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DIFFERENTIAL EQUATION SOLVER SIMULATOR FOR RUNGE-KUTTA METHODS

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Abstract: Many of problems in engineering and science is modeled by differential equations mathematically, therefore their solutions have an important role. Various methods have been developed for analytical or numerical solutions of differential equations. In proportion to the development of technology, the numerical solution methods are utilized widely. In particular, the main objectives in real time applications are to reach the correct solution as soon as possible with minimal processing and maximum precision. In the performed study, a simulator that contains Runge-Kutta based forty-eight methods was developed for numerical solution of differential equations. In the user friendly simulator which can be used also for educational purposes, the solution of defined differential equation under the specified initial condition with given step size or according to the number of points requested within the specified range can be obtained by the selected method. Solutions can be presented to the user both numerical (step values, computation time) and graphically; also the subject explanations about the methods/solutions can be given. Furthermore, the comparative solutions (performance analysis) can be implemented by the simulator. So, the users can realize the numerical solutions of differential equations with different methods by the simulator; the students learn the methods in this field visually with the aid of subject explanation and can implement step by step; the designers can choose the most appropriate method easily, effectively and accurately for their systems by the comparative analysis.

Keywords: Ordinary differential equation, Runge-Kutta methods, Simulator

Runge-Kutta Yöntemleri İçin Diferansiyel Denklem Çözüm Simülatörü

Öz: Mühendislik ve fen bilimlerine ait problemlerinin birçoğu diferansiyel denklemlerle matematiksel olarak modellenmekte, dolayısıyla çözümleri önemli yer tutmaktadır. Diferansiyel denklemlerin analitik veya sayısal çözümleri için çok sayıda yöntemler geliştirilmiştir. Teknolojinin gelişmesiyle orantılı olarak sayısal çözüm yöntemlerinden yoğun bir şekilde faydalanılmaktadır. Özellikle gerçek zamanlı uygulamalarda en az işlemle, en kısa sürede, en yüksek hassasiyetle, en doğru çözüme ulaşmak başlıca hedeflerdendir. Gerçekleştirilen çalışmada, diferansiyel denklemlerin sayısal çözümü için Runge-Kutta tabanlı 48 yöntemi içeren bir simülatör geliştirilmiştir. Kullanıcı dostu ve eğitim amaçlı da kullanılabilecek simülatörde tanımlanan diferansiyel denklemin, belirtilen başlangıç koşulu altında, verilen adım boyutu veya nokta sayısına göre, istenen aralıktaki çözümü seçilen yöntemle elde edilebilmektedir. Çözümler kullanıcıya hem sayısal (adım değerleri, hesaplama süresi vb.) hem de grafiksel olarak sunulabilmekte; bunun yanında yöntemlerle/çözümlerle ilgili konu anlatımları da verilebilmektedir. Ayrıca simülatörle karşılaştırmalı çözümler (performans analizleri) de gerçekleştirilebilmekte. Böylece simülatör ile kullanıcılar farklı yöntemlerle diferansiyel denklemlerin sayısal çözümlerini gerçekleştirebilmekte; öğrenciler bu alandaki yöntemleri konu anlatımı destekli görsel olarak öğrenip adım adım uygulayabilmekte; tasarımcılar da karşılaştırmalı analizlerle sistemleri için en uygun yöntemi kolay, etkin ve doğru bir şekilde seçebilmektedirler.

Anahtar Kelimeler: Adi diferansiyel denklem, Runge-Kutta yöntemleri, Simülatör

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1. INTRODUCTION

Many physical systems or processes in nature can be modeled mathematically with differential equations. Therefore, the analytical or numerical solutions of related equations have an important role in basic and applied sciences. With development of digital electronics and computer technology, the use of numerical solutions of differential equations, especially in real time applications has increased. Thus, studies continue about the methods have less number and type of operation, use less memory space, obtain high accuracy solutions in short time. One of the main issues in numerical analysis in the field of engineering and science education is the solution of differential equations.

Some software such as MATLAB, Maple, Mathcad, Mathematica, Scilab developed for general-purposes, contains many numerical analysis methods (18-19). In addition to these, to be able to make computations by numerical methods, web pages (20-25), mobile applications (26-29), course/lesson pages (30-32) have been developed for educational purposes.

In this study; a simulator is designed which can perform the numerical solutions of differential equations by Runge-Kutta based 48 different methods, present subject explanations about these methods, and provides performance analyses comparatively. By the user friendly simulator which can be used for education and application purposes, the numerical solutions of desired differential equations can be performed by selected methods under the specified initial conditions, defined range and step size or number of points, and the results can be seen both numerical and graphically.

2. RUNGE-KUTTA METHODS

A general first order ordinary differential equation (ODE) system under the initial condition $y_0 = y(x_0)$ can be given as

$$y'(x) = f(x, y(x)) \tag{1}$$

To obtain a second order Runge-Kutta method, Taylor series expansion can be used as

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2}y''(x) + \mathcal{O}(h^3) \tag{2}$$

and the equations

$$\left.\begin{aligned} y'(x) &\cong f(x,y) \\ y''(x) &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}.\frac{\partial y}{\partial x} = f_x + f_y.y' = f_x(x,y) + f_y(x,y).f(x,y) \end{aligned}\right\} \tag{3}$$

can be substituted in Eq. 2, as follows.

$$\left.\begin{aligned} y(x+h) &= y(x) + hf(x,y) + \frac{h^2}{2}\left[f_x(x,y) + f_y(x,y).f(x,y)\right] + \mathcal{O}(h^3) \\ y(x+h) &= y(x) + \frac{h}{2}f(x,y) + \frac{h}{2}\left[f(x,y) + hf_x(x,y) + hf_y(x,y)f(x,y)\right] + \mathcal{O}(h^3) \end{aligned}\right\} \tag{4}$$

If the multivariable Taylor expansion is used as

$$f(x+h, y+k) = f(x,y) + hf_x(x,y) + f_y(x,y)k + \cdots \tag{5}$$

then, the statement in the parenthesis of Eq. 4 can be written as follows.

$$f(x+h, y+f(x,y)) = f(x,y) + hf_x(x,y) + hf_y(x,y)f(x,y) + O(h^2) \tag{6}$$

Thus, the following equation

$$y(x+h) = y(x) + \frac{h}{2}f(x,y) + \frac{h}{2}f(x+h, y+hf(x,y)) + O(h^3) \tag{7}$$

or equivalently, the following classical second order Runge-Kutta method is obtained as a numerical method.

$$\left.\begin{array}{r} k_1 = f(x_i, y_i) \\ k_2 = f(x_i+h, y_i+hk_1) \end{array}\right\} \Rightarrow y_{i+1} = y_i + \frac{h}{2}(k_1+k_2) \tag{8}$$

By the same viewpoint, the Euler method is written as

$$y_{i+1} = y_i + h.f(x_i, y_i) \tag{9}$$

and also known as first order Runge-Kutta method (Vatansever, 2006). Many different methods have been developed with similar steps. The used methods with their iteration equations in the designed simulator are given in Table 1-5.

Table 1. Second order Runge-Kutta methods

Modified Euler (Midpoint integration) method (Chapra and Canale, 2002)	Improved Euler (Trapezoidal integration) (Heun) method (Chapra and Canale, 2002)	Ralston's 2nd order method (Ralston, 1962; Ralston and Rabinowitz, 1978)
$k_1 = f(x_i, y_i)$ $k_2 = f\left(x_i + \frac{1}{2}h,\ y_i + \frac{1}{2}k_1h\right)$ $y_{i+1} = y_i + k_2h$	$k_1 = f(x_i, y_i)$ $k_2 = f(x_i + h,\ y_i + k_1h)$ $y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$	$k_1 = f(x_i, y_i)$ $k_2 = f\left(x_i + \frac{3}{4}h,\ y_i + \frac{3}{4}k_1h\right)$ $y_{i+1} = y_i + \left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right)h$
RK2: Kopal (Kopal, 1955; Ralston, 1962)	**2nd order contraharmonic mean method (Ababneh and Rozita, 2009)**	
$k_1 = f(x_i, y_i)$ $k_2 = f\left(x_i + \frac{2}{3}h,\ y_i + \frac{2}{3}k_1h\right)$ $y_{i+1} = y_i + \left(\frac{1}{4}k_1 + \frac{3}{4}k_2\right)h$	$k_1 = f(x_i, y_i)$ $k_2 = f(x_i + h,\ y_i + k_1h)$ $y_{i+1} = y_i + \left(\frac{k_1^2 + k_2^2}{k_1 + k_2}\right)h$	

Table 2. Third order Runge-Kutta methods

3rd order Runge-Kutta method-1 (Ralston and Rabinowitz, 1978)	3rd order Runge-Kutta method-2
$k_1 = f(x_i, y_i)$ $k_2 = f\left(x_i + \frac{1}{2}h,\ y_i + \frac{1}{2}k_1h\right)$ $k_3 = f\left(x_i + h,\ y_i - k_1h + 2k_2h\right)$ $y_{i+1} = y_i + \frac{1}{6}(k_1 + 4k_2 + k_3)h$	$k_1 = f(x_i, y_i)$ $k_2 = f(x_i + h,\ y_i + k_1h)$ $k_3 = f\left(x_i + \frac{1}{2}h,\ y_i + \frac{1}{4}(k_1 + k_2)h\right)$ $y_{i+1} = y_i + \frac{1}{6}(k_1 + k_2 + 4k_3)h$
Heun's 3rd order Runge-Kutta method (Butcher, 2000)	**Ralston's 3rd order Runge-Kutta method (Ralston, 1962)**

$k_1 = f(x_i, y_i)$ $k_2 = f\left(x_i + \frac{1}{3}h,\ y_i + \frac{1}{3}k_1 h\right)$ $k_3 = f\left(x_i + \frac{2}{3}h,\ y_i + \frac{2}{3}k_2 h\right)$ $y_{i+1} = y_i + \frac{1}{4}(k_1 + 3k_3)h$	$k_1 = f(x_i, y_i)$ $k_2 = f\left(x_i + \frac{1}{2}h,\ y_i + \frac{1}{2}k_1 h\right)$ $k_3 = f\left(x_i + \frac{3}{4}h,\ y_i + \frac{3}{4}k_2 h\right)$ $y_{i+1} = y_i + \frac{1}{9}(2k_1 + 3k_2 + 4k_3)h$
3rd order harmonic mean method (Wazwaz, 1990)	***3rd order contraharmonic mean method (Ababneh and Rozita, 2009)***
$k_1 = f(x_i, y_i)$ $k_2 = f\left(x_i + \frac{2}{3}h,\ y_i + \frac{2}{3}k_1 h\right)$ $k_3 = f\left(x_i + \frac{2}{3}h,\ y_i - \frac{2}{3}k_1 h + \frac{4}{3}k_2 h\right)$ $y_{i+1} = y_i + \frac{1}{2}\left(\frac{2k_1k_2}{k_1+k_2} + \frac{2k_2k_3}{k_2+k_3}\right)h$	$k_1 = f(x_i, y_i)$ $k_2 = f\left(x_i + \frac{2}{3}h,\ y_i + \frac{2}{3}k_1 h\right)$ $k_3 = f\left(x_i + \frac{2}{3}h,\ y_i + \frac{2}{3}k_2 h\right)$ $y_{i+1} = y_i + \frac{1}{2}\left(\frac{k_1^2+k_2^2}{k_1+k_2} + \frac{k_2^2+k_3^2}{k_2+k_3}\right)h$
3rd order modified contraharmonic mean weights method (Ababneh and Rozita, 2009)	***3rd order arithmetic mean Runge-Kutta method (Wazwaz, 1994; Ahmad and Yaacob, 2013)***
$k_1 = f(x_i, y_i)$ $k_2 = f\left(x_i + \frac{2}{3}h,\ y_i + \frac{2}{3}k_1 h\right)$ $k_3 = f\left(x_i + \frac{4}{21}(3+\sqrt{2})h,\ y_i + \frac{4}{21}(3+\sqrt{2})k_2 h\right)$ $y_{i+1} = y_i + \left(\frac{1}{4}\frac{k_1^2+k_2^2}{k_1+k_2} + \frac{3}{4}\frac{k_2^2+k_3^2}{k_2+k_3}\right)h$	$k_1 = f(x_i, y_i)$ $k_2 = f\left(x_i + \frac{2}{3}h,\ y_i + \frac{2}{3}k_1 h\right)$ $k_3 = f\left(x_i + \frac{2}{3}h,\ y_i - \frac{1}{3}k_1 h + k_2 h\right)$ $y_{i+1} = y_i + \frac{1}{2}\left(\frac{k_1+k_2}{2} + \frac{k_2+k_3}{2}\right)h$
New 3rd order arithmetic mean Runge-Kutta method (Ahmad and Yaacob, 2013)	***Composite arithmetic–harmonic mean Runge–Kutta method (Ahmad and Yaacob, 2005)***
$k_1 = f(x_i, y_i)$ $k_2 = f\left(x_i + \frac{2-\sqrt{2}}{3}h,\ y_i + \frac{2-\sqrt{2}}{3}k_1 h\right)$ $k_3 = f\left(x_i + \frac{1}{3}h,\ y_i + \frac{1}{3}\frac{k_1+k_2}{2}h\right)$ $y_{i+1} = y_i + \left(\frac{4+3\sqrt{2}}{4}k_1 + \frac{-3(4+3\sqrt{2})}{4}k_2 + \frac{3(2+\sqrt{2})}{2}k_3\right)h$	$k_1 = f(x_i, y_i)$ $k_2 = f\left(x_i + \frac{3}{5}h,\ y_i + \frac{3}{5}k_1 h\right)$ $k_3 = f\left(x_i + \frac{4}{5}h,\ y_i + \frac{4}{5}(\frac{k_1+k_2}{2})h\right)$ $y_{i+1} = y_i + \frac{1}{2}\left(\frac{2k_1k_2}{k_1+k_2} + \frac{2k_2k_3}{k_2+k_3}\right)h$

Table 3. Fourth order Runge-Kutta methods

Classical 4th order Runge-Kutta (arithmetic mean) method (Chapra and Canale, 2002)	***Ralston's 4th order method (Ralston, 1962)***
$k_1 = f(x_i, y_i)$ $k_2 = f\left(x_i + \frac{1}{2}h,\ y_i + \frac{1}{2}k_1 h\right)$ $k_3 = f\left(x_i + \frac{1}{2}h,\ y_i + \frac{1}{2}k_2 h\right)$ $k_4 = f(x_i + h,\ y_i + k_3 h)$ $y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$ $= y_i + \frac{1}{3}\left(\frac{k_1+k_2}{2} + \frac{k_2+k_3}{2} + \frac{k_3+k_4}{2}\right)h$	$k_1 = f(x_i, y_i)$ $k_2 = f\left(x_i + \frac{2}{5}h,\ y_i + \frac{2}{5}k_1 h\right)$ $k_3 = f\left(x_i + \frac{3}{5}h,\ y_i - \frac{3}{20}k_1 h + \frac{3}{4}k_2 h\right)$ $k_4 = f\left(x_i + h,\ y_i + \frac{1}{44}(19k_1 - 15k_2 + 40k_3)h\right)$ $y_{i+1} = y_i + \frac{1}{72}(11k_1 + 25k_2 + 25k_3 + 11k_4)h$
3/8 Runge-Kutta method (Derr, Outlaw, Sarafyan, 1993)	***4th order harmonic mean method (Wazwaz, 1991)***

$k_1 = f(x_i, y_i)$ $k_2 = f\left(x_i + \frac{1}{3}h,\ y_i + \frac{1}{3}k_1 h\right)$ $k_3 = f\left(x_i + \frac{2}{3}h,\ y_i - \frac{3}{4}k_1 h + k_2 h\right)$ $k_4 = f\left(x_i + h,\ y_i + (k_1 - k_2 + k_3)h\right)$ $y_{i+1} = y_i + \frac{1}{8}\left(k_1 + 3k_2 + 3k_3 + k_4\right)h$	$k_1 = f(x_i, y_i)$ $k_2 = f\left(x_i + \frac{1}{2}h,\ y_i + \frac{1}{2}k_1 h\right)$ $k_3 = f\left(x_i + \frac{3}{8}h,\ y_i + \frac{1}{8}(-k_1 + 5k_2)h\right)$ $k_4 = f\left(x_i + h,\ y_i + \frac{1}{20}(-5k_1 + 7k_2 + 18k_3)h\right)$ $y_{i+1} = y_i + \frac{1}{3}\left(\frac{2k_1k_2}{k_1 + k_2} + \frac{2k_2k_3}{k_2 + k_3} + \frac{2k_3k_4}{k_3 + k_4}\right)h$
4th order Runge-Kutta method with minimum error bound (Ralston, 1962)	***4th order contraharmonic mean Runge-Kutta method (Ababneh and Rozita, 2009)***
$k_1 = f(x_i, y_i)$ $k_2 = f(x_i + 0.4h,\ y_i + 0.4k_1 h)$ $k_3 = f\left(x_i + 0.45573725h,\ y_i + (0.29697761k_1 + 0.15875964k_2)h\right)$ $k_4 = f\left(x_i + h,\ y_i + (0.21810040k_1 - 3.05096516k_2 + 3.83286476k_3)h\right)$ $y_{i+1} = y_i + (0.17476028k_1 - 0.55148066k_2 + 1.20553560k_3$ $+ 0.17118478k_4)h$	$k_1 = f(x_i, y_i)$ $k_2 = f\left(x_i + \frac{1}{2}h,\ y_i + \frac{1}{2}k_1 h\right)$ $k_3 = f\left(x_i + \frac{5}{8}h,\ y_i + \frac{1}{8}k_1 h + \frac{3}{8}k_2 h\right)$ $k_4 = f\left(x_i + h,\ y_i + \frac{1}{4}k_1 h - \frac{3}{4}k_2 h + \frac{3}{2}k_3 h\right)$ $y_{i+1} = y_i + \frac{1}{3}\left(\frac{k_1^2 + k_2^2}{k_1 + k_2} + \frac{k_2^2 + k_3^2}{k_2 + k_3} + \frac{k_3^2 + k_4^2}{k_3 + k_4}\right)h$
4th order heronian mean mehtod (Murugesan et. al., 2001)	***RK4: Butcher-Johnston (Butcher and Johnston, 1993)***
$k_1 = f(x_i, y_i)$ $k_2 = f\left(x_i + \frac{1}{2}h,\ y_i + \frac{1}{2}k_1 h\right)$ $k_3 = f\left(x_i + \frac{23}{48}h,\ y_i - \frac{1}{48}k_1 h + \frac{25}{48}k_2 h\right)$ $k_4 = f\left(x_i + h,\ y_i - \frac{1}{24}k_1 h + \frac{47}{600}k_2 h + \frac{289}{300}k_3 h\right)$ $y_{i+1} = y_i + \frac{1}{9}\left(k_1 + 2k_2 + 2k_3 + k_4 + \sqrt{\lvert k_1k_2 \rvert} + \sqrt{\lvert k_2k_3 \rvert} + \sqrt{\lvert k_3k_4 \rvert}\right)h$	$k_1 = f(x_i, y_i)$ $k_2 = f\left(x_i + \frac{3}{4}h,\ y_i + \frac{3}{4}k_1 h\right)$ $k_3 = f\left(x_i + \frac{1}{2}h,\ y_i + \frac{1}{3}k_1 h + \frac{1}{6}k_2 h\right)$ $k_4 = f\left(x_i + h,\ y_i - \frac{1}{3}k_1 h - \frac{2}{3}k_2 h + 2k_3 h\right)$ $y_{i+1} = y_i + \frac{1}{6}\left(k_1 + 4k_3 + k_4\right)h$
4th order centroidal mean method (Murugesan et. al., 2001)	***RK4: England (England, 1969)***
$k_1 = f(x_i, y_i)$ $k_2 = f(x_i + \frac{1}{2}h,\ y_i + \frac{1}{2}k_1 h)$ $k_3 = f(x_i + \frac{13}{24}h,\ y_i + \frac{1}{24}k_1 h + \frac{11}{24}k_2 h)$ $k_4 = f(x_i + h,\ y_i + \frac{1}{12}k_1 h - \frac{25}{132}k_2 h + \frac{73}{66}k_3 h)$ $y_{i+1} = y_i + \frac{2}{9}\left(\frac{k_1^2 + k_1k_2 + k_2^2}{k_1 + k_2} + \frac{k_2^2 + k_2k_3 + k_3^2}{k_2 + k_3} + \frac{k_3^2 + k_3k_4 + k_4^2}{k_3 + k_4}\right)h$	$k_1 = f(x_i, y_i)$ $k_2 = f\left(x_i + \frac{1}{2}h,\ y_i + \frac{1}{2}k_1 h\right)$ $k_3 = f\left(x_i + \frac{1}{4}h,\ y_i + \frac{1}{4}k_2 h\right)$ $k_4 = f\left(x_i + h,\ y_i - k_2 h + 2k_3 h\right)$ $y_{i+1} = y_i + \frac{1}{6}\left(k_1 + 4k_3 + k_4\right)h$
RK4: Gill (Gill, 1951)	
$k_1 = f(x_i, y_i)$ $k_2 = f\left(x_i + \frac{1}{2}h,\ y_i + \frac{1}{2}k_1 h\right)$ $k_3 = f\left(x_i + \frac{1}{2}h,\ y_i + \left[-\frac{1}{2} \pm \sqrt{\frac{1}{2}}\right]k_1 h + \left[1 \mp \sqrt{\frac{1}{2}}\right]k_2 h\right)$ $k_4 = f\left(x_i + h,\ y_i \mp \sqrt{\frac{1}{2}}k_2 h + \left[1 \pm \sqrt{\frac{1}{2}}\right]k_3 h\right)$ $y_{i+1} = y_i + \frac{1}{6}k_1 h + \frac{1}{3}\left[1 \mp \sqrt{\frac{1}{2}}\right]k_2 h + \frac{1}{3}\left[1 \pm \sqrt{\frac{1}{2}}\right]k_3 h + \frac{1}{6}k_4 h$	

Table 4. Fifth order Runge-Kutta methods

RK5: Butcher-1 (Butcher, 1964; Chapra and Canale, 2002)	***RK5: Butcher-2 (Butcher, 2009)***
$k_1 = f(x_i, y_i)$ $k_2 = f\left(x_i + \frac{1}{4}h,\ y_i + \frac{1}{4}k_1 h \right)$ $k_3 = f\left(x_i + \frac{1}{4}h,\ y_i + \frac{1}{8}k_1 h + \frac{1}{8}k_2 h \right)$ $k_4 = f\left(x_i + \frac{1}{2}h,\ y_i - \frac{1}{2}k_2 h + k_3 h \right)$ $k_5 = f\left(x_i + \frac{3}{4}h,\ y_i + \frac{3}{16}k_1 h + \frac{9}{16}k_4 h \right)$ $k_6 = f\left(x_i + h,\ y_i - \frac{3}{7}k_1 h + \frac{2}{7}k_2 h + \frac{12}{7}k_3 h - \frac{12}{7}k_4 h + \frac{8}{7}k_5 h \right)$ $y_{i+1} = y_i + \frac{1}{90}(7k_1 + 32k_3 + 12k_4 + 32k_5 + 7k_6)h$	$k_1 = f(x_i, y_i)$ $k_2 = f\left(x_i + \frac{1}{3}h,\ y_i + \frac{1}{3}k_1 h \right)$ $k_3 = f\left(x_i + \frac{3}{4}h,\ y_i - \frac{123}{256}k_1 h + \frac{315}{256}k_2 h \right)$ $k_4 = f\left(x_i + \frac{1}{5}h,\ y_i + \frac{193}{750}k_1 h - \frac{189}{1250}k_2 h + \frac{176}{1875}k_3 h \right)$ $k_5 = f\left(x_i + \frac{2}{3}h,\ y_i - \frac{26}{81}k_1 h + \frac{7}{15}k_2 h - \frac{304}{4455}k_3 h - \frac{175}{297}k_4 h \right)$ $k_6 = f\left(x_i + h,\ y_i + \frac{151}{150}k_1 h - \frac{351}{250}k_2 h + \frac{304}{4125}k_3 h - \frac{5}{77}k_4 h + \frac{243}{175}k_5 h \right)$ $y_{i+1} = y_i + \left(\frac{1}{24}k_1 + \frac{125}{336}k_4 + \frac{27}{56}k_5 + \frac{5}{48}k_6 \right)h$
RK5: Abraham-Bolarin (Abraham and Bolarin, 2011)	***RK5: Kutta (Butcher, 1996)***
$k_1 = f(x_i, y_i)$ $k_2 = f(x_i + h,\ y_i + k_1 h)$ $k_3 = f\left(x_i + \frac{1}{2}h,\ y_i + \frac{181}{906}k_1 h - \frac{545}{727}k_2 h \right)$ $k_4 = f\left(x_i + \frac{1}{5}h,\ y_i - \frac{409}{583}k_1 h + \frac{387}{691}k_2 h + \frac{14}{41}k_3 h \right)$ $k_5 = f\left(x_i + \frac{1}{4}h,\ y_i - \frac{208}{809}k_1 h + \frac{43}{954}k_2 h + \frac{215}{609}k_3 h - \frac{233}{575}k_4 h \right)$ $k_6 = f\left(x_i + \frac{3}{4}h,\ y_i - \frac{625}{828}k_1 h + \frac{16}{55}k_2 h + \frac{267}{805}k_3 h + \frac{257}{189}k_4 h - \frac{117}{245}k_5 h \right)$ $y_{i+1} = y_i + \left(\frac{7}{90}k_1 + \frac{7}{90}k_2 + \frac{2}{15}k_3 + \frac{16}{45}k_5 + \frac{16}{45}k_6 \right)h$	$k_1 = f(x_i, y_i)$ $k_2 = f\left(x_i + \frac{1}{5}h,\ y_i + \frac{1}{5}k_1 h \right)$ $k_3 = f\left(x_i + \frac{2}{5}h,\ y_i + \frac{2}{5}k_2 h \right)$ $k_4 = f\left(x_i + h,\ y_i + \frac{9}{4}k_1 h - 5k_2 h + \frac{15}{4}k_3 h \right)$ $k_5 = f\left(x_i + \frac{3}{5}h,\ y_i - \frac{63}{100}k_1 h + \frac{9}{5}k_2 h - \frac{13}{20}k_3 h + \frac{2}{25}k_4 h \right)$ $k_6 = f\left(x_i + \frac{4}{5}h,\ y_i - \frac{6}{25}k_1 h + \frac{4}{5}k_2 h + \frac{2}{15}k_3 h + \frac{8}{75}k_4 h \right)$ $y_{i+1} = y_i + \left(\frac{17}{144}k_1 + \frac{25}{36}k_3 + \frac{1}{72}k_4 - \frac{25}{72}k_5 + \frac{25}{48}k_6 \right)h$
RK5: Nyström (Butcher, 1996)	***RK5: Luther-Konen-1 (Luther and Konen, 1965)***
$k_1 = f(x_i, y_i)$ $k_2 = f\left(x_i + \frac{1}{3}h,\ y_i + \frac{1}{3}k_1 h \right)$ $k_3 = f\left(x_i + \frac{4}{25}h,\ y_i + \frac{6}{25}k_2 h \right)$ $k_4 = f\left(x_i + h,\ y_i + \frac{1}{4}k_1 h - 3k_2 h + \frac{15}{4}k_3 h \right)$ $k_5 = f\left(x_i + \frac{2}{3}h,\ y_i + \frac{2}{27}k_1 h + \frac{10}{9}k_2 h - \frac{50}{81}k_3 h + \frac{8}{81}k_4 h \right)$ $k_6 = f\left(x_i + \frac{4}{5}h,\ y_i + \frac{2}{25}k_1 h + \frac{12}{25}k_2 h + \frac{2}{15}k_3 h + \frac{8}{75}k_4 h \right)$ $y_{i+1} = y_i + \left(\frac{23}{192}k_1 + \frac{125}{192}k_3 - \frac{27}{64}k_5 + \frac{125}{192}k_6 \right)h$	$k_1 = f(x_i, y_i)$ $k_2 = f\left(x_i + \frac{1}{3}h,\ y_i + \frac{1}{3}k_1 h \right)$ $k_3 = f\left(x_i + \frac{2}{5}h,\ y_i + \frac{1}{25}(4k_1 + 6k_2)h \right)$ $k_4 = f\left(x_i + h,\ y_i + \frac{1}{4}(k_1 - 12k_2 + 15k_3)h \right)$ $k_5 = f\left(x_i + \frac{2}{3}h,\ y_i + \frac{1}{81}(6k_1 + 90k_2 - 50k_3 + 8k_4)h \right)$ $k_6 = f\left(x_i + \frac{4}{5}h,\ y_i + \frac{1}{75}(6k_1 + 36k_2 + 10k_3 + 8k_4)h \right)$ $y_{i+1} = y_i + \frac{1}{192}(23k_1 + 125k_2 - 81k_5 + 125k_6)h$
RK5: Luther-Konen-2 (Luther and Konen, 1965)	

$$k_1 = f(x_i, y_i)$$
$$k_2 = f\left(x_i + \frac{4}{11}h,\ y_i + \frac{4}{11}k_1 h \right)$$
$$k_3 = f\left(x_i + \frac{2}{5}h,\ y_i + \frac{1}{50}(9k_1 + 11k_2)\,h \right)$$
$$k_4 = f\left(x_i + \frac{1}{2}h,\ y_i + \frac{1}{4}(-11k_1 + 15k_3)h \right)$$
$$k_5 = f\left(x_i + \frac{6-\sqrt{6}}{10}h,\ y_i + \frac{1}{600}\left[(81+9\sqrt{6})k_1 + (255-55\sqrt{6})k_3 + (24-14\sqrt{6})k_4\right]h \right)$$
$$k_6 = f\left(x_i + \frac{6+\sqrt{6}}{10}h,\ y_i + \frac{1}{600}\left[(81-9\sqrt{6})k_1 + (255+55\sqrt{6})k_3 + (24+14\sqrt{6})k_4\right]h \right)$$
$$y_{i+1} = y_i + \frac{1}{36}\left[4k_1 + (16+\sqrt{6})k_5 + (16-\sqrt{6})k_6\right]h$$

RK5: Luther-Konen-3 (Luther and Konen, 1965)

$$k_1 = f(x_i, y_i)$$
$$k_2 = f\left(x_i + \frac{1}{2}h,\ y_i + \frac{1}{2}k_1 h \right)$$
$$k_3 = f\left(x_i + \frac{5-\sqrt{5}}{10}h,\ y_i + \frac{1}{10}\left[2k_1 + (3-\sqrt{5})k_2\right]h \right)$$
$$k_4 = f\left(x_i + \frac{1}{2}h,\ y_i + \frac{1}{4}(k_1 + k_2)h \right)$$
$$k_5 = f\left(x_i + \frac{5+\sqrt{5}}{10}h,\ y_i + \frac{1}{20}\left[(1-\sqrt{5})k_1 - 4k_2 + (5+3\sqrt{5})k_3 + 8k_4\right]h \right)$$
$$k_6 = f\left(x_i + h,\ y_i + \frac{1}{4}\left[(\sqrt{5}-1)k_1 + (2\sqrt{5}-2)k_2 + (5-\sqrt{5})k_3 - 8k_4 + (10-2\sqrt{5})k_5\right]h \right)$$
$$y_{i+1} = y_i + \frac{1}{12}(k_1 + 5k_3 + 5k_5 + k_6)h$$

RK5: Luther-Konen-4 (Luther and Konen, 1965)

$$k_1 = f(x_i, y_i)$$
$$k_2 = f\left(x_i + h,\ y_i + k_1 h\right)$$
$$k_3 = f\left(x_i + \frac{1}{2}h,\ y_i + \frac{1}{8}(3k_1 + k_2)h \right)$$
$$k_4 = f\left(x_i + h,\ y_i + \frac{1}{2}(-k_1 - k_2 + 4k_3)h \right)$$
$$k_5 = f\left(x_i + \frac{5-\sqrt{15}}{10}h,\ y_i + \frac{1}{100}\left[-\sqrt{15}k_1 - 10k_2 + (60-8\sqrt{15})k_3 - \sqrt{15}k_4\right]h \right)$$
$$k_6 = f\left(x_i + \frac{5+\sqrt{15}}{10}h,\ y_i + \frac{1}{20}\left[(-6-\sqrt{15})k_1 - 2k_2 + 12k_3 + (6-\sqrt{15})k_4 + 4\sqrt{15}k_5\right]h \right)$$
$$y_{i+1} = y_i + \frac{1}{18}(8k_3 + 5k_5 + 5k_6)h$$

RK5: Luther-1 (Luther, 1966)	***RK5: Luther-2 (Luther, 1966)***

$$
\begin{aligned}
k_1 &= f(x_i, y_i)\\
k_2 &= f\left(x_i + h,\ y_i + k_1 h\right)\\
k_3 &= f\left(x_i + h,\ y_i + \frac{1}{2}(k_1 + k_2)h\right)\\
k_4 &= f\left(x_i + \frac{1}{4}h,\ y_i + \frac{1}{64}(14k_1 + 5k_2 - 3k_3)h\right)\\
k_5 &= f\left(x_i + \frac{1}{2}h,\ y_i + \frac{1}{96}(-12k_1 - 12k_2 + 8k_3 + 64k_4)h\right)\\
k_6 &= f\left(x_i + \frac{3}{4}h,\ y_i + \frac{1}{64}(-9k_2 + 5k_3 + 16k_4 + 36k_5)h\right)\\
y_{i+1} &= y_i + \frac{1}{90}\left(7k_1 + 7k_3 + 32k_4 + 12k_5 + 32k_6\right)h
\end{aligned}
$$

$$
\begin{aligned}
k_1 &= f(x_i, y_i)\\
k_2 &= f\left(x_i + h,\ y_i + k_1 h\right)\\
k_3 &= f\left(x_i + \frac{1}{2}h,\ y_i + \frac{1}{8}(3k_1 + k_2)h\right)\\
k_4 &= f\left(x_i + h,\ y_i + \frac{1}{2}(-k_1 - k_2 + 4k_3)h\right)\\
k_5 &= f\left(x_i + \frac{1}{4}h,\ y_i + \frac{1}{64}(4k_1 - 5k_2 + 20k_3 - 3k_4)h\right)\\
k_6 &= f\left(x_i + \frac{3}{4}h,\ y_i + \frac{1}{64}(12k_1 + 9k_2 - 12k_3 + 7k_4 + 32k_5)h\right)\\
y_{i+1} &= y_i + \frac{1}{90}\left(7k_1 + 12k_3 + 7k_4 + 32k_5 + 32k_6\right)h
\end{aligned}
$$

RK5: Luther-3 (Luther, 1966)

$$
\begin{aligned}
k_1 &= f(x_i, y_i)\\
k_2 &= f\left(x_i + h,\ y_i + k_1 h\right)\\
k_3 &= f\left(x_i + \tfrac{1}{2}h,\ y_i + \tfrac{1}{8}(3k_1 + k_2 h\right)\\
k_4 &= f\left(x_i + h,\ y_i + \tfrac{1}{2}(-k_1 - k_2 + 4k_3)h\right)\\
k_5 &= f\left(x_i + \tfrac{5-\sqrt{5}}{10}h,\ y_i + \tfrac{1}{100}\left[(25 - 7\sqrt{5})k_1 + (5 - 5\sqrt{5})k_2 + (20 + 4\sqrt{5})k_3 - 2\sqrt{5}k_4\right]h\right)\\
k_6 &= f\left(x_i + \tfrac{5\mp\sqrt{5}}{10}h,\ y_i + \tfrac{1}{20}\left[(3 + \sqrt{5})k_1 + (1 + \sqrt{5})k_2 + (4 - 4\sqrt{5})k_3 + 2k_4 + 4\sqrt{5}k_5\right]h\right)\\
y_{i+1} &= y_i + \tfrac{1}{12}\left(k_1 + k_4 + 5k_5 + 5k_6\right)h
\end{aligned}
$$

RK5: Butcher-3 (Butcher, 1994, 1969) | ***RK5: Butcher-4 (Butcher, 1994, 1969)***

$$
\begin{aligned}
k_1 &= f(x_i, y_i)\\
k_2 &= f\left(x_i + \frac{1}{5}h,\ y_i + \frac{1}{5}k_1 h\right)\\
k_3 &= f\left(x_i + \frac{2}{5}h,\ y_i + \frac{2}{5}k_2 h\right)\\
k_4 &= f\left(x_i + \frac{3}{4}h,\ y_i + \frac{75}{64}k_1 h - \frac{9}{4}k_2 h + \frac{117}{64}k_3 h\right)\\
k_5 &= f\left(x_i + h,\ y_i - \frac{37}{36}k_1 h + \frac{7}{3}k_2 h - \frac{3}{4}k_3 h + \frac{4}{9}k_4 h\right)\\
y_{i+1} &= y_i + \left(\frac{19}{144}k_1 + \frac{25}{48}k_3 + \frac{2}{9}k_4 + \frac{1}{8}k_5\right)h
\end{aligned}
$$

$$
\begin{aligned}
k_1 &= f(x_i, y_i)\\
k_2 &= f\left(x_i + \frac{1}{5}h,\ y_i + \frac{1}{5}k_1 h\right)\\
k_3 &= f\left(x_i + \frac{2}{5}h,\ y_i + \frac{2}{5}k_2 h\right)\\
k_4 &= f\left(x_i + \frac{1}{2}h,\ y_i + \frac{3}{16}k_1 h + \frac{5}{16}k_3 h\right)\\
k_5 &= f\left(x_i + h,\ y_i + \frac{1}{4}k_1 h - \frac{5}{4}k_3 h + 2k_4 h\right)\\
y_{i+1} &= y_i + \left(\frac{1}{6}k_1 + \frac{2}{3}k_4 + \frac{1}{6}k_5\right)h
\end{aligned}
$$

RK5: Butcher-5 (Butcher, 1994, 1969)

$$
\begin{aligned}
k_1 &= f(x_i, y_i)\\
k_2 &= f\left(x_i + \frac{1}{5}h,\ y_i + \frac{1}{5}k_1 h\right)\\
k_3 &= f\left(x_i + \frac{2}{5}h,\ y_i + \frac{2}{5}k_2 h\right)\\
k_4 &= f\left(x_i + \frac{3}{4}h,\ y_i + \frac{161}{192}k_1 h - \frac{19}{12}k_2 h + \frac{287}{192}k_3 h\right)\\
k_5 &= f\left(x_i + h,\ y_i - \frac{27}{28}k_1 h + \frac{19}{7}k_2 h - \frac{291}{196}k_3 h + \frac{36}{49}k_4 h\right)\\
y_{i+1} &= y_i + \left(\frac{7}{48}k_1 + \frac{475}{1008}k_3 + \frac{2}{7}k_4 + \frac{7}{72}k_5\right)h
\end{aligned}
$$

RK5: Butcher-6 (Butcher, 2010)

$$k_1 = f(x_i, y_i)$$
$$k_2 = f\left(x_i + \frac{1}{10}h,\ y_i + \frac{1}{10}k_1h \right)$$
$$k_3 = f\left(x_i + \frac{1}{5}h,\ y_i + \frac{11}{260}k_1h + \frac{41}{260}k_2h \right)$$
$$k_4 = f\left(x_i + \frac{3}{10}h,\ y_i - \frac{3}{170}k_1h + \frac{63}{340}k_2h + \frac{9}{68}k_3h \right)$$
$$k_5 = f\left(x_i + \frac{3}{4}h,\ y_i + \frac{13035}{544}k_1h - \frac{75447}{1088}k_2h + \frac{9009}{136}k_3h - \frac{1287}{64}k_4h \right)$$
$$k_6 = f\left(x_i + h,\ y_i - \frac{165709}{918}k_1h + \frac{733747}{1326}k_2h - \frac{638689}{1122}k_3h + \frac{32053}{162}k_4h - \frac{5320}{11583}k_5h \right)$$
$$y_{i+1} = y_i + \left(\frac{5}{54}k_1 + \frac{250}{567}k_4 + \frac{32}{81}k_5 + \frac{1}{14}k_6 \right)h$$

Table 5. Sixth order Runge-Kutta methods

RK6: Butcher-1 (Butcher, 1964)

$$k_1 = f(x_i, y_i)$$
$$k_2 = f\left(x_i + \frac{1}{3}h,\ y_i + \frac{1}{3}k_1h \right)$$
$$k_3 = f\left(x_i + \frac{2}{3}h,\ y_i + \frac{2}{3}k_2h \right)$$
$$k_4 = f\left(x_i + \frac{1}{3}h,\ y_i + \frac{1}{12}k_1h + \frac{1}{3}k_2h - \frac{1}{12}k_3h \right)$$
$$k_5 = f\left(x_i + \frac{1}{2}h,\ y_i - \frac{1}{16}k_1h + \frac{9}{8}k_2h - \frac{3}{16}k_3h - \frac{3}{8}k_4h \right)$$
$$k_6 = f\left(x_i + \frac{1}{2}h,\ y_i + \frac{9}{8}k_2h - \frac{3}{8}k_3h - \frac{3}{4}k_4h + \frac{1}{2}k_5h \right)$$
$$k_7 = f\left(x_i + h,\ y_i + \frac{9}{44}k_1h - \frac{9}{11}k_2h + \frac{63}{44}k_3h + \frac{18}{11}k_4h - \frac{16}{11}k_6h \right)$$
$$y_{i+1} = y_i + \left(\frac{11}{120}k_1 + \frac{27}{40}k_3 + \frac{27}{40}k_4 - \frac{4}{15}k_5 - \frac{4}{15}k_6 + \frac{11}{120}k_7 \right)h$$

RK6: Butcher-2 (Butcher, 1964)

$$k_1 = f(x_i, y_i)$$
$$k_2 = f\left(x_i + h,\ y_i + k_1h \right)$$
$$k_3 = f\left(x_i + \frac{2}{3}h,\ y_i + \frac{4}{9}k_1h + \frac{2}{9}k_2h \right)$$
$$k_4 = f\left(x_i + \frac{1}{3}h,\ y_i + \frac{11}{36}k_1h + \frac{1}{9}k_2h - \frac{1}{12}k_3h \right)$$
$$k_5 = f\left(x_i - \frac{1}{3}h,\ y_i + \frac{151}{36}k_1h + \frac{29}{9}k_2h - \frac{7}{4}k_3h - 6k_4h \right)$$
$$k_6 = f\left(x_i + \frac{4}{3}h,\ y_i - \frac{112}{9}k_1h - \frac{116}{9}k_2h + \frac{32}{3}k_3h + 18k_4h - 2k_5h \right)$$
$$k_7 = f\left(x_i + h,\ y_i - \frac{5}{4}k_1h - \frac{29}{23}k_2h + \frac{397}{276}k_3h + \frac{152}{69}k_4h - \frac{10}{69}k_5h + \frac{1}{69}k_6h \right)$$
$$y_{i+1} = y_i + \left(\frac{23}{160}k_1 + \frac{29}{80}k_3 + \frac{29}{80}k_4 - \frac{1}{160}k_5 - \frac{1}{160}k_6 + \frac{23}{160}k_7 \right)h$$

RK6: Butcher-3 (Butcher, 1964)

$$k_1 = f(x_i, y_i)$$
$$k_2 = f\left(x_i + \frac{1}{2}h,\ y_i + \frac{1}{2}k_1h\right)$$
$$k_3 = f\left(x_i + \frac{2}{3}h,\ y_i + \frac{2}{9}k_1h + \frac{4}{9}k_2h\right)$$
$$k_4 = f\left(x_i + \frac{1}{3}h,\ y_i + \frac{7}{36}k_1h + \frac{2}{9}k_2h - \frac{1}{12}k_3h\right)$$
$$k_5 = f\left(x_i + \frac{5}{6}h,\ y_i - \frac{35}{144}k_1h - \frac{55}{36}k_2h + \frac{35}{48}k_3h + \frac{15}{8}k_4h\right)$$
$$k_6 = f\left(x_i + \frac{1}{6}h,\ y_i - \frac{1}{360}k_1h - \frac{11}{36}k_2h - \frac{1}{8}k_3h + \frac{1}{2}k_4h + \frac{1}{10}k_5h\right)$$
$$k_7 = f\left(x_i + h,\ y_i - \frac{41}{260}k_1h + \frac{23}{13}k_2h + \frac{43}{156}k_3h - \frac{118}{39}k_4h + \frac{32}{195}k_5h + \frac{80}{39}k_6h\right)$$
$$y_{i+1} = y_i + \left(\frac{13}{200}k_1 + \frac{11}{40}k_3 + \frac{11}{40}k_4 + \frac{4}{25}k_5 + \frac{4}{25}k_6 + \frac{13}{200}k_7\right)h$$

RK6: Butcher-4 (Butcher, 1964)

$$k_1 = f(x_i, y_i)$$
$$k_2 = f\left(x_i + \tfrac{5\mp\sqrt{5}}{10}h,\ y_i + \tfrac{5\mp\sqrt{5}}{10}k_1h\right)$$
$$k_3 = f\left(x_i + \tfrac{5\pm\sqrt{5}}{10}h,\ y_i \mp \tfrac{\sqrt{5}}{10}k_1h + \tfrac{5\pm2\sqrt{5}}{10}k_2h\right)$$
$$k_4 = f\left(x_i + \tfrac{5\mp\sqrt{5}}{10}h,\ y_i + \tfrac{-15\pm7\sqrt{5}}{20}k_1h + \tfrac{-1\pm\sqrt{5}}{4}k_2h + \tfrac{15\mp7\sqrt{5}}{10}k_3h\right)$$
$$k_5 = f\left(x_i + \tfrac{5\pm\sqrt{5}}{10}h,\ y_i + \tfrac{5\pm\sqrt{5}}{60}k_1h + \tfrac{1}{6}k_3h + \tfrac{15\pm7\sqrt{5}}{60}k_4h\right)$$
$$k_6 = f\left(x_i + \tfrac{5\mp\sqrt{5}}{10}h,\ y_i + \tfrac{5\pm\sqrt{5}}{60}k_1h + \tfrac{9\mp5\sqrt{5}}{12}k_3h + \tfrac{1}{6}k_4h + \tfrac{-5\pm3\sqrt{5}}{10}k_5h\right)$$
$$k_7 = f\left(x_i + h,\ y_i + \tfrac{1}{6}k_1h + \tfrac{-55\pm25\sqrt{5}}{12}k_3h + \tfrac{-25\mp7\sqrt{5}}{12}k_4h + (5\mp2\sqrt{5})k_5h + \tfrac{5\pm\sqrt{5}}{2}k_6h\right)$$
$$y_{i+1} = y_i + \frac{1}{12}(k_1 + 5k_5 + 5k_6 + k_7)h$$

RK6: Luther (Luther, 1967)

$$k_1 = f(x_i, y_i)$$
$$k_2 = f(x_i + h,\ y_i + k_1h)$$
$$k_3 = f\left(x_i + \frac{1}{2}h,\ y_i + \frac{1}{8}(3k_1 + k_2)h\right)$$
$$k_4 = f\left(x_i + \frac{2}{3}h,\ y_i + \frac{1}{27}(8k_1 + 2k_2 + 8k_3)h\right)$$
$$k_5 = f\left(x_i + \frac{7-\sqrt{21}}{14}h,\ y_i + \frac{1}{392}\left[3(3\sqrt{21}-7)k_1 - 8(7-\sqrt{21})k_2 + 48(7-\sqrt{21})k_3 - 3(21-\sqrt{21})k_4\right]h\right)$$
$$k_6 = f\left(x_i + \frac{7+\sqrt{21}}{14}h,\ y_i + \frac{1}{1960}\left[-5(231+51\sqrt{21})k_1 - 40(7+\sqrt{21})k_2 - 320\sqrt{21}k_3 + 3(21+121\sqrt{21})k_4 + 392(6+\sqrt{21})k_5\right]h\right)$$
$$k_7 = f\left(x_i + h,\ y_i + \frac{1}{180}\left[15(22+7\sqrt{21})k_1 + 120k_2 + 40(7\sqrt{21}-5)k_3 - 63(3\sqrt{21}-2)k_4 - 14(49+9\sqrt{21})k_5 + 70(7-\sqrt{21})k_6\right]h\right)$$
$$y_{i+1} = y_i + \frac{1}{180}(9k_1 + 64k_3 + 49k_5 + 49k_6 + 9k_7)h$$

3. THE DESIGNED SIMULATOR

The main screenshot and descriptions of the simulator designed by using MATLAB (Mathworks, 2007) are given in Figure 1. The ordinary differential equations, initial condition, solutions range and step size or number of points selectively, are entered to the simulator. The solution of the differential equation is implemented according to the selected method, and the solution values/steps are also listed. In addition, the solution of the differential equation in the specified range is drawn graphically. The solution range is also determined by the user. By using menu options of the simulator (see Table 6), the results can be saved or printed; single or comparative solutions can be created; program settings (drawing shape of data, graphical data representation formats, etc.) can be made (see Table 7) and help topics can be accessed. In addition, the simulator calculate the solutions for the entered ODE using three methods selected by the user and presents the obtained results comparatively (solution values, computation/elapsed times etc.) to the user on the separate windows both as numerical and graphical. As an example, the solution steps of the simulator by using a second order Runge-Kutta method are summarized with a flowchart given in Figure 2. Similar steps are carried out with the other methods.

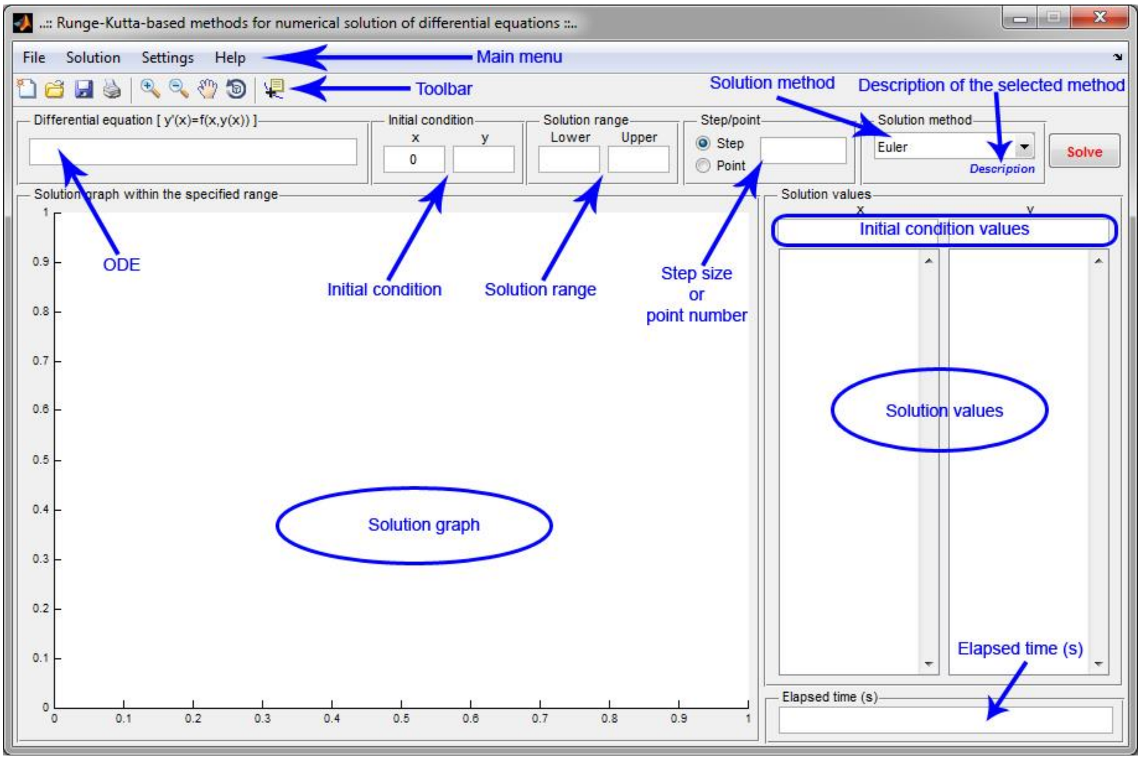


Figure 1:
Main screen of designed simulator

Table 6. Main menu and toolbar of designed simulator

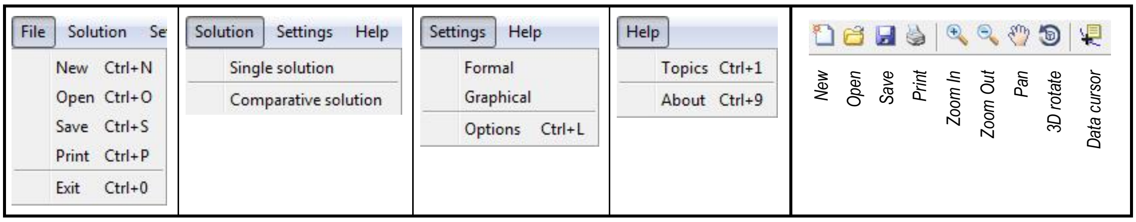
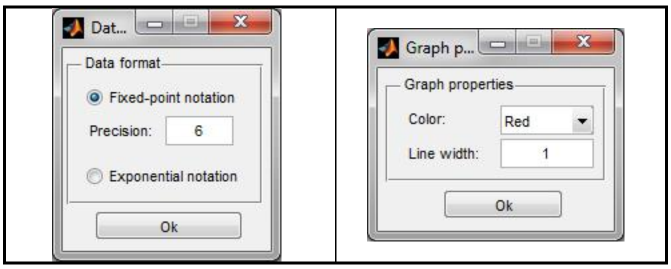


Table 7. Settings menu items of designed simulator



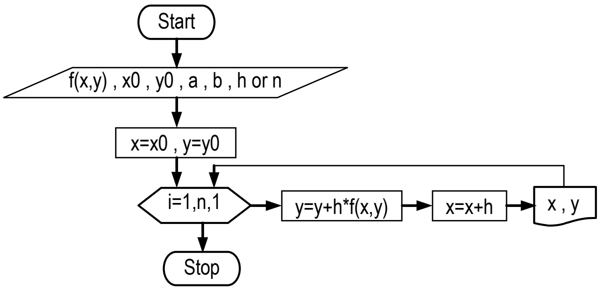


Figure 2:
Flowchart for Euler's method

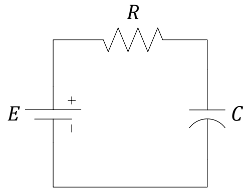


Figure 3:
RC circuit

In the first simulation, the following differential equation belong to a RC circuit given in Figure 3 is used.

$$y'(t) + \frac{1}{RC} y(t) = \frac{1}{RC} E \tag{10}$$

The screenshot of the solution of the differential equation of the RC circuit, for $R = 1M\Omega$, $C = 1\mu F$, $E = 12V$ values, under the initial condition $y(0) = 0$ within solution range $0 - 10s$ with 3rd order Runge-Kutta method-1 method is given in Figure 4. The comparative error results for the first simulation are given in Table 8.

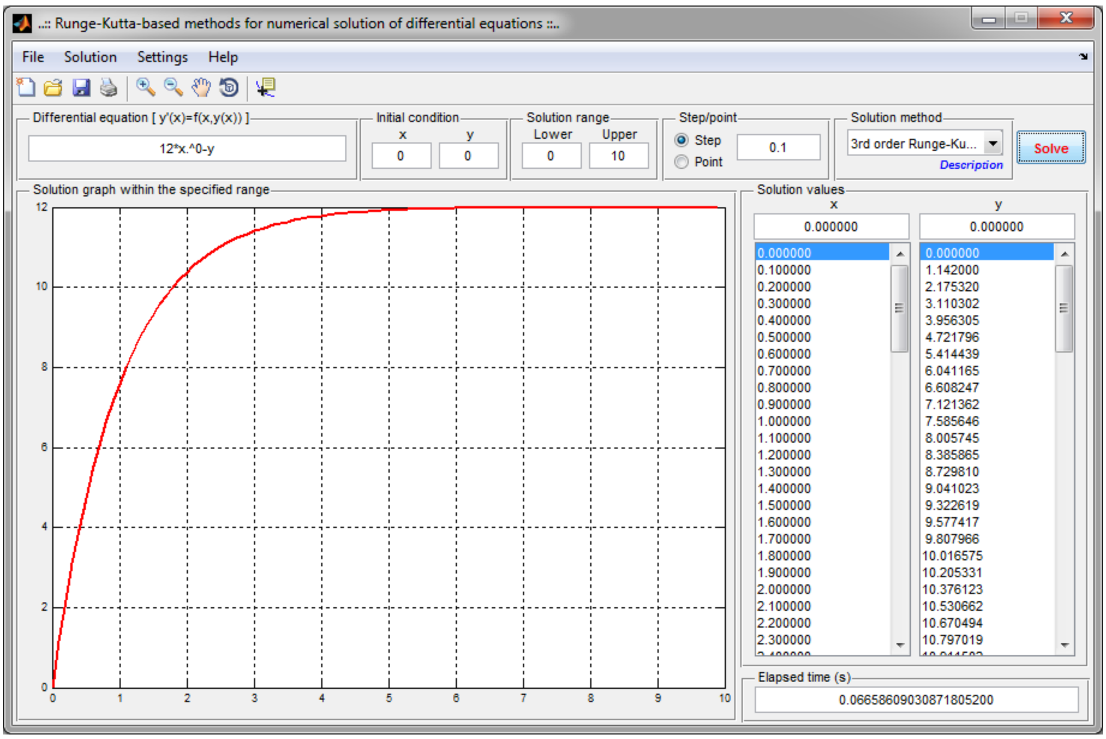


Figure 4:
Screenshot of single solution simulation

Table 8. Comparative results for first simulation

Method	Average value of		Standard deviation of	
	Absolute errors	Relative errors	Absolute errors	Relative errors
Euler	0.0609767184	0.0082205416	0.0768299455	0.0128024671
Modified Euler (Midpoint integration) method	-0.0021570225	-0.0002869302	0.0026589173	0.0004359286
Improved Euler (Trapezoidal integration) (Heun) method	-0.0021570225	-0.0002869302	0.0026589173	0.0004359286
Ralston's 2nd order method	-0.0021570225	-0.0002869302	0.0026589173	0.0004359286
RK2: Kopal method	-0.0021570225	-0.0002869302	0.0026589173	0.0004359286
2nd order contraharmonic mean method	0.0013306927	0.0001771378	0.0016422934	0.0002694863
3rd order Runge-Kutta method-1	0.0000540983	0.0000071995	0.0000667367	0.0000109475
3rd order Runge-Kutta method-2	0.0000540983	0.0000071995	0.0000667367	0.0000109475
Heun's 3rd order Runge-Kutta method	0.0000540983	0.0000071995	0.0000667367	0.0000109475
Ralston's 3rd order Runge-Kutta method	0.0000540983	0.0000071995	0.0000667367	0.0000109475
3rd order harmonic mean method	0.0000147783	0.0000019667	0.0000182306	0.0000029905
3rd order contraharmonic mean method	0.0000829636	0.0000110410	0.0001023467	0.0000167890
3rd order modified contraharmonic mean weights method	-0.0189165634	-0.0025077231	0.0231838016	0.0037854139
3rd order arithmetic mean Runge-Kutta method	0.0000540983	0.0000071995	0.0000667367	0.0000109475
New 3rd order arithmetic mean Runge-Kutta method	0.0000540983	0.0000071995	0.0000667367	0.0000109475
Composite arithmetic–harmonic mean RK method	-0.0020314427	-0.0002702324	0.0025042261	0.0004105798
Classical 4th order Runge-Kutta (arithmetic mean) method	-0.0000010856	-0.0000001445	0.0000013392	0.0000002197
Ralston's 4th order method	-0.0000010856	-0.0000001445	0.0000013392	0.0000002197
3/8 Runge-Kutta method	0.0196920531	0.0026313832	0.0244580546	0.0040319106
4th order harmonic mean method	-0.0000042995	-0.0000005722	0.0000053039	0.0000008700
4th order Runge-Kutta method with minimum error bound	-0.0000010850	-0.0000001444	0.0000013385	0.0000002196
4th order contraharmonic mean Runge-Kutta method	0.0000023531	0.0000003131	0.0000029027	0.0000004762
4th order heronian mean mehtod	-0.0000015779	-0.0000002100	0.0000019464	0.0000003193
RK4: Butcher-Johnston	0.0206272296	0.0027568893	0.0256278712	0.0042257593
4th order centroidal mean method	-0.0000001007	-0.0000000134	0.0000001242	0.0000000204
RK4: England	-0.0000010856	-0.0000001445	0.0000013392	0.0000002197
RK4: Gill	-0.0000010856	-0.0000001445	0.0000013392	0.0000002197
RK5: Butcher-1	-0.0000000026	-0.0000000003	0.0000000032	0.0000000005
RK5: Butcher-2	0.0677120446	0.0091420692	0.0855173678	0.0142758941
RK5: Abraham-Bolarin	0.0393374558	0.0052784758	0.0491924400	0.0081504478
RK5: Kutta	0.0000000181	0.0000000024	0.0000000224	0.0000000037
RK5: Nyström	0.0050385725	0.0006712337	0.0062263951	0.0010226416
RK5: Luther-Konen-1	0.0129487114	0.0017278525	0.0160451400	0.0026405330
RK5: Luther-Konen-2	-0.0000367737	-0.0000048938	0.0000453634	0.0000074412
RK5: Luther-Konen-3	0.0000000181	0.0000000024	0.0000000224	0.0000000037
RK5: Luther-Konen-4	0.0000000457	0.0000000061	0.0000000564	0.0000000093
RK5: Luther-1	0.0000000595	0.0000000079	0.0000000734	0.0000000120
RK5: Luther-2	0.0000000457	0.0000000061	0.0000000564	0.0000000093
RK5: Luther-3	0.0099076608	0.0013212230	0.0122639745	0.0020167274
RK5: Butcher-3	0.0000013702	0.0000001823	0.0000016903	0.0000002773
RK5: Butcher-4	0.0000000181	0.0000000024	0.0000000224	0.0000000037
RK5: Butcher-5	-0.0000013339	-0.0000001775	0.0000016455	0.0000002699
RK5: Butcher-6	-0.0000000001	-0.0000000000	0.0000000001	0.0000000000
RK6: Butcher-1	-0.0000000009	-0.0000000001	0.0000000011	0.0000000002
RK6: Butcher-2	-0.0000000009	-0.0000000001	0.0000000011	0.0000000002
RK6: Butcher-3	-0.0006294366	-0.0000837549	0.0007763026	0.0001273227
RK6: Butcher-4	-0.0040174643	-0.0005342046	0.0049490680	0.0008110252
RK6: Luther	0.0184501311	0.0046785007	0.0952041412	0.0146319521

In the second simulation performed, the following ODE was used.

$$y'(t) + \frac{1}{2}y(t) = 10e^{-\frac{(x-2)^2}{0,01}} \qquad (11)$$

The screenshot for the comparative solution of the ODE given above is given in Figure 5.

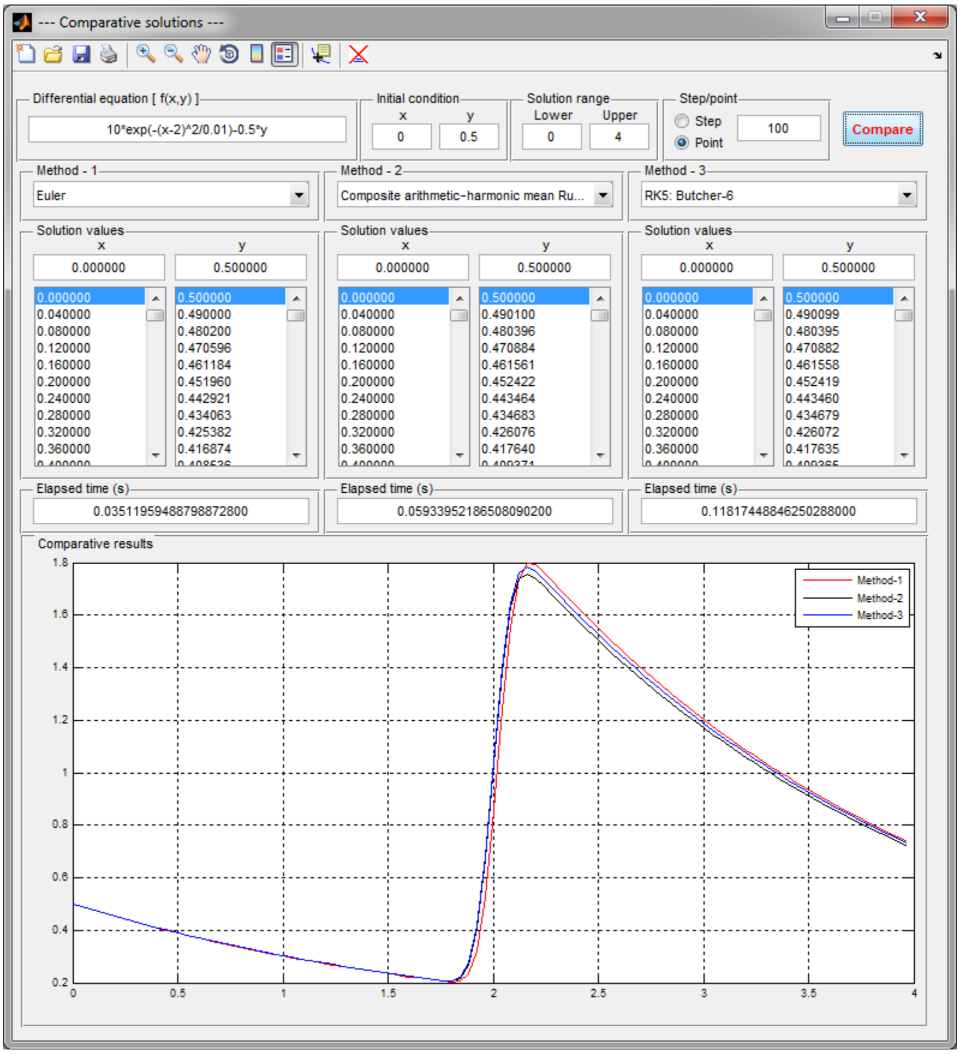


Figure 5:
Screenshot of comparative solution simulation

4. CONCLUSIONS

In the study, a general purpose simulator which can make numerical solution of ordinary differential equations with Runge-Kutta-based methods and be used in different technical fields (basic and applied sciences, engineering sciences, etc.) has been developed. In the simulator which can be used also for educational purposes; numerical solution of ODE defined/entered by the user is obtained by selected method using the specified initial condition and within the specified range, and the obtained results can be shown as both numerical and graphical. In the simulator which contains 48 different methods, the subject explanations are also included. In addition, comparative numerical solutions of ODE can also be studied with the methods chosen by the user. Thus, the users can easily show the effectiveness and efficiency of the methods by using the simulator which presents comparative numerical solutions of ODEs, and the use of the simulator can facilitate the choice (for using, for learning etc.) of the method for the students or those who are going to use the methods in their applications. As a future work, it is planned to expand the simulator by adding the other methods presented for numerical solution of ODEs, and to present it for online use by preparing in the form of a web page.

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