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ON SOME DIFFERENTIAL PROPERTIES OF FUNCTIONS IN GENERALIZED GRAND SOBOLEV-MORREY SPACES

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Abstract. In this paper we introduce a generalized grand Sobolev-Morrey spaces. Some differential and differential-difference properties of functions from this spaces are proved by means of the integral representation.

1. Introduction and Preliminary Notes

Note that the grand Lebesgue spaces $L_{p)}(G)$ $(|G| < \infty)$ introduced in [4] by T. Iwaniec and C. Sbordone. After a vast amount of research about grand Lebesgue, small Lebesgue, grand Lebesgue-Morrey, grand grand Lebesgue-Morrey, grand grand Sobolev-Morrey, small small Sobolev-Morrey, grand grand Nikolskii Morrey and generalized grand Lebesgue-Morrey spaces has been introduced and studied by many mathematicians (see, e.g. [2,3], [5]- [14]) etc.

In this paper we construct a generalized grand Sobolev-Morrey spaces $W^{l}_{p),\phi}(G)$ and we study some differential properties with help of the method of integral representation of functions in view of embedding theory. Let $G \subset \mathbb{R}^n$ and $B \subset G$ be any Lebesgue measurable set, $l \in \mathbb{N}^n$, $p \in [1, \infty)$, and let $\phi(\cdot, |B|)$ be a function on $[0, p-1)$ which is a positive bounded and satisfies $\phi(0, |B|) = \phi(|B|)$. And also $\phi(\varepsilon, \cdot)$ is a positive bounded function defined on $(0, h_0]$ and h_0 is a fixed positive number.

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Definition 1. *Denote by $W^l_{p),\phi}(G)$ a space of locally summable functions f on G having the generalized derivatives $D^{l_i}_i f$ ($l_i > 0$ are integers $i = 1, 2, \ldots, n$) with the finite norm*

$$\|f\|_{W^l_{p),\phi}(G)} = \|f\|_{p),\phi,G} + \sum_{i=1}^{n} \left\| D^{l_i}_i f \right\|_{p),\phi,G}, \tag{1}$$

where

$$\|f\|_{p),\phi;G} = \|f\|_{L_{p),\phi}(G)} = \sup_{\substack{0 \le \varepsilon < p-1, \\ B \subset G}} \phi(\varepsilon, |B|) \left(\int_B |f(x)|^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}}. \tag{2}$$

Here $|B|$ is the Lebesgue measure of B.

Note that

(1) If $\phi(\varepsilon, |B|) = \left(\frac{\varepsilon^\theta}{|B|^{\lambda+n}}\right)^{\frac{1}{p-\varepsilon}}, \theta > 0$, then $L_{p),\phi}(G) \equiv L^\theta_{p),\lambda}(G)$; in case $\theta = 1$, then $L_{p),\phi}(G) \equiv L_{p),\lambda}(G)$;

(2) If $\phi(\varepsilon, |B|) = \left(\frac{\varepsilon^\theta}{|B|^n}\right)^{\frac{1}{p-\varepsilon}}, \theta > 0$, then $L_{p),\phi}(G) \equiv L^\theta_{p)}(G)$; in case $\theta = 1, L_{p),\phi}(G) \equiv L_{p)}(G)$; $(B \equiv B(x, r))$;

(3) If $\phi(\varepsilon, |B|) = \left(\frac{\varepsilon^\theta}{r^{|\chi|a+|\chi|}}\right)^{\frac{1}{p-\varepsilon}}, \theta > 0$, then $L_{p),\phi}(G) \equiv L^\theta_{p),\chi,a}(G)$; in case $\theta = 1$, then $L_{p),\phi}(G) \equiv L_{p),\chi,a}(G)$;

(4) If $\phi(\varepsilon, |B|) = \left(\frac{1}{|B|^\lambda}\right)^{\frac{1}{p}}$, then $L_{p),\phi}(G) \equiv L_{1,\lambda}(G)$;

Observe some properties of $L_{p),\phi}(G)$ and $W^l_{p),\phi}(G)$.

(1) The following embedding hold:

$$L_{p),\phi}(G) \to L_{p)}(G), W^l_{p),\phi}(G) \to W^l_{p)}(G),$$

i.e.,
$\|f\|_{p);G} \le \|f\|_{p);\phi;G}$ and $\|f\|_{W^l_{p)}(G)} \le \|f\|_{W^l_{p),\phi}(G)}$
where

$$\|f\|_{W^l_{p)}G} \le \|f\|_{p),G} + \sum_{i=1}^{n} \left\| D^{l_i}_i f \right\|_{p),G}$$

$$\|f\|_{p),G} = \|f\|_{L_{p)}(G)} = \sup_{0 \le \varepsilon < p-1,} \phi(\varepsilon, |G|) \left(\int_G |f(x)|^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}}.$$

Indeed,

$$\|f\|_{p),\phi,G} = \sup_{\substack{0 \le \varepsilon < p-1 \\ B \subset G}} \phi(\varepsilon, |B|) \left(\int_G |f(x)|^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}}$$

$$\geq \sup_{0\leq\varepsilon<p-1} \phi(\varepsilon,|G|)\left(\int_G |f(x)|^{p-\varepsilon}\,dx\right)^{\frac{1}{p-\varepsilon}} = \|f\|_{p),G}\,.$$

(1) $L_{p),\phi}(G)$ and $W^l_{p),\phi}(G)$ are complete.

The proof of completeness properties of these spaces is similar to [10].

It can be shown that, for $f \in W^l_{p-\varepsilon}(G)$ has the integral representation $(x \in U \subset G)$

$$D^\nu f(x) = f_h^{(\nu)}(x) + \sum_{i=1}^n \int_0^h \int_{R^n} L_i^{(\nu)}\left(\frac{y}{\psi(v)}\right) D_i^{l_i} f(x+y) \times$$

$$\times \prod_{j=1}^n \left(\psi_j(v)\right)^{-1-\nu_j} \left(\psi_i(v)\right)^{-1+l_i} \psi_i'(v)\,dydv, \tag{3}$$

$$f_h^{(\nu)}(x) = \prod_{j=1}^n \left(\psi_j(v)\right)^{-1-\nu_j} \int_{R^n} f(x+y)\,\Omega^{(\nu)}\left(\frac{y}{\psi(h)}\right) dy, \tag{4}$$

where $\frac{y}{\psi(v)} = \left(\frac{y_1}{\psi_1(v)}, \frac{y_2}{\psi_2(v)}, \ldots, \frac{y_n}{\psi_n(v)}\right)$, $\psi_i(v) = \;\; i = 1,2,\ldots,n$ is arbitrary differentiable non-decreasing functions defined for $0 < v \leq h \leq h_0$, $\lim\limits_{v\to+0} \psi_i(v) = 0$, $L_i(\cdot)\,,\,\Omega(\cdot) \in C_0^\infty(R^n)\,S(M) = \;\operatorname{supp} M \subset I_{\psi(h_0)} = \left\{y : |y_j| < \psi_j(h_0)\,, j = 1,2,\ldots,n\right\}$ and the ψ horn $x + V = x + \bigcup_{0<h\leq h_0}\left\{y : \frac{y}{\psi(h)} \in S(\Omega)\right\}$ is the support of the representation (3), (4) and $\nu = (\nu_1,\ldots,\nu_n)\,,\,\nu_j \geq 0$ are integers $(j = 1,2,\ldots,n)$.

Lemma 1. *Let $1 < p < q \leq r \leq \infty$, $0 < \eta, v \leq h \leq h_0$, $\nu = (\nu_1,\ldots,\nu_n)\,,\,\nu_j \geq 0$ be integers $(j = 1,2,\ldots,n)\,,\;\; \varphi \in L_{p),\phi}(G)$ and*

$$R_\eta^i = \int_0^\eta \prod_{j=1}^n \left(\psi_j(v)\right)^{-\nu_j - \frac{1}{p-\varepsilon} + \frac{1}{q-\varepsilon}} \left(\psi_i(v)\right)^{-1+l_i} \psi_i'(v)\,dv, \tag{5}$$

$$A_\eta^i(x) = \int_0^\eta \prod_{j=1}^n \left(\psi_j(v)\right)^{-1-\nu_j} \left(\psi_i(v)\right)^{-1+l_i} \psi_i^1(v) \int_{R^n} \varphi(x+y)\,K\left(\frac{y}{\psi(v)}\right) dydv, \tag{6}$$

$$A_{\eta,h}^i(x) = \int_\eta^h \prod_{j=1}^n \left(\psi_j(v)\right)^{-1-\nu_j} \left(\psi_i(v)\right)^{-1+l_i} \psi_i'(v) \int_{R^n} \varphi(x+y)\,K\left(\frac{y}{\psi(v)}\right) dydv. \tag{7}$$

Then

$$\left\|A_\eta^i\right\|_{q-\varepsilon,U} \leq c_1 \|\varphi\|_{p),\phi,G} \left(\phi(\varepsilon,|U|)\right)^{-\frac{p-\varepsilon}{q-\varepsilon}} \left(\phi(\varepsilon,|B|)\right)^{-1+\frac{p-\varepsilon}{q-\varepsilon}} \left|R_\eta^i\right| \;\; \left(R_\eta^i < \infty\right) \tag{8}$$

$$\left\|A_{\eta\,h}^i\right\|_{q-\varepsilon,U} \leq c_2 \|\varphi\|_{p),\phi,G} \left(\phi(\varepsilon,|U|)\right)^{-\frac{p-\varepsilon}{q-\varepsilon}} \left(\phi(\varepsilon,|B|)\right)^{-1+\frac{p-\varepsilon}{q-\varepsilon}} \left|R_{\eta,h}^i\right|, \tag{9}$$

where $R_{\eta,h}^i = \int_\eta^h \prod_{j=1}^n \left(\psi_j(v)\right)^{-\nu_j - \frac{1}{p-\varepsilon} + \frac{1}{q-\varepsilon}} \left(\psi_i(v)\right)^{-1+l_i} \psi_i'(v)\,dv$, and U is an open set containing in the domain G.

Proof. Applying the generalized Minkowski inequality, we deduce

$$\left\|A_\eta^i\right\|_{q-\varepsilon,U} \leq \int_0^\eta \prod_{j=1}^n \left(\psi_j\left(\upsilon\right)\right)^{-1-\nu_j} \left(\psi_i\left(\upsilon\right)\right)^{-1+l_i} \psi_i'\left(\upsilon\right) \left\|F\left(\cdot,\upsilon\right)\right\|_{q-\varepsilon,U} d\upsilon, \quad (10)$$

for every

$$F\left(x,\upsilon\right) = \int_{R^n} \varphi\left(x+y\right) K\left(\frac{y}{\psi(\upsilon)}\right) dy. \quad (11)$$

Estimate of the norm $\left\|F\left(\cdot,\upsilon\right)\right\|_{q-\varepsilon,U}$. From Hölders inequality $(q \leq r)$ we obtain

$$\left\|F\left(\cdot,\upsilon\right)\right\|_{q-\varepsilon,U} \leq \left\|F\left(\cdot,\upsilon\right)\right\|_{r-\varepsilon,U} \left|U\right|^{\frac{1}{q-\varepsilon}-\frac{1}{r-\varepsilon}}. \quad (12)$$

Let X be the characteristic function of $S\left(K\right)$. It is obvious that

$$\left\|\varphi K\right\| = \left(\left|\varphi\right|^{p-\varepsilon} \left|K\right|^s\right)^{\frac{1}{r-\varepsilon}} \left(\left|\varphi\right|^{p-\varepsilon} X\right)^{\frac{1}{p-\varepsilon}-\frac{1}{r-\varepsilon}} \left(\left|K\right|^s\right)^{\frac{1}{s}-\frac{1}{r-\varepsilon}},$$

where $\frac{1}{s} = 1 - \frac{1}{p-\varepsilon} + \frac{1}{r-\varepsilon}$.
And applying again Hölders inequality
$\left(\frac{1}{r-\varepsilon} + \left(\frac{1}{p-\varepsilon} - \frac{1}{r-\varepsilon}\right) + \left(\frac{1}{s} - \frac{1}{r-\varepsilon}\right) = 1\right)$ we have

$$\left\|F\left(\cdot,\upsilon\right)\right\|_{r-\varepsilon,U} \leq \sup_{x\in U} \left(\int_{R^n} \left|\varphi\left(x+y\right)\right|^{p-\varepsilon} X\left(\frac{y}{\psi}\right) dy\right)^{\frac{1}{p-\varepsilon}-\frac{1}{r-\varepsilon}}$$

$$\times \sup_{y\in\upsilon} \left(\int_U \left|\varphi\left(x+y\right)\right|^{p-\varepsilon} dx\right)^{\frac{1}{r-\varepsilon}} \left(\int_{R^n} \left|K\left(\frac{y}{\psi}\right)\right|^s dy\right)^{\frac{1}{s}}. \quad (13)$$

For every $x \in U$ we have

$$\int_{R^n} \left|\varphi\left(x+y\right)\right|^{p-\varepsilon} X\left(\frac{y}{\psi}\right) dy \leq \int_{I_{\psi(\upsilon)}} \left|\varphi\left(x+y\right)\right|^{p-\varepsilon} dy \leq \left\|\varphi\right\|_{p-\varepsilon,I_{\psi(\upsilon)}}^{p-\varepsilon}$$

$$\leq \left\|\varphi\right\|_{p),\phi,G}^{p-\varepsilon} \left(\phi\left(\varepsilon,\left|I_{\psi(\upsilon)}\right)\right)\right|^{-(p-\varepsilon)}. \quad (14)$$

For

$$\int_U \left|\varphi\left(x+y\right)\right|^{p-\varepsilon} dx \leq \left\|\varphi\right\|_{p-\varepsilon,U}^{p-\varepsilon} \leq \left\|\varphi\right\|_{p),\phi,U}^{p-\varepsilon} \left|\phi\left(\varepsilon,\left|U\right|\right)\right|^{-(p-\varepsilon)}$$

$$\leq \left\|\varphi\right\|_{p),\phi,G}^{p-\varepsilon} \left(\phi\left(\varepsilon,\left|U\right|\right)\right)^{-(p-\varepsilon)}, \quad (15)$$

$$\int_{R^n} \left|K\left(\frac{y}{\psi}\right)\right|^s dy = \prod_{j=1}^n \psi_j\left(\upsilon\right) \left\|K\right\|_s^s. \quad (16)$$

It follows from (12)-(16) for $r = q$ that

$$\left\|F\left(\cdot,\upsilon\right)\right\|_{q-\varepsilon,U} \leq \left\|\varphi\right\|_{p),\phi,G} \left|\phi\left(\varepsilon,\left|I_{\psi(\upsilon)}\right|\right)\right|^{-1+\frac{p-\varepsilon}{q-\varepsilon}} \phi\left(\varepsilon,\left|U\right|\right)^{-\frac{p-\varepsilon}{q-\varepsilon}} \left\|K\right\|_s \left|\psi\left(\upsilon\right)\right|^{\frac{1}{s}}. \quad (17)$$

Unseating this inequality in (10) we have

$$\left\|A_\eta^i\right\|_{q-\varepsilon,U} \leq c \left\|\varphi\right\|_{p),\phi,G} \left(\phi\left(\varepsilon,\left|U\right|\right)\right)^{-\frac{p-\varepsilon}{q-\varepsilon}} \left(\phi\left(\varepsilon,\left|B\right|\right)\right)^{-1+\frac{p-\varepsilon}{q-\varepsilon}} \left|R_\eta^i\right| \left(R_\eta^i < \infty\right) \quad (18)$$

□

2. Main Results

Now we will prove two theorems on the properties of the functions from spaces $W^l_{p),\phi}(G)$.

Theorem 1. *Let $G \subset R^n$ be an open set such that it satisfies the horn condition, $1 \le p < \infty$, $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$, $\nu_j \ge 0$ be integers $(j = 1, 2, \ldots, n)$, $R^i_h < \infty$ $(i = 1, 2, \ldots, n)$ and $f \in W^l_{p),\phi}(G)$.*

Then $D^\nu : W^l_{p),\phi}(G) \to L_{q-\varepsilon}(G)$ holds for any $\varepsilon\,(0 \le \varepsilon < p-1)$.

Moreover, the following inequality is valid

$$\|D^\nu f\|_{q-\varepsilon,G} \le c(\varepsilon)\left(\|f\|_{p),\phi;G} + \sum_{i=1}^n \left|R^i_h\right| \left\|D_i^{l_i} f\right\|_{p),\phi;G}\right). \tag{19}$$

In particular, if

$$R^{i,0}_h = \int_0^h \prod_{j=1}^n \left(\psi_j(v)\right)^{-\nu_j - \frac{1}{p-\varepsilon}} \left(\psi_i(v)\right)^{-1+l_i} \psi_i'(v)\,dv < \infty,$$

$i = 1, 2, \ldots, n$, then $D^\nu f(x)$ is continuous on G and

$$\sup_{x\in G} \|D^\nu f(x)\| \le c(\varepsilon)\left(\|f\|_{p),\phi;G} + \sum_{i=1}^n \left|R^{i,0}_h\right| \left\|D_i^{l_i} f\right\|_{p),\phi;G}\right), \tag{20}$$

where $0 < h \le h_0$, h_0 is fixed positive number, $c(\varepsilon) = C \cdot (\phi(\varepsilon, |B|))^{-1+\frac{p-\varepsilon}{q-\varepsilon}}$ and C is a constant independent of f, h and ε.

Proof. Under the conditions of our theorem, the weak derivatives $D^\nu f$ exists. Since $p < q$ and $W^l_{p),\phi}(G) \to W^l_{p)}(G) \to W^l_{p-\varepsilon}(G)$ $(p - \varepsilon > 1)$. Then $D^\nu f$ exists on G (for all $B \subseteq I_{\psi(h_0)} \subset G$) has the integral representation

$$\begin{gathered} D^\nu f(x) = f_h^{(\nu)}(x) + \textstyle\sum_{i=1}^n \int_0^h \int_{R^n} L_i^{(\nu)}\left(\frac{y}{B}\right) \times \\ \times D_i^{l_i} f(x+y) \prod_{j=1}^n \left(\psi_j(v)\right)^{-1-\nu_j} \left(\psi_i(v)\right)^{-1-\nu_i} \psi_i'(v)\,dvdy, \end{gathered} \tag{21}$$

where

$$f_h^{(\nu)}(x) = \prod_{j=1}^n \left(\psi_j(h)\right)^{-1-\nu_j} \int_{R^n} f(x+y)\,\Omega^{(\nu)}\left(\frac{y}{B}\right) dy, \tag{22}$$

$0 < h \le h_0$, L_i and $\Omega \in C_0^\infty(R^n)$, $i = 1, 2, \ldots, n$, and $\frac{y}{B} = \left(\frac{y_1}{\left|B^{(1)}\right|}, \frac{y_2}{\left|B^{(2)}\right|}, \ldots, \frac{y_n}{\left|B^{(n)}\right|}\right)$, $B^{(i)} = \left\{x : x = \left(x_1^0, x_2^0, \ldots, x_i^0, x_i, x_{i+1}^0, \ldots x_n^0\right)\right\}$ i.e., $B^{(i)} = proj_{x_i} B$. The representation (21), (22) carrier is contained in the set $x + V \subset G$. Hence, using Minkowski's inequality we arrive

$$\|D^{\nu} f\|_{q-\varepsilon,G} \leq \left\|f_h^{(\nu)}\right\|_{q-\varepsilon,G} + \sum_{i=1}^{n} \left\|F_h^l\right\|_{q-\varepsilon,G}. \tag{23}$$

By (17) for $U = G,\ \varphi = f,\ K = \Omega^{(v)}, I_{\psi(h)} = B$, we have

$$\left\|f_h^{(\nu)}\right\|_{q-\varepsilon,G} \leq c\,\|f\|_{p),\phi,G}\,|\phi(\varepsilon,|B|)|^{-1+\frac{p-\varepsilon}{q-\varepsilon}}\,|\phi(\varepsilon,|U|)|^{-\frac{p-\varepsilon}{q-\varepsilon}} \leq c_1(\varepsilon)\,\|f\|_{p),\phi,G}$$

By (8) for $U = G,\ \varphi = D_i^{l_i} f,\ K = L_i^{(\nu)}, I_{\psi(v)} = B$, $\eta = h$ we have

$$\left\|F_h^i\right\|_{q-\varepsilon,G} \leq c(\varepsilon)\left\|D_i^{l_i} f\right\|_{p),\phi,G}\left|R_h^i\right|.$$

Consequently,

$$\|D^{\nu} f\|_{q-\varepsilon,G} \leq C(\varepsilon)\left(\|f\|_{p),\phi;G} + \sum_{i=1}^{n}\left|R_h^i\right|\left\|D_i^{l_i} f\right\|_{p),\phi;G}\right). \tag{24}$$

Now let

$$R_{h,0}^i = \int_0^h \prod_{j=1}^{n}\left(\psi_j(v)\right)^{-\nu_j-\frac{1}{p-\varepsilon}}\left(\psi_i(v)\right)^{-1+l_i}\psi_i'(v)\,dv < \infty\,(i = 1,2,\ldots,n).$$

We show that $D^v f$ is continuous on G. By (23) and (24) for $q = \infty$ we obtain:

$$\left\|D^{\nu} f - f_h^{(\nu)}\right\|_{\infty,G} \leq C(\varepsilon)\sum_{i=1}^{n}\left|R_h^i\right|\left\|D_i^{l_i} f\right\|_{p),\phi;G}.$$

It follows that the left-hand part of the last inequality tends to zero as $h \to 0$. Since $f_h^{(\nu)}$ is continuous on G, in our case the convergence in $L_{\infty}(G)$ coincides with uniform convergence; consequently, $D^{\nu} f$ is continuous on G.

Thus the theorem is proved. □

Let γ be an n dimensional vector.

Theorem 2. *Suppose that the domain G the parameters p, q and vector v satisfy the condition of Theorem 1. If $R_h^i < \infty$ $(i = 1,2,\ldots,n)$, then $D^v f$ satisfies the Hölder condition on G in the metric of $L_{q-\varepsilon}$, more exactly*

$$\|\Delta(\gamma,G)\,D^{\nu} f\|_{q-\varepsilon,G} \leq c(\varepsilon)\,\|f\|_{W^l_{p),\phi}(G)}\left|R_{h,\gamma}^1\right|. \tag{25}$$

If $R_h^i < \infty$ $(i = 1,2,\ldots,n)$, then

$$\sup_{x\in G}\|\Delta(\gamma,G)\,D^{\nu} f(x)\| \leq c(\varepsilon)\,\|f\|_{W^l_{p),\phi}(G)}\left|R_{h,\gamma}^{1,0}\right|, \tag{26}$$

where

$$R_{h,\gamma}^1 = \max_i\left\{|\gamma|,\ |\gamma|\,|R_h^i|,\ \ |\gamma|\,|R_{h,\gamma}^i|\right\}$$

and

$$R^{1,0}_{h,\gamma} = \max_i \left\{ |\gamma|\,, \ |\gamma|\,|R^{i,0}_{h}|, \ \ |\gamma|\,|R^{i,0}_{h,\gamma}| \right\}.$$

Proof. By Lemma 8.6 of [1], there is a domain $G_\sigma \subset G$ $(G = \xi\rho\,(x)\,, \ \xi > 0, \ \rho\,(x) = = dist\,(x, \partial G)\,, x \in G)$ and $|\gamma| < \sigma$. Then, for every $x \in G_\sigma$ then the line segment joining the points x and $x+\gamma$ is contained in G. Identities (21), (22) are valid for all points of the segment with some kernels. After simple transformations, we have

$$|\Delta\,(\gamma, G)\, D^\nu f\,(x)| \le \prod_{j=1}^{n} \left(\psi_j\,(h)\right)^{-1-\nu_j} \int_{R^n} |f\,(x+y)| \left| \Omega^{(\nu)} \left(\frac{y-\gamma}{B}\right) - \Omega^{(\nu)}\left(\frac{y}{B}\right)\right| dy$$

$$+ \sum_{i=1}^{n} \left\{ \int_0^{|\gamma|} \prod_{j=1}^{n} \left(\psi_j\,(\upsilon)\right)^{-1-\nu_j} \left(\psi_j\,(\upsilon)\right)^{-1+l_i} \int_{R^n} \left(\left| D_i^{l_i} f\,(x+\gamma+y)\right| + \left| D_i^{l_i} f\,(x+y)\right| \right) \right.$$

$$\times \left| L_i^{(\nu)}\left(\frac{y}{B}\right)\right| \psi_i'(\upsilon) d\upsilon dy + \int_{|\gamma|}^{h} \prod_{j=1}^{n} \left(\psi_j\,(\upsilon)\right)^{-1-\nu_j} \left(\psi_i\,(\upsilon)\right)^{-1+l_i}$$

$$\times \int_{R^n} \left| D_i^{l_i} f\,(x+y)\right| \left| L_i^{(\nu)}\left(\frac{y-\gamma}{B}\right) - L_i^{(\nu)}\left(\frac{y}{B}\right)\right| \psi_i'(\upsilon) d\upsilon dy$$

$$= A\,(x,\gamma) + \sum_{i=1}^{n} \left(A_1\,(x,\gamma) + A_2\,(x,\gamma)\right), \tag{27}$$

where $0 < h \le h_0$. We also assume that $|\gamma| < h$ consequently $|\gamma| \le \min(\sigma, h)$. If $x \in G \backslash G_\sigma$, then by definition $\Delta\,(\gamma, G)\, D^\nu f\,(x) = 0$. By (27)

$$\|\Delta\,(\gamma, G)\, D^\nu f\|_{q-\varepsilon, G} = \|\Delta\,(\gamma, G)\, D^\nu f\|_{q-\varepsilon, G_\sigma} \le \|A\,(\cdot,\gamma)\|_{q-\varepsilon, G_\sigma}$$

$$+ \sum_{i=1}^{n} \left(\|A_1\,(\cdot,\gamma)\|_{q-\varepsilon, G_\sigma} + \|A_2\,(\cdot,\gamma)\|_{q-\varepsilon, G_\sigma} \right).$$

Note that

$$\left|\Omega^{(\nu)}\left(\frac{y-\gamma}{B}\right) - \Omega^{(\nu)}\left(\frac{y}{B}\right)\right| \le \left| \int_0^{|\gamma|} \frac{d}{d\xi} \Omega^{(\nu)} \left(\left(y - \xi \frac{\gamma}{|\gamma|}\right) : B \right) d\xi \right|$$

$$\le \sum_{j=1}^{n} \left| B^{(j)} \right|^{-1} \int_0^{|\gamma|} \left| D_j \Omega^{(\nu)}\left((y - \xi e_\gamma) : B\right)\right| d\xi, \ \ e_\gamma = \frac{\gamma}{|\gamma|}.$$

Therefore,

$$A\,(x,\gamma) \le \prod_{j=1}^{n} \left(\psi_j\,(\upsilon)\right)^{-1-\nu_j} \sum_{j=1}^{n} \left| B^{(j)}\right|^{-1} \times$$

$$\times \int_0^{|\gamma|} d\xi \int_{R^n} |f\,(x+\xi e_j + y)| \left| D_j \Omega^{(\nu)}\left(\frac{y}{B}\right)\right| dy. \tag{28}$$

Similarly,

$$A_2(x,\gamma) \leq \sum_{j=1}^{n} \left|B^{(j)}\right|^{-1} \int_0^{|\gamma|} d\xi \int_{|\gamma|}^{h} \prod_{j=1}^{n} \left(\psi_j(\upsilon)\right)^{-1-\nu_j} \left(\psi_i(\upsilon)\right)^{-1+l_i} \left(\psi_i'(\upsilon)\right) d\upsilon$$

$$\times \int_{R^n} \left|D_i^{l_i} f\left(x+\xi e_j+y\right)\right| \left|D_j L_i^{(\nu)}\left(\frac{y}{B}\right)\right| dy, \tag{29}$$

Using (17) for $U=G,\ \varphi=f,\ \eta=|\gamma|,\ K=\Omega^{(\nu)}$, we obtain

$$\|A(\cdot,\gamma)\|_{q-\varepsilon,G} \leq c_1(\varepsilon)\,|\gamma|\,\|f\|_{p),\phi;G}\,, \tag{30}$$

with the help of (8) for $U=G,\ \varphi=D_i^{l_i}f,\ \eta=|\gamma|,\ K=L_i^{(\nu)}$ we obtain

$$\|A_1(\cdot,\gamma)\|_{q-\varepsilon,G} \leq c_2(\varepsilon)\,|\gamma| \left\|D_i^{l_i} f\right\|_{p),\phi;G} \left|R_h^i\right|, \tag{31}$$

and from (9) for $U=G,\ \varphi=D_i^{l_i}f,\ \eta=|\gamma|,\ K=L_i^{(\nu)}$ we obtain

$$\|A_2(\cdot,\gamma)\|_{q-\varepsilon,G} \leq c_3(\varepsilon)\, R_{h,\gamma}^i \left\|D_i^{l_i} f\right\|_{p),\phi;G}. \tag{32}$$

It follows from (27), (30)-(32) that

$$\|\Delta(\gamma,G)\, D^\nu f\|_{q-\varepsilon,G} \leq c(\varepsilon)\, \|f\|_{W^l_{p),\phi;G}(G)} \left|R_{h,\gamma}^1\right|,$$

where

$$R_{h,\gamma}^1 = \max_i \left\{|\gamma|,\ |\gamma|\,|R_h^i|,\ \ |\gamma|\,|R_{h,\gamma}^i|\right\}.$$

Suppose now that $|\gamma| \geq \min(\sigma,T)$. Then

$$\|\Delta(\gamma,G)\, D^\nu f\|_{q-\varepsilon,G} \leq 2\,\|D^\nu f\|_{q-\varepsilon,G} \leq c(\sigma,h)\, \|D^\nu f\|_{q-\varepsilon,G}\, |R_{h,\gamma}|\,.$$

Estimating $\|D^\nu f\|_{q-\varepsilon,G}$ by means of (21) we obtain the sought inequality in this case as well. Thus the theorem is proved . □

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