak{g}_1$:
Let $F = \sum_{i,j} a_{ij} e^{ij}$ be a $(0,2)$ tensor in the algebra $\mathfrak{g}_{3,5} \oplus \mathfrak{g}_1$. From the equations $dF(e_1,e_2,e_3) = 0$, $dF(e_1,e_3,e_4) = 0$ and $dF(e_2,e_3,e_4) = 0$, we get $\beta a_{12} = 0$, $\beta a_{14} = a_{24}$ and $\beta a_{24} = -a_{14}$, respectively. However, this imply $\beta^2 = -1$, which is contradiction since β is real number. So, **there is no almost Kaehlerian structure in** $\mathfrak{g}_{3,5} \oplus \mathfrak{g}_1$.
Assume, the tensor S given with (1.3) in the algebra $\mathfrak{g}_{3,5} \oplus \mathfrak{g}_1$ vanishes. After long but direct calculations of $S(e_i,e_j)$'s , one can see that the structures of the form

$$J(e_1) = ae_2, \quad J(e_2) = -ae_1, \quad J(e_3) = f(e_4), \quad J(e_4) = -fe_3,$$

for $a^2 = f^2 = 1$, have Nijenhuis tensor that vanishes $(\beta \neq 0)$. Thus, There exist Hermitian structures in the algebra $\mathfrak{g}_{3,5} \oplus \mathfrak{g}_1$.

3. Conclusion

In this paper, the existences of Hermitian structures and almost Kaehlerian structures on four dimensional indecomposable real Lie algebras are investigated and so, the possible structures are stated as examples.

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Competing interests

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Authors contributions

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