Advances in the Theory of Nonlinear Analysis and its Applications 6 (2022) No. 4, 528–537.
https://doi.org/10.31197/atnaa.1170948
Available online at www.atnaa.org
Research Article

Advances in the Theory of Nonlinear Analysis and its Applications

ISSN: 2587-2648

Peer-Reviewed Scientific Journal

# Several recursive and closed-form formulas for some specific values of partial Bell polynomials

Wei-Shih Du[a], Dongkyu Lim[b,c], Feng Qi[c,d,e]

[a]*Department of Mathematics, National Kaohsiung Normal University, Kaohsiung 82444, Taiwan*
[b]*Department of Mathematics Education, Andong National University, Andong 36729, Republic of Korea*
[c]*Corresponding author*
[d]*Institute of Mathematics, Henan Polytechnic University, Jiaozuo 454010, Henan, China*
[e]*Independent Researcher, Dallas, TX 75252-8024, USA*

*Dedicated to Professor Huan-Nan Shi on the occasion of his 75th birthday*

**Abstract**

In this paper, the authors derive several recursive and closed-form formulas for some specific values of partial Bell polynomials.

*Keywords:* specific value, partial Bell polynomial, special sequence, recursive formula, closed-form formula, Gauss' hypergeometric function, open problem.
*2010 MSC:* Primary 11B83; Secondary 33C05.

## 1. Motivations

Let $\mathbb{N} = \{1, 2, \dots\}$ be the set of all natural numbers.

*Email addresses:* wsdu@mail.nknu.edu.tw (Wei-Shih Du), dgrim84@gmail.com, dklim@anu.ac.kr (Dongkyu Lim), qifeng618@gmail.com, qifeng618@hotmail.com, qifeng618@qq.com (Feng Qi)

*Received September 5, 2022; Accepted: September 29, 2022; Online: October 9, 2022.*

The partial Bell polynomials $B_{n,k}$ for $n \ge k \ge 0$ were defined in [2, Definition 11.2] and [3, p. 134, Theorem A] by

$$B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{1 \le i \le n-k+1, \\ \ell_i \in \{0\} \cup \mathbb{N}, \\ \sum_{i=1}^{n-k+1} i\ell_i = n, \\ \sum_{i=1}^{n-k+1} \ell_i = k}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left(\frac{x_i}{i!}\right)^{\ell_i}.$$

One can also call the quantities $B_{n,k}$ the Bell polynomials of the second kind.

The so-called *ordinary* partial Bell polynomial $B^{\circ}_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ are defined in [10] (see also [6]) by the relation

$$B^{\circ}_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \frac{k!}{n!} B_{n,k}(1!x_1, 2!x_2, \dots, (n-k+1)!x_{n-k+1}) \tag{1}$$

or, equivalently,

$$B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \frac{n!}{k!} B^{\circ}_{n,k}\left(\frac{x_1}{1!}, \frac{x_2}{2!}, \dots, \frac{x_{n-k+1}}{(n-k+1)!}\right). \tag{2}$$

On 10 February 2022, Frank Oertel (f.oertel@email.de) asked the following problem in an e-mail to Qi.

> Let $n, k \in \mathbb{N}$ such that $n \ge k$. What is the (value of the) following *ordinary* partial Bell polynomial $B^{\circ}_{n,k}(x_1, x_2, \dots, x_{n-k+1})$? where
>
> $$x_i = \frac{[(2i-1)!!]^2}{[(2i)!!]^2(i+1)}, \quad i = 1, 2, \dots, n-k+1.$$
>
> In other words, what is the value of
>
> $$B^{\circ}_{n,k}\left(\frac{1}{8}, \frac{3}{64}, \frac{25}{1024}, \dots, \frac{[(2(n-k)+1)!!]^2}{[(2(n-k+1))!!]^2(n-k+2)}\right)? \tag{3}$$

Essentially, by the relation (1) or (2), Oertel's problem is equivalent to compute the specific values

$$B_{n,k}\left(\frac{1}{8}, \frac{3}{32}, \frac{75}{512}, \dots, \frac{(n-k+1)!}{n-k+2}\left[\frac{(2n-2k+1)!!}{(2n-2k+2)!!}\right]^2\right), \quad n \ge k \in \mathbb{N}. \tag{4}$$

In this paper, we will consider the above problem and provide several solutions to it.

## 2. Recursive formulas for specific values of partial Bell polynomials

In this section, we will derive recursive formulas for specific values expressed in (3) and (4).

**Theorem 1.** *For $k, n \in \mathbb{N}$ such that $n \ge k$, we have*

$$B_{n,k}\left(\frac{1}{8}, \frac{3}{32}, \frac{75}{512}, \dots, \frac{(n-k+1)!}{n-k+2}\left[\frac{(2n-2k+1)!!}{(2n-2k+2)!!}\right]^2\right) = (-1)^k \frac{n!}{k!} \sum_{m=1}^{k} (-1)^m \binom{k}{m} b_{m,n} \tag{5}$$

*and*

$$B^{\circ}_{n,k}\left(\frac{1}{8}, \frac{3}{64}, \frac{25}{1024}, \dots, \frac{[(2(n-k)+1)!!]^2}{[(2(n-k+1))!!]^2(n-k+2)}\right) = (-1)^k \sum_{m=1}^{k} (-1)^m \binom{k}{m} b_{m,n}, \tag{6}$$

*where $b_{m,0} = 1$ and*

$$b_{m,n} = \frac{1}{n} \sum_{q=0}^{n-1} \left[\frac{(2(n-q)-1)!!}{(2(n-q))!!}\right]^2 \frac{(n-q)m-q}{n-q+1} b_{m,q} \tag{7}$$

*for $m, n \in \mathbb{N}$.*

*Proof.* From (1), it follows that

$$B^{\circ}_{n,k}\left(\frac{1}{8},\frac{3}{64},\frac{25}{1024},\dots,\frac{[(2(n-k)+1)!!]^2}{[(2(n-k+1))!!]^2(n-k+2)}\right)\\ =\frac{k!}{n!}B_{n,k}\left(\frac{1}{8},\frac{3}{32},\frac{75}{512},\dots,\frac{(n-k+1)!}{n-k+2}\left[\frac{(2n-2k+1)!!}{(2n-2k+2)!!}\right]^2\right) \tag{8}$$

for $n \ge k \ge 0$.

Employing the formula

$$B_{n,k}\left(\frac{x_2}{2},\frac{x_3}{3},\dots,\frac{x_{n-k+2}}{n-k+2}\right)=\frac{n!}{(n+k)!}B_{n+k,k}(0,x_2,x_3,\dots,x_{n+1})$$

in [3, p. 136], we acquire

$$B_{n,k}\left(\frac{1}{8},\frac{3}{32},\frac{75}{512},\dots,\frac{(n-k+1)!}{n-k+2}\left[\frac{(2n-2k+1)!!}{(2n-2k+2)!!}\right]^2\right)\\ =\frac{n!}{(n+k)!}B_{n+k,k}\left(0,\frac{1}{4},\frac{9}{32},\dots,n!\left[\frac{(2n-1)!!}{(2n)!!}\right]^2\right). \tag{9}$$

Making use of the formula

$$\frac{1}{k!}\left(\sum_{m=1}^{\infty}x_m\frac{t^m}{m!}\right)^k=\sum_{n=k}^{\infty}B_{n,k}(x_1,x_2,\dots,x_{n-k+1})\frac{t^n}{n!}$$

for $k \ge 0$ in [3, p. 133] yields

$$\sum_{n=0}^{\infty}B_{n+k,k}\left(0,\frac{1}{4},\frac{9}{32},\dots,n!\left[\frac{(2n-1)!!}{(2n)!!}\right]^2\right)\frac{t^{n+k}}{(n+k)!}=\frac{1}{k!}\left(\sum_{m=2}^{\infty}\left[\frac{(2m-3)!!}{(2m-2)!!}\right]^2\frac{t^m}{m}\right)^k$$

which can be simplified as

$$\sum_{n=0}^{\infty}\frac{n!}{(n+k)!}B_{n+k,k}\left(0,\frac{1}{4},\frac{9}{32},\dots,n!\left[\frac{(2n-1)!!}{(2n)!!}\right]^2\right)\frac{t^n}{n!}=\frac{1}{k!}\left(\sum_{m=1}^{\infty}\left[\frac{(2m-1)!!}{(2m)!!}\right]^2\frac{t^m}{m+1}\right)^k.$$

This implies that

$$B_{n+k,k}\left(0,\frac{1}{4},\frac{9}{32},\dots,n!\left[\frac{(2n-1)!!}{(2n)!!}\right]^2\right)=\binom{n+k}{n}\lim_{t\to0}\frac{\mathrm{d}^n}{\mathrm{d}\,t^n}\left(\sum_{m=1}^{\infty}\left[\frac{(2m-1)!!}{(2m)!!}\right]^2\frac{t^m}{m+1}\right)^k$$

which is equivalent to

$$B_{\ell,k}\left(0,\frac{1}{4},\frac{9}{32},\dots,(\ell-k)!\left[\frac{(2\ell-2k-1)!!}{(2\ell-2k)!!}\right]^2\right)=\binom{\ell}{k}\lim_{t\to0}\frac{\mathrm{d}^{\ell-k}}{\mathrm{d}\,t^{\ell-k}}\left(\sum_{m=1}^{\infty}\left[\frac{(2m-1)!!}{(2m)!!}\right]^2\frac{t^m}{m+1}\right)^k \tag{10}$$

for $\ell \ge k \ge 0$.

For $\alpha,\beta\in\mathbb{C}$ and $\gamma\in\mathbb{C}\setminus\{0,-1,-2,\dots\}$, Gauss' hypergeometric function ${}_2F_1(\alpha,\beta;\gamma;z)$ can be defined by the series

$${}_2F_1(\alpha,\beta;\gamma;z)=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{z^n}{n!}. \tag{11}$$

The series in (11) absolutely and uniformly converges on the open unit disc $|z| < 1$. If $\Re(\alpha + \beta - \gamma) < 0$, the series in (11) absolutely converges over the unit circle $|z| = 1$. For details and more information, please refer to [9, pp. 64–66, Section 3.7] and [19, Chapter 5].

By the definition (11), we acquire that

$$\sum_{m=0}^{\infty}\left[\frac{(2m-1)!!}{(2m)!!}\right]^2\frac{t^m}{m+1} = {}_2F_1\left(\frac{1}{2},\frac{1}{2};2;t\right), \quad |t| \le 1.$$

Then we have

$$\left(\sum_{m=1}^{\infty}\left[\frac{(2m-1)!!}{(2m)!!}\right]^2\frac{t^m}{m+1}\right)^k = (-1)^k + \sum_{m=1}^{k}(-1)^{k-m}\binom{k}{m}\left[{}_2F_1\left(\frac{1}{2},\frac{1}{2};2;t\right)\right]^m, \quad |t| \le 1.$$

In [4, p. 18], it is given that

$$\left(\sum_{k=0}^{\infty}a_kx^k\right)^n = \sum_{k=0}^{\infty}c_kx^k,$$

where

$$c_0 = a_0^n, \quad c_m = \frac{1}{ma_0}\sum_{k=1}^{m}(kn-m+k)a_kc_{m-k}$$

for $m, n \ge 1$. Then, for $m \ge 1$, we have

$$\left[{}_2F_1\left(\frac{1}{2},\frac{1}{2};2;t\right)\right]^m = \sum_{j=0}^{\infty}b_{m,j}t^j, \tag{12}$$

where $b_{m,0} = 1$ and

$$b_{m,j} = \frac{1}{j}\sum_{q=1}^{j}\left[\frac{(2q-1)!!}{(2q)!!}\right]^2\frac{qm-j+q}{q+1}b_{m,j-q} = \frac{1}{j}\sum_{q=0}^{j-1}\left[\frac{(2(j-q)-1)!!}{(2(j-q))!!}\right]^2\frac{(j-q)m-q}{j-q+1}b_{m,q}$$

for $m, j \ge 1$. Therefore, when $\ell > k \ge 1$, we arrive at

$$\begin{aligned}
&\lim_{t\to0}\frac{\mathrm{d}^{\ell-k}}{\mathrm{d}\,t^{\ell-k}}\left(\sum_{m=1}^{\infty}\left[\frac{(2m-1)!!}{(2m)!!}\right]^2\frac{t^m}{m+1}\right)^k\\
&=\lim_{t\to0}\frac{\mathrm{d}^{\ell-k}}{\mathrm{d}\,t^{\ell-k}}\left(\sum_{m=1}^{k}(-1)^{k-m}\binom{k}{m}\left[{}_2F_1\left(\frac{1}{2},\frac{1}{2};2;t\right)\right]^m\right)\\
&=\lim_{t\to0}\frac{\mathrm{d}^{\ell-k}}{\mathrm{d}\,t^{\ell-k}}\left[\sum_{m=1}^{k}(-1)^{k-m}\binom{k}{m}\sum_{j=0}^{\infty}b_{m,j}t^j\right]\\
&=\lim_{t\to0}\frac{\mathrm{d}^{\ell-k}}{\mathrm{d}\,t^{\ell-k}}\sum_{j=0}^{\infty}\left[\sum_{m=1}^{k}(-1)^{k-m}\binom{k}{m}b_{m,j}\right]t^j\\
&=\lim_{t\to0}\sum_{j=\ell-k}^{\infty}\left[\sum_{m=1}^{k}(-1)^{k-m}\binom{k}{m}b_{m,j}\right]\langle j\rangle_{\ell-k}t^{j-\ell+k}\\
&=\langle \ell-k\rangle_{\ell-k}\sum_{m=1}^{k}(-1)^{k-m}\binom{k}{m}b_{m,\ell-k}
\end{aligned}$$

$$= (\ell-k)!\sum_{m=1}^{k}(-1)^{k-m}\binom{k}{m}b_{m,\ell-k},$$

where

$$\langle z\rangle_k = \prod_{\ell=0}^{k-1}(z-\ell) = \begin{cases} z(z-1)\cdots(z-k+1), & k\ge 1\\ 1, & k=0\end{cases}$$

is the falling factorial of $z\in\mathbb{C}$. Substituting this limit into (10) and simplifying lead to

$$B_{\ell,k}\left(0,\frac{1}{4},\frac{9}{32},\dots,(\ell-k)!\left[\frac{(2\ell-2k-1)!!}{(2\ell-2k)!!}\right]^2\right) = \frac{\ell!}{k!}\sum_{m=1}^{k}(-1)^{k-m}\binom{k}{m}b_{m,\ell-k} \tag{13}$$

for $\ell > k \ge 1$. Making use of (13) in (9) reveals (5) for $n,k\ge 1$. Substituting (5) into (8) yields (6) for $n,k\ge 1$. The proof of Theorem 1 is complete. □

## 3. Closed-form formulas for specific values of partial Bell polynomials

From the recursive relation (7), we acquire the following six specific values

$$b_{m,1}=\frac{m}{8},\quad b_{m,2}=\frac{m(m+5)}{128},\quad b_{m,3}=\frac{m(m^2+15m+59)}{3072},$$
$$b_{m,4}=\frac{m(m^3+30m^2+311m+1128)}{98304},$$
$$b_{m,5}=\frac{m(m^4+50m^3+965m^2+8590m+30084)}{3932160},$$

and

$$b_{m,6}=\frac{m(m^5+75m^4+2305m^3+36495m^2+299914m+1033350)}{188743680}.$$

These specific values hint us that the quantity $b_{m,n}$ for $m\in\mathbb{N}$ and $n\ge 0$ should be a polynomial of $m$ with degree $n$.

**Theorem 2.** *Let*

$$b_{m,n}=\sum_{\ell=1}^{n}\alpha_{n,\ell}m^\ell \tag{14}$$

*for $m,n\in\mathbb{N}$. Then*

$$b_{m,n}=\frac{1}{2^{3n}n!}m^n+\frac{5}{2^{3n+1}(n-2)!}m^{n-1}+\frac{1}{2^{3n+3}}\frac{75n^3-214n^2+117n+22}{3(n-1)!}m^{n-2}+\sum_{\ell=1}^{n-3}\alpha_{n,\ell}m^\ell.$$

*Proof.* Substituting the sum (14) into (7) and simplifying give

$$\begin{aligned} b_{m,n+1} &= \sum_{\ell=1}^{n+1}\alpha_{n+1,\ell}m^\ell\\ &= \left[\frac{(2n+1)!!}{(2n+2)!!}\right]^2\frac{m}{n+2}+\frac{1}{n+1}\sum_{q=1}^{n}\left[\frac{(2(n-q)+1)!!}{(2(n-q)+2)!!}\right]^2\frac{(n-q+1)m-q}{n-q+2}\sum_{\ell=1}^{q}\alpha_{q,\ell}m^\ell\\ &= \left[\frac{(2n+1)!!}{(2n+2)!!}\right]^2\frac{m}{n+2}+\frac{1}{n+1}\sum_{\ell=1}^{n}\left(\sum_{q=\ell}^{n}\left[\frac{(2(n-q)+1)!!}{(2(n-q)+2)!!}\right]^2\frac{(n-q+1)m-q}{n-q+2}\alpha_{q,\ell}\right)m^\ell \end{aligned}$$

$$
\begin{aligned}
&= \left[\frac{(2n+1)!!}{(2n+2)!!}\right]^2 \frac{m}{n+2} - \frac{1}{n+1}\sum_{\ell=1}^{n}\left(\sum_{q=\ell}^{n}\left[\frac{(2(n-q)+1)!!}{(2(n-q)+2)!!}\right]^2 \frac{q}{n-q+2}\alpha_{q,\ell}\right]\Bigg) m^{\ell} \\
&\quad + \frac{1}{n+1}\sum_{\ell=1}^{n}\left(\sum_{q=\ell}^{n}\left[\frac{(2(n-q)+1)!!}{(2(n-q)+2)!!}\right]^2 \frac{n-q+1}{n-q+2}\alpha_{q,\ell}\right) m^{\ell+1} \\
&= \left[\frac{(2n+1)!!}{(2n+2)!!}\right]^2 \frac{m}{n+2} - \frac{1}{n+1}\left(\sum_{q=1}^{n}\left[\frac{(2(n-q)+1)!!}{(2(n-q)+2)!!}\right]^2 \frac{q}{n-q+2}\alpha_{q,1}\right]\Bigg) m \\
&\quad - \frac{1}{n+1}\sum_{\ell=2}^{n}\left(\sum_{q=\ell}^{n}\left[\frac{(2(n-q)+1)!!}{(2(n-q)+2)!!}\right]^2 \frac{q}{n-q+2}\alpha_{q,\ell}\right]\Bigg) m^{\ell} \\
&\quad + \frac{1}{n+1}\sum_{\ell=2}^{n+1}\left(\sum_{q=\ell-1}^{n}\left[\frac{(2(n-q)+1)!!}{(2(n-q)+2)!!}\right]^2 \frac{n-q+1}{n-q+2}\alpha_{q,\ell-1}\right) m^{\ell} \\
&= \left(\frac{1}{n+2}\left[\frac{(2n+1)!!}{(2n+2)!!}\right]^2 - \frac{1}{n+1}\sum_{q=1}^{n}\left[\frac{(2(n-q)+1)!!}{(2(n-q)+2)!!}\right]^2 \frac{q}{n-q+2}\alpha_{q,1}\right) m \\
&\quad + \frac{1}{n+1}\sum_{\ell=2}^{n}\left(\sum_{q=\ell-1}^{n}\left[\frac{(2(n-q)+1)!!}{(2(n-q)+2)!!}\right]^2 \frac{n-q+1}{n-q+2}\alpha_{q,\ell-1}\right. \\
&\quad \left. - \sum_{q=\ell}^{n}\left[\frac{(2(n-q)+1)!!}{(2(n-q)+2)!!}\right]^2 \frac{q}{n-q+2}\alpha_{q,\ell}\right) m^{\ell} + \frac{\alpha_{n,n}}{8(n+1)} m^{n+1}.
\end{aligned}
$$

Hence, we acquire

$$
\begin{aligned}
\alpha_{n+1,1} &= \frac{1}{n+2}\left[\frac{(2n+1)!!}{(2n+2)!!}\right]^2 - \frac{1}{n+1}\sum_{q=1}^{n}\left[\frac{(2(n-q)+1)!!}{(2(n-q)+2)!!}\right]^2 \frac{q}{n-q+2}\alpha_{q,1}, \\
\alpha_{n+1,\ell} &= \frac{1}{n+1}\left(\sum_{q=\ell-1}^{n}\left[\frac{(2(n-q)+1)!!}{(2(n-q)+2)!!}\right]^2 \frac{n-q+1}{n-q+2}\alpha_{q,\ell-1} - \sum_{q=\ell}^{n}\left[\frac{(2(n-q)+1)!!}{(2(n-q)+2)!!}\right]^2 \frac{q}{n-q+2}\alpha_{q,\ell}\right)
\end{aligned}
$$

for $2 \le \ell \le n$, and

$$
\alpha_{n+1,n+1} = \frac{\alpha_{n,n}}{8(n+1)}. \tag{15}
$$

Consecutively recursing (15) gives

$$
\alpha_{n,n} = \frac{1}{8n}\frac{1}{8(n-1)}\frac{1}{8(n-2)}\cdots\frac{\alpha_{1,1}}{8\times 2} = \frac{1}{2^{3n}n!}, \quad n \ge 1. \tag{16}
$$

Making use of the explicit formula (16) leads to

$$
\begin{aligned}
\alpha_{n+1,n} &= \frac{1}{n+1}\left(\frac{1}{8}\alpha_{n,n-1} + \frac{3}{32}\alpha_{n-1,n-1} - \frac{n}{8}\alpha_{n,n}\right) \\
&= \frac{1}{n+1}\left[\frac{1}{8}\alpha_{n,n-1} + \frac{3}{32}\frac{1}{8^{n-1}(n-1)!} - \frac{n}{8}\frac{1}{8^n n!}\right] \\
&= \frac{1}{8(n+1)}\left[\alpha_{n,n-1} + \frac{5}{8^n(n-1)!}\right].
\end{aligned}
$$

Consequently, we conclude

$$
\alpha_{n,n-1} = \frac{5}{2^{3n+1}(n-2)!}, \quad n \ge 2. \tag{17}
$$

Employing (16) and (17) results in

$$\begin{aligned}\alpha_{n+1,n-1} &= \frac{1}{8(n+1)}\left[\alpha_{n,n-2} + \frac{3}{4}\alpha_{n-1,n-2} + \frac{75}{128}\alpha_{n-2,n-2} - n\alpha_{n,n-1} - \frac{3(n-1)}{8}\alpha_{n-1,n-1}\right]\\ &= \frac{1}{8(n+1)}\left[\alpha_{n,n-2} + \frac{25n+9}{2^{3n+1}(n-2)!}\right],\end{aligned}$$

from which we can derive

$$\alpha_{n,n-2} = \frac{1}{2^{3n+3}}\frac{75n^3 - 214n^2 + 117n + 22}{3(n-1)!}, \quad n \ge 3.$$

The proof of Theorem 2 is complete. □

## 4. Several remarks and two open problems

In this section, we list several remarks on our main results and pose two open problems.

### 4.1. Several remarks

**Remark 1.** The specific values of partial Bell polynomials have been investigated in the papers [5, 8, 12, 15] and many related references cited therein.

**Remark 2.** The first few values of $\alpha_{n,\ell}$ defined in (14) are

$$\begin{aligned}
&\alpha_{1,1} = \frac{1}{8};\\
&\alpha_{2,1} = \frac{5}{128}, \quad \alpha_{2,2} = \frac{1}{128};\\
&\alpha_{3,1} = \frac{59}{3072}, \quad \alpha_{3,2} = \frac{5}{1024}, \quad \alpha_{3,3} = \frac{1}{3072};\\
&\alpha_{4,1} = \frac{47}{4096}, \quad \alpha_{4,2} = \frac{311}{98304}, \quad \alpha_{4,3} = \frac{5}{16384}, \quad \alpha_{4,4} = \frac{1}{98304};\\
&\alpha_{5,1} = \frac{2507}{327680}, \quad \alpha_{5,2} = \frac{859}{393216}, \quad \alpha_{5,3} = \frac{193}{786432}, \quad \alpha_{5,4} = \frac{5}{393216}, \quad \alpha_{5,5} = \frac{1}{3932160}.
\end{aligned}$$

### 4.2. The first open problem

In [19, p. 109], the following special cases of Gauss' hypergeometric function ${}_2F_1(\alpha,\beta;\gamma;z)$ are listed:

$$\begin{aligned}
&{}_2F_1(a,b;b;z) = \frac{1}{(1-z)^a}, && {}_2F_1(1,1;2;z) = -\frac{\ln(1-z)}{z},\\
&{}_2F_1\left(\frac{1}{2},1;\frac{3}{2};z^2\right) = \frac{1}{2z}\ln\frac{1+z}{1-z}, && {}_2F_1\left(\frac{1}{2},1;\frac{3}{2};-z^2\right) = \frac{\arctan z}{z},\\
&{}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};z^2\right) = \frac{\arcsin z}{z}, && {}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};-z^2\right) = \frac{\ln\left(z+\sqrt{1+z^2}\right)}{z}.
\end{aligned}$$

Lemma 2.6 in the paper [18] reads that, for $0 \ne |t| < 1$ and $n = 1, 2, \dots$,

$${}_2F_1\left(\frac{1-n}{2}, \frac{2-n}{2}; 1-n; \frac{1}{t^2}\right) = \frac{t}{2^n\sqrt{t^2-1}}\left[\left(1+\frac{\sqrt{t^2-1}}{t}\right)^n - \left(1-\frac{\sqrt{t^2-1}}{t}\right)^n\right].$$

Corollary 4.1 in the paper [17] states that, for $n = 0, 1, 2, \dots$,

$${}_2F_1\left(n+\frac{1}{2}, n+1; n+\frac{3}{2}; -1\right) = \frac{(2n+1)!!}{(2n)!!}\frac{\pi}{4} + \frac{2n+1}{2^{2n}}\sum_{k=1}^{n}(-1)^k\binom{2n-k}{n}\frac{2^{k/2}}{k}\sin\frac{3k\pi}{4}.$$

In the paper [1], see also [14, Section 6], the following formula was discussed and obtained:

$$ {}_2F_1\left(1,2;\frac{1}{2};\frac{z}{4}\right)=\frac{2(z+8)}{(4-z)^2}+\frac{24\sqrt{z}}{(4-z)^{5/2}}\arcsin\frac{\sqrt{z}}{2},\quad |z|<4. $$

Motivated by the proof of Theorem 1 and the above examples, we now pose a problem: can one find an elementary function $f(t)$ such that

$$ {}_2F_1\left(\frac{1}{2},\frac{1}{2};2;t\right)=f(t),\quad |t|\le 1? \tag{18} $$

This problem was first announced at `https://mathoverflow.net/q/423800`. Currently, this problem is still open. However, Professor Emeritus Gerald A. Edgar (Ohio State University, `https://stackexchange.com/users/503194/gerald-edgar`) answered to this problem at `https://mathoverflow.net/a/423802` on 1 June 2022 as follows.

> Maple does it in terms of complete elliptic integrals $K$ and $E$ as
>
> $$ {}_2F_1\left(\frac{1}{2},\frac{1}{2};2;t\right)=\frac{4}{\pi}\frac{(t-1)K(\sqrt{t}\,)+E(\sqrt{t}\,)}{t}. \tag{19} $$
>
> But that does not show it is elementary. In fact, I suspect it is not elementary.
>
> Recall the known formulas
>
> $$ K(\sqrt{t}\,)=\frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;t\right) $$
>
> and
>
> $$ E(\sqrt{t}\,)=\frac{\pi}{2}\,{}_2F_1\left(-\frac{1}{2},\frac{1}{2};1;t\right). $$
>
> By themselves, they are not elementary. The equation (19) should follow from these two and a contiguous formula
>
> $$ c\,{}_2F_1(a-1,b;c;x)+c(x-1)\,{}_2F_1(a,b;c;x)+(b-c)x\,{}_2F_1(a,b;c+1;x)=0 $$
>
> for the hypergeometrics.

We believe that Edgar's suspect, that is, the function $f(t)$ in the equation (18) is not elementary, should be true.

### *4.3. The second open problem*

Can one establish a general and closed-form formula for the sequence $\alpha_{n,\ell}$ generated in (14).

If this problem were solved, then we would obtain a general and closed-form formula for coefficients $b_{m,j}$ in the series expansion (12) of powers of Gauss' hypergeometric function ${}_2F_1\left(\frac{1}{2},\frac{1}{2};2;t\right)$. Then it would be very interesting and significant in mathematics. However, basing on Edgar's suspect mentioned above, we suspect that there is no general and closed-form expression for the sequence $\alpha_{n,\ell}$. For more information on series expansions of powers of functions, please refer to the papers [5, 6, 7, 11, 13, 16] and many references cited therein.

**Remark 3.** On 7 October 2022, Frank Oertel (f.oertel@email.de) commented on the problem (18) in an e-mail to the third author as follows.

Regarding your problem (18) (which also gives me a headache...), please observe that at least the following fact holds:

$$\frac{1}{\pi}\int_0^{2\pi}\arcsin(x\cos t)\cos t\,\mathrm{d}\,t = x\,{}_2F_1\left(\frac{1}{2},\frac{1}{2},2;x^2\right)$$

for all $x \in [-1,1]$, implied by the Maclaurin series representation of the function arcsin and the well-known fact that

$$\int_0^{2\pi}\cos^{2(n+1)} t\,\mathrm{d}\,t = \frac{2\sqrt{\pi}}{(n+1)!}\Gamma\left(n+\frac{3}{2}\right) = \frac{2\sqrt{\pi}}{\Gamma(n+2)}\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)$$

for all $n \in \{0\} \cup \mathbb{N}$.

(Cf. also related tricky proofs in `https://zbmath.org/?q=an%3A0646.46019`.)

**Availability of data and material** Data sharing is not applicable to this article as no new data were created or analyzed in this study.

**Competing interests** The authors declare that they have no any conflict of competing interests.

**Authors' contributions** All authors contributed equally to the manuscript and read and approved the final manuscript.

**Funding** The first author Wei-Shih Du was partially supported by Grant No. MOST 111-2115-M-017-002 of the Ministry of Science and Technology of the Republic of China. The corresponding author Dongkyu Lim was partially supported by the National Research Foundation of Korea under Grant NRF-2021R1C1C1010902, Republic of Korea.

**Acknowledgements** The authors appreciate anonymous referees for their careful reading and valuable comments on the original version of this paper.

## References

[1] T. Amdeberhan, X. Guan, L. Jiu, V. H. Moll, and C. Vignat, *A series involving Catalan numbers: Proofs and demonstrations*, Elem. Math. **71** (2016), no. 3, 109–121; available online at `https://doi.org/10.4171/EM/306`.

[2] C. A. Charalambides, *Enumerative Combinatorics*, CRC Press Series on Discrete Mathematics and its Applications, Chapman & Hall/CRC, Boca Raton, FL, 2002.

[3] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, Revised and Enlarged Edition, D. Reidel Publishing Co., 1974; available online at `https://doi.org/10.1007/978-94-010-2196-8`.

[4] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Translated from the Russian, Translation edited and with a preface by Daniel Zwillinger and Victor Moll, Eighth edition, Revised from the seventh edition, Elsevier/Academic Press, Amsterdam, 2015; available online at `https://doi.org/10.1016/B978-0-12-384933-5.00013-8`.

[5] B.-N. Guo, D. Lim, and F. Qi, *Maclaurin's series expansions for positive integer powers of inverse (hyperbolic) sine and tangent functions, closed-form formula of specific partial Bell polynomials, and series representation of generalized logsine function*, Appl. Anal. Discrete Math. **17** (2023), no. 1, in press; available online at `https://doi.org/10.2298/AADM210401017G`.

[6] B.-N. Guo, D. Lim, and F. Qi, *Series expansions of powers of arcsine, closed forms for special values of Bell polynomials, and series representations of generalized logsine functions*, AIMS Math. **6** (2021), no. 7, 7494–7517; available online at `https://doi.org/10.3934/math.2021438`.

[7] Y. Hong, B.-N. Guo, and F. Qi, *Determinantal expressions and recursive relations for the Bessel zeta function and for a sequence originating from a series expansion of the power of modified Bessel function of the first kind*, CMES Comput. Model. Eng. Sci. **129** (2021), no. 1, 409–423; available online at `https://doi.org/10.32604/cmes.2021.016431`.

[8] S. Jin, B.-N. Guo, and F. Qi, *Partial Bell polynomials, falling and rising factorials, Stirling numbers, and combinatorial identities*, CMES Comput. Model. Eng. Sci. **132** (2022), no. 3, 781–799; available online at `https://dx.doi.org/10.32604/cmes.2022.019941`.

[9] Y. L. Luke, *The Special Functions and Their Approximations*, Vol. I, Mathematics in Science and Engineering, Vol. 53, Academic Press, New York-London, 1969.

[10] F. Oertel, *Grothendieck's inequality and completely correlation preserving functions—a summary of recent results and an indication of related research problems*, arXiv (2020), available online at https://arxiv.org/abs/2010.00746v2.

[11] F. Qi, *Explicit formulas for partial Bell polynomials, Maclaurin's series expansions of real powers of inverse (hyperbolic) cosine and sine, and series representations of powers of Pi*, Research Square (2021), available online at https://doi.org/10.21203/rs.3.rs-959177/v3.

[12] F. Qi, *Taylor's series expansions for real powers of two functions containing squares of inverse cosine function, closed-form formula for specific partial Bell polynomials, and series representations for real powers of Pi*, Demonstr. Math. **55** (2022), no. 1, in press; available online at https://doi.org/10.1515/dema-2022-0157.

[13] F. Qi, C.-P. Chen, and D. Lim, *Several identities containing central binomial coefficients and derived from series expansions of powers of the arcsine function*, Results Nonlinear Anal. **4** (2021), no. 1, 57–64; available online at https://doi.org/10.53006/rna.867047.

[14] F. Qi and B.-N. Guo, *Integral representations of the Catalan numbers and their applications*, Mathematics **5** (2017), no. 3, Article 40, 31 pages; available online at https://doi.org/10.3390/math5030040.

[15] F. Qi, D.-W. Niu, D. Lim, and B.-N. Guo, *Closed formulas and identities for the Bell polynomials and falling factorials*, Contrib. Discrete Math. **15** (2020), no. 1, 163–174; available online at https://doi.org/10.11575/cdm.v15i1.68111.

[16] F. Qi and P. Taylor, *Several series expansions for real powers and several formulas for partial Bell polynomials of sinc and sinhc functions in terms of central factorial and Stirling numbers of second kind*, arXiv (2022), available online at https://arxiv.org/abs/2204.05612v4.

[17] F. Qi and M. D. Ward, *Closed-form formulas and properties of coefficients in Maclaurin's series expansion of Wilf's function composited by inverse tangent, square root, and exponential functions*, arXiv (2022), available online at https://arxiv.org/abs/2110.08576v2.

[18] F. Qi, Q. Zou, and B.-N. Guo, *The inverse of a triangular matrix and several identities of the Catalan numbers*, Appl. Anal. Discrete Math. **13** (2019), no. 2, 518–541; available online at https://doi.org/10.2298/AADM190118018Q.

[19] N. M. Temme, *Special Functions: An Introduction to Classical Functions of Mathematical Physics*, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1996; available online at http://dx.doi.org/10.1002/9781118032572.