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# Global Stability and Bifurcation Analysis in a Discrete-Time Two Predator-One Prey Model with Michaelis-Menten Type Prey Harvesting

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**Abstract**
This article studies a discrete-time Leslie-Gower two predator-one prey system with Michaelis-Menten type prey harvesting. Positivity and boundedness of the model solution are investigated. Existence and stability of fixed points are examined. Using an iteration scheme and the comparison principle of difference equations, we find out the sufficient condition for global stability of the positive fixed point. It is shown that the sufficient criterion for Neimark-Sacker bifurcation can be developed. It is observed that the system behaves in a chaotic manner when a specific set of system parameters is chosen, which are regulated by a hybrid control method. Examples are provided to illustrate our conclusions.

**Keywords:** Bifurcation, Chaos control, Leslie-Gower, Michaelis-Menten type harvesting, Predator-prey model, Stability.
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## 1. Introduction

In the real world, the interaction between prey and their predator create a major interest to the researchers to explore the dynamics of the system. Most of the existing predator-prey models come from the Lotka-Volterra system. The Lotka-Volterra models cannot justify all the predator-prey interaction. For example, when the size of the prey decreases, then the predator will search for other prey. This fact motivated Leslie to form an appropriate model known as Leslie-Gower predator-prey system to investigate the behaviour of the system. Several studies have been done on modified Leslie-Gower model with various aspects [1]-[3].

In spite of the vast research over the last few years, the knowledge about the effect of non-linear Michaelis-Menten type of harvest on one prey-two predator models is insufficient. We observe that the ecological system is often perturbed by the growing human needs for more food and more energy. For example, the fish population has decreased due to the rapid progress of fishing technology and substantial growth in human populations. Therefore, the exploitation of renewable resources, which associates immediately to sustainable development. Clark [4, 5] introduced harvesting of species through mathematical models. There are three types of harvesting namely constant rate, proportionate and Michaelis-Menten type found in the literatures [6]-[9]. Out of these, non-linear harvesting is more realistic and exhibits saturation effects with respect to both the stock abundance and effort level. Das et al. [10] analysed a prey-predator model considering Michaeli-Menten type harvesting on both the populations. They discussed boundedness, local and global stability of the proposed system. Gupta and Chandra [8] followed the similar type of harvesting in prey and derived different bifurcations such as transcritical, saddle-node, Hopf and Bogdanov-Takens in the Leslie-Gower prey-predator model. Hu and Cao [11] discussed stability and bifurcation for a predator-prey system with Michaelis-Menten type predator harvesting. Ang and Safuan [12] investigated the dynamical behaviour of an intraguild prey-predator fishery model with the non-linear harvesting of prey species.

Mathematical models followed by differential equations are reasonable for the species in which populations are overlapped. In case of non-overlapping generations, discrete-time models governed by difference equations are more appropriate than the differential equations. In real ecosystem, a discrete time system can be seen, for example, fish populations reproduce at specific timed moments or for insect populations, for which non-overlapping generations are occurring. Moreover, discrete-time models also allow more efficient computational results for numerical simulations and exhibit a rich dynamics as compared to the continuous ones [13]-[16]. Even discrete time models can admit chaotic dynamics [13, 14]. More interesting and significant results on discrete prey-predator models can be seen in [17]-[21]. Ajaz et al. [22] investigated the dynamical behaviour of a modified Leslie-Gower prey-predator model with harvesting in prey population and showed the existence and directions of period doubling and Neimark-Sacker at positive fixed point and also indicated chaos control when chaos emerge through bifurcation. Khan et al. [23] discussed a discrete-time Michaelis-Menten type prey harvesting in the modified Leslie-Gower predator-prey model and obtained the conditions for the existence of flip and Neimark-Sacker bifurcations. Chen et al. [24] studied a discrete Leslie-Gower predator-prey model with Michaelis-Menten prey harvesting and observed that the system can exhibit fold, flip and Neimark-Sacker bifurcations by the application of center manifold theorem and bifurcation theory.

The above studies are mainly confined into two species models. However, it is a common fact that several predators compete for a prey in the real world. To our knowledge, there is limited works that highlight discrete-time non-linear harvesting in the modified Leslie-Gower Holling type II two-predator one-prey model.

Now we first present a model which is a modified Leslie-Gower two predator- one prey system with Michaelis-Menten type prey harvesting:

$$\begin{aligned}
\frac{dx}{dt} &= x\left(r_1 - ax - \frac{c_1 y}{h_1 + x} - \frac{c_2 z}{h_2 + x} - \frac{qE}{d_1 E + d_2 x}\right),\\
\frac{dy}{dt} &= y\left(r_2 - \frac{f_1 y}{h_1 + x}\right),\\
\frac{dz}{dt} &= z\left(r_3 - \frac{f_2 z}{h_2 + x}\right),
\end{aligned} \tag{1.1}$$

where $x$, $y$ and $z$ denote the densities of prey, the first predator and the second predator respectively. $r_1, r_2, r_3$ stands for the intrinsic growth rate of the prey and two predators respectively. $a$ represents the intra-specific competition among the the prey species. $c_1$ and $c_2$ denote the per-capita reduction of prey $x$. $f_1$ and $f_2$ carry the same meaning as of $c_1$ and $c_2$. $h_1$ and $h_2$ signifies the environmental protection for predator $y$ and $z$ respectively. In the prey harvesting term $\frac{qEx}{d_1E+d_2x}$, $q$ is the catchability coefficient, $d_1$ and $d_2$ are the degree of competition in the harvesting business and handling time respectively. $E$ describes the harvesting effort.

For qualitative analysis, including global stability, bifurcation analysis and chaos control for a discrete analogue of system (1.1), a piecewise constant argument is introduced to describe the following exponential form of nonlinear difference equations:

$$\begin{aligned}
x_{n+1} &= x_n \exp\left\{r_1 - ax_n - \frac{c_1 y_n}{h_1 + x_n} - \frac{c_2 z_n}{h_2 + x_n} - \frac{qE}{d_1 E + d_2 x_n}\right\},\\
y_{n+1} &= y_n \exp\left\{r_2 - \frac{f_1 y_n}{h_1 + x_n}\right\},\\
z_{n+1} &= z_n \exp\left\{r_3 - \frac{f_2 z_n}{h_2 + x_n}\right\}
\end{aligned} \tag{1.2}$$

where $x_n$, $y_n$ and $z_n$ represent the densities of prey and both the predator at generation $n \in \mathbb{N}$ respectively.

The rest of the paper is formatted as follows. Positivity and boundedness of solutions are presented in Section 2. The existence and stability of the interior fixed point are discussed in Section 3. Global stability criterion is derived in Section 4. Neimark-Sacker bifurcation and flip bifurcation are described in Section 5. Chaos control mechanism is presented in Section 6. Numerical examples are given in Section 7. Section 8 concludes the paper.

## 2. Positivity and Boundedness of Solutions

In this section, we discuss positivity and boundedness of solutions of system (1.2). The first lemma follows immediately from the system structure and its proof is omitted.

**Lemma 2.1.** *Solutions of system (1.2) with positive initial conditions remain positive.*

To prove the boundedness of solutions of system (1.2), we require the following lemma:

**Lemma 2.2.** *(see [25]) Suppose that $x_m$ satisfies $x_0 > 0$ and $x_{m+1} \leq x_m exp[\alpha(1-\beta x_m)]$ for $m \in [m_1, \infty)$ where $\beta$ is a positive constant. Then $limsup_{n\to\infty} x_m \leq \frac{1}{\alpha\beta} exp(\alpha - 1)$.*

We now state the theorem which ensures that every positive solution of system (1.2) is uniformly bounded.

**Theorem 2.3.** *Every positive solution $\{(x_n, y_n, z_n)\}$ of system (1.2) is uniformly bounded.*

*Proof.* Assume that $\{(x_n, y_n, z_n)\}$ be an arbitrary positive solution of system (1.2). From the first equation of system (1.2), we get

$$x_{n+1} \leq x_n \exp(r_1 - a x_n), n = 0, 1, 2, \ldots.$$

Assume that $x_0 > 0$, then following Lemma 2.2, we get $\limsup_{n\to\infty} x_n \leq \frac{1}{a}\exp(r_1 - 1) := M_1$. From the second equation of system (1.2),

$$y_{n+1} \leq y_n \exp(r_2 - \frac{f_1}{h_1 + M_1} y_n), n = 0, 1, 2, \ldots.$$

It follows from Lemma 2.2 that $\limsup_{n\to\infty} y_n \leq \frac{h_1+M_1}{f_1}\exp(r_2 - 1) := M_2$ whenever $y_0 > 0$. Assume that $z_0 > 0$. From the third equation of system (1.2), we get

$$z_{n+1} \leq z_n \exp(r_3 - \frac{f_2}{h_2 + M_1} z_n).$$

Applying again Lemma 2.2, we get

$$\limsup_{n\to\infty} z_n \leq \frac{h_2 + M_1}{f_2}\exp(r_3 - 1) := M_3.$$

Then it follows that $\limsup_{n\to\infty}(x_n, y_n, z_n) \leq M$, where $M = \max\{M_1, M_2, M_3\}$.
This completes the proof. □

## 3. Existence of Fixed Points

In this section, we determine the fixed points and their dynamics. Evidently, system (1.1) has at most twelve non-negative fixed points $E_0 = (0,0,0)$. If $q < r_1 d_1$ then the fixed point $E_1 = (\bar{x}, 0, 0)$ exists uniquely where

$$\bar{x} = \frac{r_1 d_2 - a d_1 E + \sqrt{(r_1 d_2 - a d_1 E)^2 - 4 a d_2 E (q - r_1 d_1)}}{2 a d_2}.$$

If $q > r_1 d_1, r_1 d_2 > a d_1 E$ and $(r_1 d_2 - a d_1 E)^2 - 4 a d_2 E (q - r_1 d_1) > 0$ then multiple fixed points exist $E_{1\pm} = (\bar{x}_\pm, 0, 0)$ where

$$\bar{x}_\pm = \frac{r_1 d_2 - a d_1 E \pm \sqrt{(r_1 d_2 - a d_1 E)^2 - 4 a d_2 E (q - r_1 d_1)}}{2 a d_2}.$$

There always exists $E_2 = (0, \frac{r_2 h_1}{f_1}, 0)$ and $E_3 = (0, 0, \frac{r_3 h_2}{f_2})$. If $q f_1 + d_1 c_1 r_2 < d_1 r_1 f_1$ then there exists a unique fixed point $E_{12} = (\hat{x}, \hat{y}, 0)$ where

$$\hat{x} = \frac{d_2(r_1 f_1 - c_1 r_2) - a f_1 d_1 E + \sqrt{(d_2(r_1 f_1 - c_1 r_2) - a f_1 d_1 E)^2 - 4 a f_1 d_2 E (q f_1 + d_1 c_1 r_2 - d_1 r_1 f_1)}}{2 a f_1 d_2}$$

and

$$\hat{y} = \frac{r_2(h_1 + \hat{x})}{f_1}.$$

If $q f_1 + d_1 c_1 r_2 > d_1 r_1 f_1, r_1 f_1 d_2 > c_1 r_2 d_2 + a f_1 d_1 E$ and $\{d_2(r_1 f_1 - c_1 r_2) - a f_1 d_1 E\}^2 > 4 a f_1 d_2 E (q f_1 + d_1 c_1 r_2 - d_1 r_1 f_1)$ then there exists multiple fixed points $E_{12\pm} = (\hat{x}_\pm, \hat{y}_\pm, 0)$ where

$$\hat{x}_\pm = \frac{d_2(r_1 f_1 - c_1 r_2) - a f_1 d_1 E \pm \sqrt{(d_2(r_1 f_1 - c_1 r_2) - a f_1 d_1 E)^2 - 4 a f_1 d_2 E (q f_1 + d_1 c_1 r_2 - r_1 f_1 d_1)}}{2 a f_1 d_2}$$

and

$$\hat{y}_{\pm} = \frac{r_2(h_1+\hat{x}_{\pm})}{f_1}.$$

If $qf_2 + d_1c_2r_3 < d_1r_1f_2$ then there exists a unique fixed point $E_{13} = (\tilde{x}, 0, \tilde{z})$ where

$$\tilde{x} = \frac{d_2(r_1f_2 - c_2r_3) - af_2d_1E + \sqrt{(d_2(r_1f_2 - c_2r_3) - af_2d_1E)^2 - 4af_2d_2E(qf_2 + d_1c_2r_3 - d_1r_1f_2)}}{2af_2d_2}$$

and

$$\tilde{y} = \frac{r_3(h_2+\tilde{x})}{f_2}.$$

If $qf_2 + d_1c_2r_3 > d_1r_1f_2, r_1f_2d_2 > c_2r_3d_2 + af_2d_1E$ and $\{d_2(r_1f_2 - c_2r_3) - af_2d_1E\}^2 > 4af_2d_2E(qf_2 + d_1c_2r_3 - d_1r_1f_2)$ then there exists multiple fixed points $E_{13\pm} = (\tilde{x}_{\pm}, 0, \tilde{z}_{\pm})$ where

$$\tilde{x}_{\pm} = \frac{d_2(r_1f_2 - c_2r_3) - af_2d_1E \pm \sqrt{(d_2(r_1f_2 - c_2r_3) - af_2d_1E)^2 - 4af_2d_2E(qf_2 + d_1c_2r_3 - r_1f_2d_1)}}{2af_2d_2}$$

and

$$\tilde{z}_{\pm} = \frac{r_3(h_2+\tilde{x}_{\pm})}{f_2}.$$

There exists a unique fixed point $E_{23} = (0, \frac{r_2h_1}{f_1}, \frac{r_3h_2}{f_2})$. To determine the positive fixed point $E^* = (x^*, y^*, z^*)$ , we have to solve the following system of equations:

$$x = x(r_1 - ax - \frac{c_1y}{h_1+x} - \frac{c_2z}{h_2+x} - \frac{qE}{d_1E+d_2x}), \tag{3.1}$$

$$y = y(r_2 - \frac{f_1y}{h_1+x}), \tag{3.2}$$

$$z = z(r_3 - \frac{f_2z}{h_2+x}). \tag{3.3}$$

where $x^*, y^*$ and $z^*$ are the positive solutions of equations (3.1), (3.2) and (3.3). Solving (3.2) and (3.3) we get $y = \frac{r_2(h_1+x)}{f_1}$ and $z = \frac{r_3(h_2+x)}{f_2}$ and substituting the value of $y$ and $z$ in (3.1), we obtain the following equation:

$$Ax^2 + Bx + C = 0 \tag{3.4}$$

where

$$A = f_1f_2ad_2, B = f_1f_2ad_2E - d_2(r_1f_1f_2 - c_1r_2f_2 - c_2r_3f_1), C = E\{f_1f_2q + d_1(c_1r_2f_2 + c_2r_3f_1) - d_1r_1f_1f_2\}$$

If $C < 0$ then there exists a unique positive root $x^*$ of equation (3.4). In that case there exists a unique fixed point $E^* = (x^*, y^*, z^*)$ where

$$x^* = \frac{-B + \sqrt{B^2 - 4AC}}{2A}, y^* = \frac{r_2(h_1+x^*)}{f_1}$$

and

$$z^* = \frac{r_3(h_2+x^*)}{f_2}.$$

If $B < 0, C > 0$ and $B^2 > 4AC$ then there exists multiple fixed points $E^*_{\pm} = (x^*_{\pm}, y^*_{\pm}, z^*_{\pm})$ where

$$x^*_{\pm} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}, y^*_{\pm} = \frac{r_2(h_1+x^*_{\pm})}{f_1}$$

and

$$z^*_{\pm} = \frac{r_3(h_2+x^*_{\pm})}{f_2}.$$

### 3.1 Stability of fixed points

To investigate the local stability of the fixed points of system (1.2), we require the following lemma.

**Lemma 3.1.** *([26]) Consider the cubic equation*

$$\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3 = 0 \tag{3.5}$$

*where $p_1$, $p_2$ and $p_3$ are real numbers. Then necessary and sufficient conditions that all the roots of equation (3.5) lie in an open disk $|\lambda| < 1$ are $|p_1 + p_3| < 1 + p_2, |p_1 - 3p_3| < 3 - p_2$ and $p_3^2 + p_2 - p_3 p_1 < 1$.*

The Jacobian matrix $J(E_0)$ for system (1.2) is given by

$$J(E_0) = \begin{pmatrix} \exp(r_1 - \frac{q}{d_1}) & 0 & 0 \\ 0 & \exp r_2 & 0 \\ 0 & 0 & \exp r_3 \end{pmatrix}.$$

Then it follows from $J(E_0)$ that $E_0$ is an unstable fixed point for system (1.2). Again

$$J(E_1) = \begin{pmatrix} 1 - a\bar{x} + \frac{qEd_2\bar{x}}{(d_1E + d_2\bar{x})^2} & -\frac{c_1\bar{x}}{h_1 + \bar{x}} & -\frac{c_2\bar{x}}{h_2 + \bar{x}} \\ 0 & \exp r_2 & 0 \\ 0 & 0 & \exp r_3 \end{pmatrix}.$$

From $J(E_1)$, we conclude that that $E_1$ is an unstable fixed point for system (1.2). Similarly, it can be shown that $E_{1\pm}$ are also unstable. Now

$$J(E_2) = \begin{pmatrix} \exp(r_1 - \frac{c_1 r_2}{f_1} - \frac{q}{d_1}) & 0 & 0 \\ \frac{r_1^2}{f_1} & 1 - r_2 & 0 \\ 0 & 0 & \exp r_3 \end{pmatrix}.$$

It is obvious from $J(E_2)$ that $E_2$ is an unstable fixed point for system (1.2). For $E_3$,

$$J(E_3) = \begin{pmatrix} \exp(r_1 - \frac{c_2 r_3}{f_2} - \frac{q}{d_1}) & 0 & 0 \\ 0 & \exp r_2 & 0 \\ \frac{r_3^2}{f_2} & 0 & 1 - r_3 \end{pmatrix}.$$

Again we see that from $J(E_3)$ that $E_3$ is an unstable fixed point for system (1.2). For $E_{12}$,

$$J(E_{12}) = \begin{pmatrix} 1 - \hat{x}(a - \frac{c_1\hat{y}}{(h_1+\hat{x})^2} - \frac{qEd_2}{(d_1E + d_2\hat{x})^2}) & -\frac{c_1\hat{x}}{h_1+\hat{x}} & -\frac{c_2\hat{x}}{h_2+\hat{x}} \\ \frac{f_1\hat{y}^2}{(h_1+\hat{x})^2} & 1 - \frac{\hat{y}f_1}{h_1+\hat{x}} & 0 \\ 0 & 0 & \exp r_3 \end{pmatrix}.$$

Again we see that from $J(E_{12})$ that $E_{12}$ is an unstable fixed point for system (1.2). Similarly, it can be shown that $E_{12\pm}$ are also unstable. For $E_{13}$,

$$J(E_{13}) = \begin{pmatrix} 1 - \tilde{x}(a - \frac{c_2\tilde{z}}{(h_2+\tilde{x})^2} - \frac{qEd_2}{(d_1E + d_2\tilde{x})^2}) & -\frac{c_1\tilde{x}}{h_1+\tilde{x}} & -\frac{c_2\tilde{x}}{h_2+\tilde{x}} \\ 0 & \exp r_2 & 0 \\ \frac{\tilde{z}^2 f_2}{(h_2+\tilde{x})^2} & 0 & 1 - \frac{f_2\tilde{z}}{h_2+\tilde{x}} \end{pmatrix}.$$

It is clear from $J(E_{13})$ that $E_{13}$ is an unstable fixed point for system (1.2). Similarly, it can be shown that $E_{13\pm}$ are also unstable. Now

$$J(E_{23}) = \begin{pmatrix} \exp(r_1 - \frac{c_1 r_2}{f_1} - \frac{c_2 r_3}{f_2} - \frac{q}{d_1}) & 0 & 0 \\ \frac{r_2^2}{f_1} & 1 - r_2 & 0 \\ \frac{r_3^2}{f_2} & 0 & 1 - r_3 \end{pmatrix}.$$

If $r_1 < \frac{c_1 r_2 f_2 d_1 + c_2 r_2 f_1 d_1 + q f_1 f_2}{f_1 f_2 d_1}, r_2 < 2$ and $r_3 < 2$ then it follows from $J(E_{23})$ that $E_{23}$ is locally asymptotically stable fixed point for system (1.2). Let $E^* = (x^*, y^*, z^*)$ be the unique interior fixed point of system (1.2). The Jacobian matrix for (1.2) at $E^*$ is given by

$$J(x^*, y^*, z^*) = \begin{pmatrix} a_{11} & -\frac{c_1 x^*}{h_1 + x^*} & -\frac{c_2 x^*}{h_2 + x^*} \\ \frac{f_1 y^{*2}}{(h_1 + x^*)^2} & 1 - r_2 & 0 \\ \frac{f_2 z^{*2}}{(h_2 + x^*)^2} & 0 & 1 - r_3 \end{pmatrix}$$

where

$$a_{11} = 1 - ax^* + \frac{qEd_2x^*}{(d_1E + d_2x^*)^2} + \frac{c_2x^*z^*}{(h_2 + x^*)^2} + \frac{c_1x^*y^*}{(h_1 + x^*)^2}.$$

The characteristic polynomial of $J(E^*)$ is given by

$$P(\lambda) = \lambda^3 + p_1\lambda^2 + p_2\lambda + p_3 \tag{3.6}$$

where

$$\begin{aligned} p_1 &= r_2 + r_3 - 2 - a_{11}, \\ p_2 &= a_{11}(2 - r_2 - r_3) + (1 - r_2)(1 - r_3) + \frac{c_1f_1x^*y^{*2}}{(h_1 + x^*)^3} + \frac{c_2f_2x^*z^{*2}}{(h_2 + x^*)^3}, \\ p_3 &= a_{11}(1 - r_2)(r_3 - 1) + \frac{c_1f_1x^*y^{*2}(r_3 - 1)}{(h_1 + x^*)^3} + \frac{c_2f_2x^*z^{*2}(r_2 - 1)}{(h_2 + x^*)^3}. \end{aligned} \tag{3.7}$$

We now use Lemma 3.1 to investigate stability of $E^*$.

**Lemma 3.2.** *Assume that $C < 0$ holds. Then, the fixed point $E^*$ is locally asymptotically stable if and only if the following conditions are satisfied:*

$$|p_1 + p_3| < 1 + p_2, |p_1 - 3p_3| < 3 - p_2$$

*and $p_3^2 + p_2 - p_3p_1 < 1$ where $p_1$, $p_2$ and $p_3$ are defined in (3.7).*

**Remark 3.3.** *In case of multiple fixed points $E^*_{\pm} = (x^*_{\pm}, y^*_{\pm}, z^*_{\pm})$ , we can find similar type of conditions as in Lemma 3.2.*

## 4. Global Stability

In this section, we will utilize the process of iteration scheme and the comparison principle of difference equations to investigate the global stability of the positive fixed point of system (1.2). To establish global stability result, we require the following lemmas:

**Lemma 4.1.** *([27]) Let $f(u) = u exp(\delta - \eta u)$, where $\delta$ and $\eta$ are positive constants. Then $f(u)$ is nondecreasing for $u \in (0, \frac{1}{\eta}]$.*

**Lemma 4.2.** *([27]) Assume that the sequence $u_n$ satisfies*

$$u_{n+1} = u_n exp(\delta - \eta u_n), n = 1, 2, 3, ...$$

*where $\delta$ and $\eta$ are positive constants and $u_0 > 0$. Then, (i) If $\delta < 2$, then $lim_{n\to\infty} u_n = \frac{\delta}{\eta}$.*
*(ii) If $\delta \leq 1$, then $u_n \leq \frac{1}{\eta}, n = 2, 3, ....$*

**Lemma 4.3.** *[28] Suppose that functions $f, g : \mathbb{Z}_+ \times [0, \infty)$ satisfy $f(n, x) \leq g(n, x)$ $(f(n, x) \geq g(n, x))$ for $n \in \mathbb{Z}_+$ and $g(n, x)$ is nondecreasing with respect to x. If $u_n$ are the nonnegative solutions of the difference equations*

$$x_{n+1} = f(n, x_n), u_{n+1} = g(n, u_n)$$

*respectively, and $x_0 \leq u_0$ $(x_0 \geq u_0)$ then $x_n \leq u_n$ $(x_n \geq u_n)$ for all $n \geq 0$.*

**Theorem 4.4.** *Assume that $C < 0$, $\frac{c_1r_2h_2f_2d_1(ah_1+r_1)+c_2r_3h_1f_1d_1(ah_2+r_1)+qh_1h_2f_1f_2}{d_1} < r_1 < 1$, $\frac{f_1}{h_1} < r_2 < 1$ and $\frac{f_2}{h_2} < r_3 < 1$ hold. Then, the fixed point $E^*(x^*, y^*, z^*)$ of system (1.2) is globally asymptotically stable.*

*Proof.* Assume that $(x_n, y_n, z_n)$ is any solution of system (1.2) with initial values $x_0 > 0, y_0 > 0, z_0 > 0$. Let

$$U_1 = \text{limsup}_{n\to\infty} x_n,\ V_1 = \text{liminf}_{n\to\infty} x_n,$$

$$U_2 = \text{limsup}_{n\to\infty} y_n,\ V_2 = \text{liminf}_{n\to\infty} y_n,$$

$$U_3 = \text{limsup}_{n\to\infty} z_n,\ V_3 = \text{liminf}_{n\to\infty} z_n.$$

In the following, we will prove that $U_1 = V_1 = x^*, U_2 = V_2 = y^*, U_3 = V_3 = z^*$.
First we show that $U_1 \leq M_1^x, U_2 \leq M_1^y, U_3 \leq M_1^z$. From the first equation of system (1.2), we get

$$x_{n+1} \leq x_n \exp(r_1 - a x_n), n = 0, 1, 2, ...$$

Considering the auxiliary equation

$$u_{n+1} = u_n \exp(r_1 - a u_n) \tag{4.1}$$

by Lemma 4.2 (ii), because of $r_1 \leq 1$, we get $u_n \leq \frac{1}{a}$ for all $n \geq 2$. By Lemma 4.1, we obtain $f(u) = u\exp(r_1 - au)$ is nondecreasing for $u \in (0, \frac{1}{a}]$. Thus from Lemma 4.3, we get $x_n \leq u_n$ for all $n \geq 2$, where $u_n$ is the solution of equation (4.1) with initial value $u_2 = x_2$. By Lemma 4.2 (i), we get

$$U_1 = \text{limsup}_{n\to\infty} x_n \leq \lim_{n\to\infty} u_n = \frac{r_1}{a} \triangleq M_1^x.$$

Hence, for any sufficiently small $\varepsilon > 0$, there exists a $n_1 > 2$ such that if $n \geq n_1$, then $x_n \leq M_1^x + \varepsilon$. From the second equation of system (1.2), we obtain,

$$y_{n+1} \leq y_n \exp(r_2 - \frac{f_1}{h_1 + M_1^x + \varepsilon} y_n), n = 0, 1, 2, ...$$

Again considering the auxiliary equation

$$u_{n+1} = u_n \exp(r_2 - \frac{f_1}{h_1 + M_1^x + \varepsilon} u_n) \tag{4.2}$$

by Lemma 4.2 (ii), because of $r_2 \leq 1$, we get $u_n \leq \frac{h_1 + M_1^x + \varepsilon}{f_1}$ for all $n \geq 2$. By Lemma 4.1, we obtain $f(u) = u\exp(r_2 - \frac{f_1}{h_1 + M_1^x + \varepsilon} u)$ is nondecreasing for $u \in (0, \frac{h_1 + M_1^x + \varepsilon}{f_1}]$. Thus from Lemma 4.3, we get $x_n \leq u_n$ for all $n \geq 2$, where $u_n$ is the solution of Eq. (4.2) with initial value $u_2 = x_2$. By Lemma 4.2 (i), we get

$$U_2 = \text{limsup}_{n\to\infty} x_n \leq \lim_{n\to\infty} u_n = \frac{r_2(h_1 + M_1^x + \varepsilon)}{f_1} \triangleq M_1^y.$$

Hence, for any sufficiently small $\varepsilon > 0$, there exists a $n_2 > n_1$ such that if $n \geq n_2$, then $y_n \leq M_1^y + \varepsilon$. Similarly, from the third equation of system (1.2) for $r_3 < 1$, we obtain

$$U_3 = \text{limsup}_{n\to\infty} z_n \leq \lim_{n\to\infty} u_n = \frac{r_3(h_2 + M_1^x + \varepsilon)}{f_2} \triangleq M_1^z.$$

Hence, for any sufficiently small $\varepsilon > 0$, there exists $n_3 > n_2$ such that for $n \geq n_3, z_n \leq M_1^z + \varepsilon$. Next we show that $V_1 \geq N_1^x, V_2 \geq N_1^y, V_3 \geq N_1^z$. From the first equation of system (1.2), we have

$$x_{n+1} \geq x_n \exp[a - a x_n - \frac{c_1(M_1^y + \varepsilon)}{h_1} - \frac{c_2(M_1^z + \varepsilon)}{h_2} - \frac{q}{d_1}], n \geq n_3.$$

Consider the auxiliary equation

$$u_{n+1} = u_n \exp[r_1 - a u_n - \frac{c_1(M_1^y + \varepsilon)}{h_1} - \frac{c_2(M_1^z + \varepsilon)}{h_2} - \frac{q}{d_1}]. \tag{4.3}$$

Since we have $r_1 - \frac{c_1(M_1^y + \varepsilon)}{h_1} - \frac{c_2(M_1^z + \varepsilon)}{h_2} - \frac{q}{d_1} < 1$, by Lemma 4.2 (ii), we have, $u_n \leq \frac{1}{a}$ for $n \geq n_3$. By Lemma 4.1, we obtain $f(u) = u\exp(r_1 - \frac{c_1(M_1^y + \varepsilon)}{h_1} - \frac{c_2(M_1^z + \varepsilon)}{h_2} - \frac{q}{d_1} - au)$ is nondecreasing for $u \in (0, \frac{1}{a}]$. Thus from Lemma 4.3, we get $x_n \geq u_n$ for all $n \geq n_3$. By Lemma 4.2 (i), we get

$$V_1 = \text{liminf}_{n\to\infty} x_n \geq \lim_{n\to\infty} u_n = \frac{1}{a}[r_1 - \frac{c_1(M_1^y + \varepsilon)}{h_1} - \frac{c_2(M_1^z + \varepsilon)}{h_2} - \frac{q}{d_1}].$$

From the arbitrariness of $\varepsilon > 0$, we have

$$V_1 \geq N_1^x = \frac{1}{a}[r_1 - \frac{c_1(M_1^y + \varepsilon)}{h_1} - \frac{c_2(M_1^z + \varepsilon)}{h_2} - \frac{q}{d_1}].$$

Hence for any sufficiently small $\varepsilon > 0$, there exists $n_4 > n_3$ such that for $n \geq n_4, x_n \geq N_1^x - \varepsilon$. From the second equation of system (1.2), we have

$$y_{n+1} \geq y_n \exp[r_2 - \frac{f_1}{h_1} y_n], n \geq n_4.$$

By the same way, we can get

$$V_2 = \liminf_{n\to\infty} y_n \geq \lim_{n\to\infty} u_n = \frac{r_2 h_1}{f_1}.$$

From the arbitrariness of $\varepsilon > 0$, we have,

$$V_2 \geq N_1^y = \frac{r_2 h_1}{f_1}.$$

Hence for any sufficiently small $\varepsilon > 0$, there exists $n_5 > n_4$ such that for $n \geq n_5, y_n \geq N_1^y - \varepsilon$. Similarly, from the third equation of system (1.2), we have

$$z_{n+1} \geq z_n \exp[r_3 - \frac{f_2}{h_2} z_n], n \geq n_5.$$

with

$$V_3 = \liminf_{n\to\infty} z_n \geq \lim_{n\to\infty} u_n = \frac{r_3 h_2}{f_2}.$$

From the arbitrariness of $\varepsilon > 0$, we have,

$$V_3 \geq N_1^z = \frac{r_3 h_2}{f_2}.$$

Hence for any sufficiently small $\varepsilon > 0$, there exists $n_6 > n_5$ such that for $n \geq n_6, z_n \geq N_1^z - \varepsilon$. Now we show that $U_1 \leq M_2^x, U_2 \leq M_2^y$ and $U_3 \leq M_2^z$, where $M_2^x \leq M_1^x, M_2^y \leq M_1^y$ and $M_2^z \leq M_1^z$ respectively. From the first equation of system (1.2) for $n > n_6$, we get

$$x_{n+1} \leq x_n \exp[r_1 - a x_n - \frac{c_1(N_1^y - \varepsilon)}{h_1 + M_1^x + \varepsilon} - \frac{c_2(N_1^z - \varepsilon)}{h_2 + M_1^x + \varepsilon} - \frac{qE}{d_1 E + d_2(M_1^x + \varepsilon)}].$$

Consider the auxiliary equation

$$u_{n+1} = u_n \exp[r_1 - a u_n - \frac{c_1(N_1^y - \varepsilon)}{h_1 + M_1^x + \varepsilon} - \frac{c_2(N_1^z - \varepsilon)}{h_2 + M_1^x + \varepsilon} - \frac{qE}{d_1 E + d_2(M_1^x + \varepsilon)}]. \tag{4.4}$$

Using the similar argument as in above, we can get

$$U_1 = \limsup_{n\to\infty} x_n \leq \frac{1}{a}[r_1 - \frac{c_1(N_1^y - \varepsilon)}{h_1 + M_1^x + \varepsilon} - \frac{c_2(N_1^z - \varepsilon)}{h_2 + M_1^x + \varepsilon} - \frac{qE}{d_1 E + d_2(M_1^x + \varepsilon)}],$$

since

$$r_1 - \frac{c_1(N_1^y - \varepsilon)}{h_1 + M_1^x + \varepsilon}) - \frac{c_2(N_1^z - \varepsilon)}{h_2 + M_1^x + \varepsilon} - \frac{qE}{d_1 E + d_2(M_1^x + \varepsilon)} \leq 1.$$

From the arbitrariness of $\varepsilon > 0$, we claim that

$$U_1 \leq M_2^x = \frac{1}{a}[r_1 - \frac{c_1(N_1^y - \varepsilon)}{h_1 + M_1^x + \varepsilon} - \frac{c_2(N_1^z - \varepsilon)}{h_2 + M_1^x + \varepsilon} - \frac{qE}{d_1 E + d_2(M_1^x + \varepsilon)}].$$

Hence for any sufficiently small $\varepsilon > 0$, there exists $n_7 > n_6$ such that for $n \geq n_7, x_n \leq M_2^x + \varepsilon$. Similarly, from the second equation of system (1.2) for $n > n_7$, we get

$$y_{n+1} \leq y_n \exp[r_2 - \frac{f_1}{h_1 + M_2^x + \varepsilon} y_n].$$

Similarly to the above argument, we get

$$U_2 \leq M_2^y = \frac{r_2(h_1 + M_2^x + \varepsilon)}{f_1}.$$

Hence for any sufficiently small $\varepsilon > 0$, there exists $n_8 > n_7$ such that for $n \geq n_8, y_n \leq M_2^y + \varepsilon$. From the third equation of system (1.2) for $n > n_8$, we get

$$z_{n+1} \leq z_n \exp[r_3 - \frac{f_2}{h_2 + M_2^x + \varepsilon} y_n].$$

Similarly to the above argument, we get

$$U_3 \leq M_2^z = \frac{r_3(h_2 + M_2^x + \varepsilon)}{f_2}.$$

Hence for any sufficiently small $\varepsilon > 0$, there exists $n_9 > n_8$ such that for $n \geq n_9, z_n \leq M_2^z + \varepsilon$. Now we show that $V_1 \geq N_2^x, V_2 \geq N_2^y$ and $V_3 \geq N_2^z$, where $N_2^x \geq N_1^x, N_2^y \geq N_1^y$ and $N_2^z \geq N_1^z$ respectively. Further, from the first equation of system (1.2) for $n > n_9$, we get

$$x_{n+1} \geq x_n \exp[r_1 - a x_n - \frac{c_1(M_2^y + \varepsilon)}{h_1 + N_1^x - \varepsilon} - \frac{c_2(M_2^z + \varepsilon)}{h_2 + N_1^x - \varepsilon} - \frac{qE}{d_1 E + d_2(N_1^x - \varepsilon)}].$$

Using a similar argument, we get

$$V_1 = \liminf_{n\to\infty} x_n \geq \frac{1}{a}[r_1 - \frac{c_1(M_2^y + \varepsilon)}{h_1 + N_1^x - \varepsilon} - \frac{c_2(M_2^z + \varepsilon)}{h_2 + N_1^x - \varepsilon} - \frac{qE}{d_1 E + d_2(N_1^x - \varepsilon)}] \leq 1.$$

From the arbitrariness of $\varepsilon > 0$, we claim that

$$V_1 \geq N_2^x = \frac{1}{a}[r_1 - \frac{c_1(M_2^y + \varepsilon)}{h_1 + N_1^x - \varepsilon} - \frac{c_2(M_2^z + \varepsilon)}{h_2 + N_1^x - \varepsilon} - \frac{qE}{d_1 E + d_2(N_1^x - \varepsilon)}].$$

Hence for any sufficiently small $\varepsilon > 0$, there exists $n_{10} > n_9$ such that for $n \geq n_{10}, x_n \geq N_2^x - \varepsilon$. Similarly, from the second equation of system (1.2) for $n > n_{10}$, we have

$$y_{n+1} \geq y_n \exp[r_2 - \frac{f_1}{h_1 + N_2^x - \varepsilon} y_n]$$

with

$$V_2 = \liminf_{n\to\infty} y_n \geq \frac{r_2(h_1 + N_2^x - \varepsilon)}{f_1}.$$

From the arbitrariness of $\varepsilon > 0$, we claim that $V_2 \geq N_2^y = \frac{r_2(h_1 + N_2^x - \varepsilon)}{f_1}$. Hence for any sufficiently small $\varepsilon > 0$, there exists $n_{11} > n_{10}$ such that for $n \geq n_{11}, y_n \geq N_2^y - \varepsilon$. Similarly, from the third equation of system (1.2) for $n > n_{11}$, we have

$$z_{n+1} \geq z_n \exp[r_3 - \frac{f_2}{h_2 + N_2^x - \varepsilon} z_n].$$

with

$$V_3 = \liminf_{n\to\infty} z_n \geq \frac{r_3(h_2 + N_2^x - \varepsilon)}{f_2}.$$

From the arbitrariness of $\varepsilon > 0$, we conclude that $V_3 \geq N_2^z = \frac{r_3(h_2 + N_2^x - \varepsilon)}{f_2}$. Hence for any sufficiently small $\varepsilon > 0$, there exists $n_{12} > n_{11}$ such that for $n \geq n_{12}, z_n \geq N_2^z - \varepsilon$. Repeating the above process, we ultimately get six sequences $\{M_n^x\}, \{M_n^y\}, \{M_n^z\}, \{N_n^x\}$,

$\{N_n^y\}$, and $\{N_n^z\}$ such that for all $n \geq 2$,

$$
\begin{aligned}
M_n^x &= \frac{1}{a}[r_1 - \frac{c_1 N_{n-1}^y}{h_1 + M_{n-1}^x} - \frac{c_2 N_{n-1}^z}{h_2 + M_{n-1}^x} - \frac{qE}{d_1 E + d_2 M_{n-1}^x}],\\
M_n^y &= \frac{r_2(h_1 + M_n^x)}{f_1},\\
M_n^z &= \frac{r_3(h_2 + M_n^x)}{f_2},\\
N_n^x &= \frac{1}{a}[r_1 - \frac{c_1 M_n^y}{h_1 + N_{n-1}^x} - \frac{c_2 M_n^z}{h_2 + N_{n-1}^x} - \frac{qE}{d_1 E + d_2 N_{n-1}^x}],\\
N_n^y &= \frac{r_2(h_1 + N_n^x)}{f_1},\\
N_n^z &= \frac{r_3(h_2 + N_n^x)}{f_2}.
\end{aligned}
\tag{4.5}
$$

Clearly, we have for any integer $n > 0$,

$$N_n^x \leq V_1 \leq U_1 \leq M_n^x, N_n^y \leq V_2 \leq U_2 \leq M_n^y, \text{and } N_n^z \leq V_3 \leq U_3 \leq M_n^z.$$

In the following, we will prove that $\{M_n^x\}, \{M_n^y\}$ and $\{M_n^z\}$ are monotonically decreasing and $\{N_n^x\}, \{N_n^y\}$ and $\{N_n^z\}$ are monotonically increasing, with the help of inductive method. Firstly, it is clear that

$$M_2^x \leq M_1^x, M_2^y \leq M_1^y, M_2^z \leq M_1^z, N_2^x \geq N_1^x, N_2^y \geq N_1^y, \text{and } N_2^z \geq N_1^z.$$

For $n = k (k \geq 2)$, we assume that

$$M_k^x \leq M_{k-1}^x, M_k^y \leq M_{k-1}^y, M_k^z \leq M_{k-1}^x, N_k^x \geq N_{k-1}^x, N_k^y \geq N_{k-1}^y, \text{and } N_k^z \geq N_{k-1}^z.$$

Now

$$
\begin{aligned}
M_{k+1}^x - M_k^x &= -\frac{1}{a}[\frac{c_1\{(N_k^y M_{k-1}^x - M_k^x N_{k-1}^y) + h_1(N_k^y - N_{k-1}^y)\}}{(h_1 + M_k^x)(h_1 + M_{k-1}^x)} + \frac{c_2\{(N_k^z M_{k-1}^x - N_{k-1}^z M_k^x) + h_2(N_k^z - N_{k-1}^z)\}}{(h_2 + M_k^x)(h_2 + M_{k-1}^x)}\\
&+ \frac{qEd_2(M_k^x - M_{k-1}^x)}{(d_1 E + d_2 M_k^x)(d_1 E + d_2 M_{k-1}^x)}] \leq 0\\
M_{k+1}^y - M_k^y &= \frac{r_2(M_{k+1}^x - M_k^x)}{f_1} \leq 0\\
M_{k+1}^z - M_k^z &= \frac{r_3(M_{k+1}^x - M_k^x)}{f_2} \leq 0\\
N_{k+1}^x - N_k^x &= -\frac{1}{a}[\frac{c_1\{(M_{k+1}^y N_{k-1}^x - M_k^y N_k^x) + h_1(M_{k+1}^y - M_k^y)\}}{(h_1 + N_k^x)(h_1 + N_{k-1}^x)} + \frac{c_2\{(M_{k+1}^z N_{k-1}^x - M_k^z N_k^x) + h_2(M_{k+1}^z - M_k^z)\}}{(h_2 + N_k^x)(h_2 + N_{k-1}^x)}\\
&+ \frac{qEd_2(N_{k-1}^x - N_k^x)}{(d_1 E + d_2 N_k^x)(d_1 E + d_2 N_{k-1}^x)}] \geq 0\\
N_{k+1}^y - N_k^y &= \frac{r_2(N_{k+1}^x - N_k^x)}{f_1} \geq 0\\
N_{k+1}^z - N_k^z &= \frac{r_3(N_{k+1}^x - N_k^x)}{f_2} \geq 0
\end{aligned}
$$

This shows that $\{M_n^x\}, \{M_n^y\}$ and $\{M_n^z\}$ are monotonically decreasing and $\{N_n^x\}, \{N_n^y\}$ and $\{N_n^z\}$ are monotonically increasing. Therefore, by the criterion of monotonic bounded, we have established that every one of this six sequences has a limit.
Let

$$\lim_{n\to\infty} M_n^x = x_1, \lim_{n\to\infty} M_n^y = x_2, \lim_{n\to\infty} M_n^z = x_3, \lim_{n\to\infty} N_n^x = y_1, \lim_{n\to\infty} N_n^y = y_2, \lim_{n\to\infty} N_n^z = y_3.$$

Passing to the limit as $n \to \infty$ in (4.5), we get

$$\begin{aligned} x_1 &= \frac{1}{a}[r_1 - \frac{c_1 y_2}{h_1 + x_1} - \frac{c_2 y_3}{h_2 + x_1} - \frac{qE}{d_1 E + d_2 x_1}], \\ x_2 &= \frac{r_2(h_1 + x_1)}{f_1}, \\ x_3 &= \frac{r_3(h_2 + x_1)}{f_2}, \\ y_1 &= \frac{1}{a}[r_1 - \frac{c_1 x_2}{h_1 + y_1} - \frac{c_2 x_3}{h_2 + y_1} - \frac{qE}{d_1 E + d_2 y_1}] \\ y_2 &= \frac{r_2(h_1 + y_1)}{f_1}, \\ y_3 &= \frac{r_3(h_2 + y_1)}{f_2}. \end{aligned} \tag{4.6}$$

It is clear that $x_1 = y_1, x_2 = y_2$ and $x_3 = y_3$. Thus we obtain $x_1 = x^*, x_2 = y^*, x_3 = z^*$ as a solution of (15). Hence, the global asymptotic stability of $(x^*, y^*, z^*)$ is obtained. This completes the proof of the theorem. □

## 5. Bifurcation Study

In this section, we discuss the parametric restrictions for obtaining Neimark-Sacker bifurcation at the interior fixed point $E^*$ of system (1.2).

### 5.1 Neimark-Sacker bifurcation

To examine Neimark-Sacker bifurcation in system (1.2), we need the following result [29].

**Lemma 5.1.** *Consider an n-dimensional discrete dynamical system $U_{k+1} = f_m(U_k)$ where $m \in \mathbb{R}$ is a bifurcation parameter. Let $U^*$ be fixed point of $f_m$ and the characteristic polynomial for Jacobian matrix $J(U^*) = (b_{ij})_{n\times n}$ of n-dimensional map $f_m(U_k)$ is given by*

$$P_m(\lambda) = \lambda^n + b_1 \lambda^{n-1} + \cdots + b_{n-1}\lambda + b_n \tag{5.1}$$

*where $b_i = b_i(m,u), i = 1,2,3,\cdots,n$ and $u$ is a control parameter or another parameter to be deduced. Let $\Delta_0^{\pm}(m,u) = 1, \Delta_1^{\pm}(m,u), \cdots, \Delta_n^{\pm}(m,u)$ be a sequence of determinants defined by $\Delta_i^{\pm}(m,u) = det(M_1 \pm M_2), i = 1,2,3,\cdots,n$ where*

$$M_1 = \begin{pmatrix} 1 & b_1 & b_2 & \cdots & b_{i-1} \\ 0 & 1 & b_1 & \cdots & b_{i-2} \\ 0 & 0 & 1 & \cdots & b_{i-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} b_{n-i+1} & b_{n-i+2} & \cdots & b_{n-1} & b_n \\ b_{n-i+2} & b_{n-i+3} & \cdots & b_n & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{n-1} & b_n & \cdots & 0 & 0 \\ b_n & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

*Moreover, the following conditions hold:*

**A1** *Eigenvalue assignment*

$$\Delta_{n-1}^{-}(m_0,u) = 0, \Delta_{n-1}^{+}(m_0,u) > 0, P_{m_0}(1) > 0, (-)^n P_{m_0}(-1) > 0, \Delta_i^{\pm}(m_0,u) > 0, i = n-3, n-5, \cdots, 1 (or\ 2),$$

*when n is even or odd, respectively.*

**A2** *Transversality condition:* $[\frac{d(\Delta_{n-1}^{-}(m,u))}{dm}]_{m=m_0} \neq 0.$

**A3** *Non-resonance condition:*

$$cos(2\pi/j) \neq \psi, or\ resonance\ condition\ cos(2\pi/j) = \psi\ where j = 3,4,5,\cdots$$

*and $\psi = 1 - 0.5P_{m_0}(1)\Delta_{n-3}^{-}(m_0,u)/\Delta_{n-2}^{+}(m_0,u)$. Then Neimark-Sacker bifurcation occurs at $m_0$.*

Now we state bifurcation result by considering $a$ as a bifurcation parameter of system (1.2).

**Theorem 5.2.** *The fixed point $E^*$ of system (1.2) admits Neimark-Sacker bifurcation if the following conditions are satisfied:*

$$\begin{aligned} 1-p_2+p_3(p_1-p_3) &= 0, \\ 1+p_2-p_3(p_1+p_3) &> 0, \\ 1+p_1+p_2+p_3 &> 0, \\ 1-p_1+p_2-p_3 &> 0 \end{aligned} \tag{5.2}$$

*where $p_1$, $p_2$ and $p_3$ are defined in (3.7).*

*Proof.* Following Lemma 4.1, we have found the following equalities and inequalities:

$$\begin{aligned} \Delta_2^-(a^*) &= 1-p_2+p_3(p_1-p_3) = 0, \\ \Delta_2^+(a^*) &= 1+p_2-p_3(p_1+p_3) > 0, \\ P_{a^*}(1) &= 1+p_1+p_2+p_3 > 0, \\ (-1)^3 P_{a^*}(-1) &= 1-p_1+p_2-p_3 > 0. \end{aligned} \tag{5.3}$$

□

## 6. Chaos Control

Here, we examine chaos control for system (1.2). It is more pertinent for model related with biological species. It is normally seen that discrete-time models are more chaotic and complicated than the continuous systems. Thus it is justifiable to execute control method to prevent any uncertainty. We primarily apply hybrid control process discussed in [30]. This technique takes a single control parameter which lies in the open unit interval. Various types of methods are available for regulating chaos in discrete systems, for example, state feed back method, pole-placement technique and hybrid control method [31]-[**?**] in which, hybrid control technique is most simple to apply. We use hybrid control technique to system (1.2) for controlling chaos developed through bifurcation. Assume that the system admits Neimark-Sacker bifurcation at its fixed point $(x^*, y^*, z^*)$, then the corresponding controlled system using the hybrid control method is given by:

$$\begin{aligned} x_{n+1} &= \rho x_n \exp\{r_1 - a x_n - \frac{c_1 y_n}{h_1+x_n} - \frac{c_2 z_n}{h_2+x_n} - \frac{qE}{d_1 E + d_2 x_n}\} + (1-\rho)x_n, \\ y_{n+1} &= \rho y_n \exp\{r_2 - \frac{f_1 y_n}{h_1+x_n}\} + (1-\rho)y_n, \\ z_{n+1} &= \rho z_n \exp\{r_3 - \frac{f_2 y_n}{h_2+x_n}\} + (1-\rho)z_n. \end{aligned} \tag{6.1}$$

where $0 < \rho < 1$ is taken as a control parameter. The Jacobian matrix of controlled system (6.1) evaluated at $E^*$ is given by

$$J(x^*, y^*, z^*) = \begin{pmatrix} 1-\rho x^*(a - \frac{c_1 y^*}{(h_1+x^*)^2} - \frac{c_2 z^*}{(h_2+x^*)^2} - \frac{qEd_2}{(d_1E+d_2x^*)^2}) & -\frac{\rho x^* c_1}{h_1+x^*} & \frac{\rho x^* c_2}{h_2+x^*} \\ \frac{\rho {y^*}^2 f_1}{(h_1+x^2)^2} & 1-\rho r_2 & 0 \\ \frac{\rho {z^*}^2 f_2}{(h_2+x^2)^2} & 0 & 1-\rho r_3 \end{pmatrix} \tag{6.2}$$

The fixed point $E^*$ of controlled system (6.1) is locally asymptotically stable if all the roots of the characteristic polynomial of (6.2) lie in an unit open disk.

## 7. Numerical Simulations

In this section, we present some numerical computations to justify our analytical results. We show the role of the intra-specific competition coefficient among the prey species, harvesting effort and the maximum value of per capita reduction rate of $y$ can attain on the discrete system visually through numerical simulations.

**Example 7.1.** *Suppose $r_1 = 0.8, r_2 = 0.5, r_3 = 0.4, c_1 = 0.01, c_2 = 0.02, h_1 = 1, h_2 = 1, d_1 = 1, d_2 = 1, f_1 = 0.2, f_2 = 0.1, a = 0.1, q = 0.1, E = 1$ for system (1.2). Then all the conditions of Theorem 4.4 are satisfied. Thus the fixed point $E^* = (6.878, 19.94, 30.72)$ is globally asymptotically stable (see Fig. 7.1). The Fig. 7.1) shows that initially all the population increases and eventually all the interacting populations get their steady states and finally become globally asymptotically stable.*

**Example 7.2.** *Suppose* $r_1 = 3.5, r_2 = 2.2, r_3 = 2, c_1 = 0.2, c_2 = 1, h_1 = 1, h_2 = 1, d_1 = 1, d_2 = 1, f_1 = 1.5, f_2 = 1, a = 0.3, q = 0.2, E = 1$ *initial points (0.5, 0.5, 0.) for system (2). Then the conditions of Lemma 3.2 are violated. Thus the fixed point* $E^* = (3.894, 7.196, 9.813)$ *is unstable. Moreover, system (1.2) admits chaotic behaviour (see 7.2(a)). In order to show the effectiveness of hybrid control method implemented in system (6.1), we choose* $\rho = 0.5$ *and other parameters are same as in Example 7.2. The 7.2(b) shows that the solutions initiating from (0.5,0.5,0.5) approaches to the fixed point* $E^* = (3.894, 7.196, 9.813)$. *i.e., the steady state for controlled system (6.1) is a sink.*

**Example 7.3.** *Suppose* $r_1 = 3, r_2 = 2.2, r_3 = 2, c_1 = 0.2, c_2 = 1, h_1 = 1, h_2 = 1, d_1 = 1, d_2 = 1, f_1 = 1, f_2 = 1, q = 0.2, E = 1$ *and initial points (0.5, 0.5, 0.5) and* $a \in (0.1, 1.5)$ *in system (1.2) with the initial condition* $(x_0, y_0, z_0) = (0.5, 0.5, 0.5)$. *When a is considered as a bifurcation parameter, then at* $a = a^* = 0.326$, *the interior fixed point* $E^* = (1.46935, 5.43257, 4.9387)$ *becomes unstable and system (1.2) undergoes Neimark-Sacker bifurcation by Theorem 5.2. Bifurcation diagrams and maximum Lyapunov exponents (MLE) respect to the parameter a of system (1.2) are depicted in Fig. 7.3. As a increases, we observe that a transition from unstable to stable.*

**Example 7.4.** *Suppose* $r_1 = 2.98, r_2 = 2.2, r_3 = 2, c_1 = 0.2, c_2 = 1, h_1 = 1, h_2 = 1, d_1 = 1, d_2 = 1, f_1 = 1, f_2 = 1, q = 0.2, a = 0.3$ *and initial points (0.5, 0.5, 0.5) and* $a \in (0.5, 1.5)$ *in system (1.2) with the initial condition* $(x_0, y_0, z_0) = (0.5, 0.5, 0.5)$. *When E is considered as a bifurcation parameter, then at* $E = E_* = 0.978$, *the interior fixed point* $E^* = (1.435, 5.373, 4.884)$ *becomes unstable and system (1.2) undergoes Neimark-Sacker bifurcation by Theorem 5.2. Bifurcation diagrams and MLE respect to the parameter E of system (1.2) are depicted in Fig. 7.4. As E increases, we observe that a transition from unstable to stable.*

**Example 7.5.** *Suppose* $r_1 = 2.98, r_2 = 2.2, r_3 = 2, c_1 = 0.2, c_2 = 1, h_1 = 1, h_2 = 1, d_1 = 1, d_2 = 1, E = 1, f_2 = 1, q = 0.2, a = 0.3$ *and initial points (0.5, 0.5, 0.5) and* $f_1 \in (0.6, 2)$ *in system (2) with the initial condition* $(x_0, y_0, z_0) = (0.5, 0.5, 0.5)$. *When* $f_1$ *is considered as a bifurcation parameter, then at* $f_1 = f_1^* = 0.998$, *the interior fixed point* $E^* = (1.534, 5.584, 5.066)$ *becomes unstable and system (1.2) undergoes Neimark-Sacker bifurcation by Theorem 5.2. Bifurcation diagrams and MLE respect to the parameter* $f_1$ *of system (1.2) are depicted in Fig. 7.5. As* $f_1$ *increases, we observe that a transition from stable to unstable and then bifurcation within a limit cycle to a periodic window and finally to chaos.*

**Example 7.6.** *Suppose* $r_1 = 5.8, r_2 = 2, r_3 = 3, c_1 = 1, c_2 = 1, h_1 = 1, h_2 = 1, d_1 = 1, d_2 = 1, E = 0.2, f_1 = 1, f_2 = 1, q = 1, a = 1$ *and initial points (0.5, 3, 4) ,we obtained two interior fixed points* $E_+^* = (0.523607, 3.047214, 4.570821)$ *and* $E_-^* = (0.0763932, 2.1527864, 3.2291796)$ *both are unstable (see Fig. 7.6). Fig. 7.6(b) represents the time series plot of system (2) when* $E = 0.28$

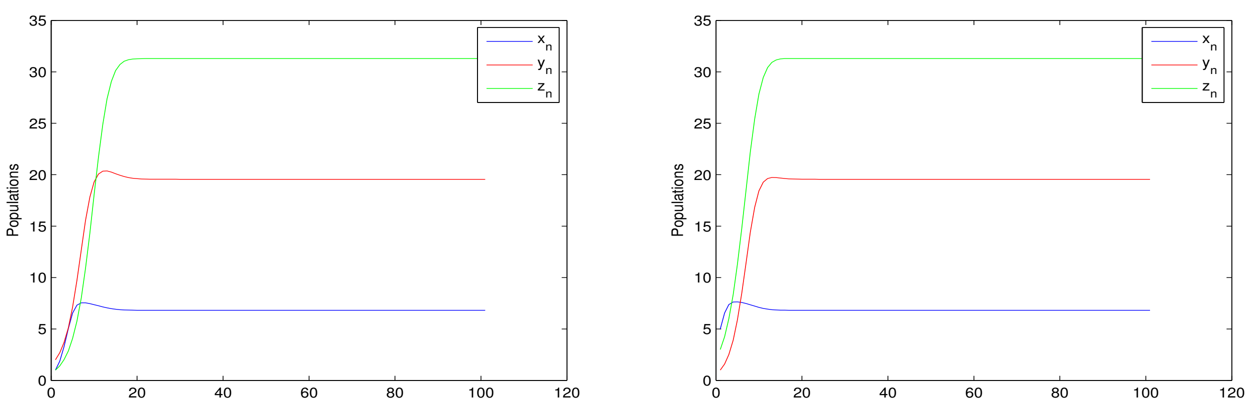


**Figure 7.1.** Time series plots of system (1.2) with parameter values $r_1 = 0.8, r_2 = 0.5, r_3 = 0.4, c_1 = 0.01, c_2 = 0.02, h_1 = 1, h_2 = 1, d_1 = 1, d_2 = 1, f_1 = 0.2, f_2 = 0.1, a = 0.1, q = 0.1, E = 1$ and initial points (1, 2, 1) and (5, 1, 3).

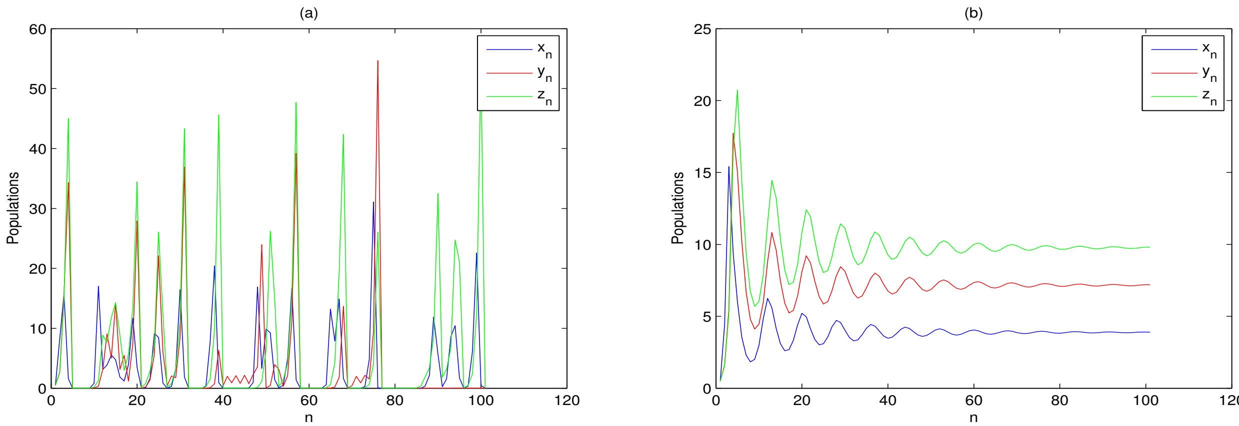


**Figure 7.2.** (a) Time series plots of system (1.2) with parameter values $r_1 = 3.5, r_2 = 2.2, r_3 = 2, c_1 = 0.2, c_2 = 1, h_1 = 1, h_2 = 1, d_1 = 1, d_2 = 1, f_1 = 1.5, f_2 = 1, a = 0.3, q = 0.2, E = 1$ with initial points (0.5, 0.5, 0.5) and (b) phase portrait of controlled system (6.1) for $\rho = 0.5$

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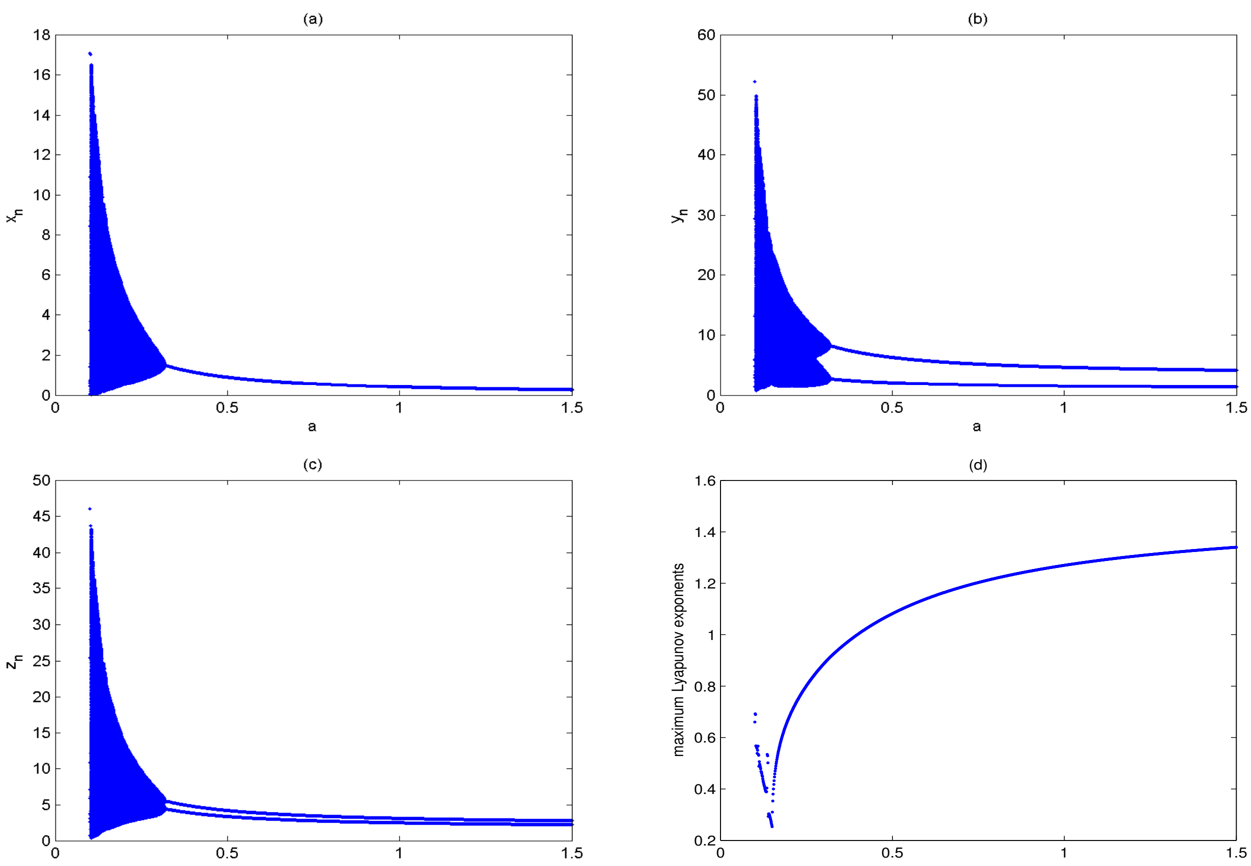


**Figure 7.3.** Bifurcation diagrams and MLE for system (1.2) with parameter values $r_1 = 3, r_2 = 2.2, r_3 = 2, c_1 = 0.2, c_2 = 1, h_1 = 1, h_2 = 1, d_1 = 1, d_2 = 1, f_1 = 1, f_2 = 1, q = 0.2, E = 1, a \in (0.1, 1.5)$ and initial point (0.5, 0.5, 0.5).

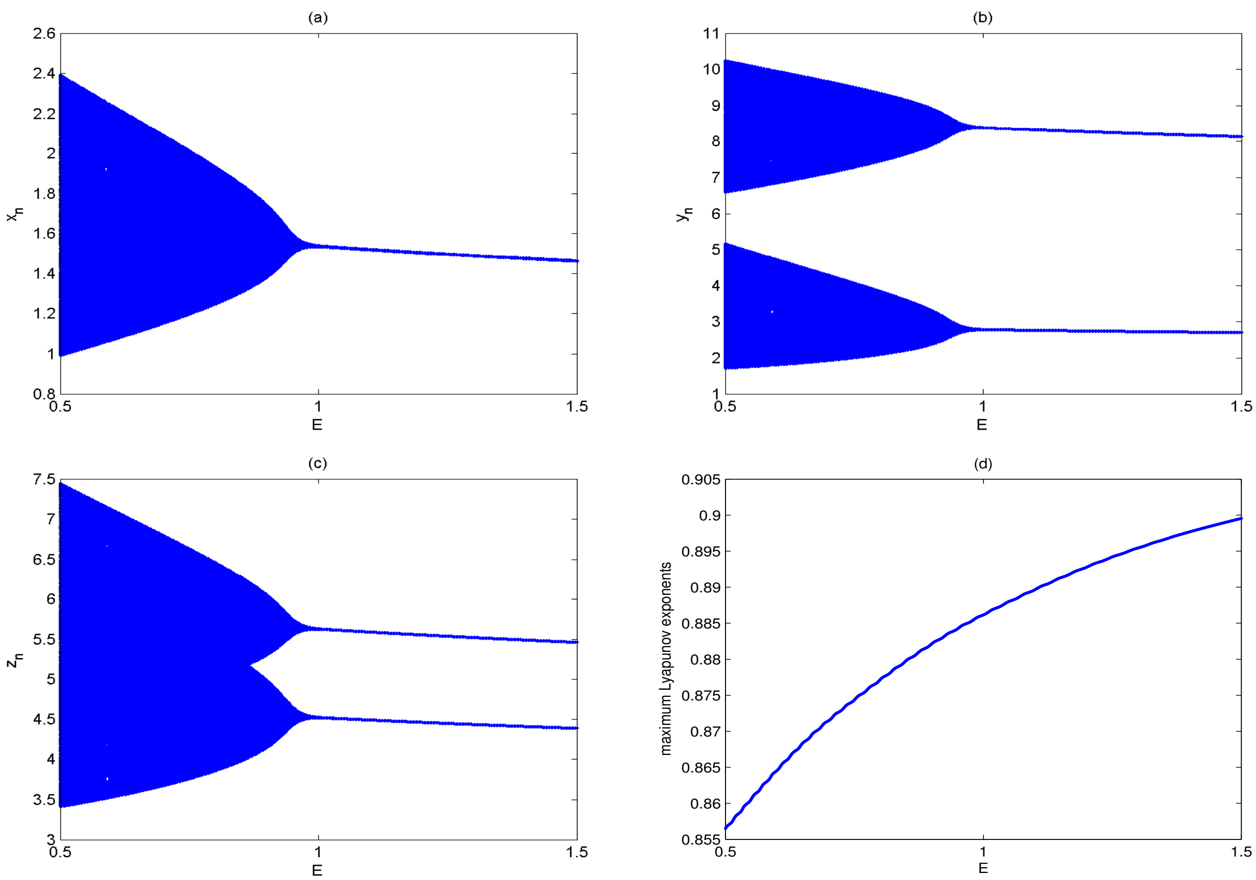


**Figure 7.4.** Bifurcation diagrams and MLE for system (1.2) with parameter values $r_1 = 2.98, r_2 = 2.2, r_3 = 2, c_1 = 0.2, c_2 = 1, h_1 = 1, h_2 = 1, d_1 = 1, d_2 = 1, f_1 = 1, f_2 = 1, q = 0.2, a = 0.3, E \in (0.5, 1.5)$ and initial point (0.5, 0.5, 0.5)

.

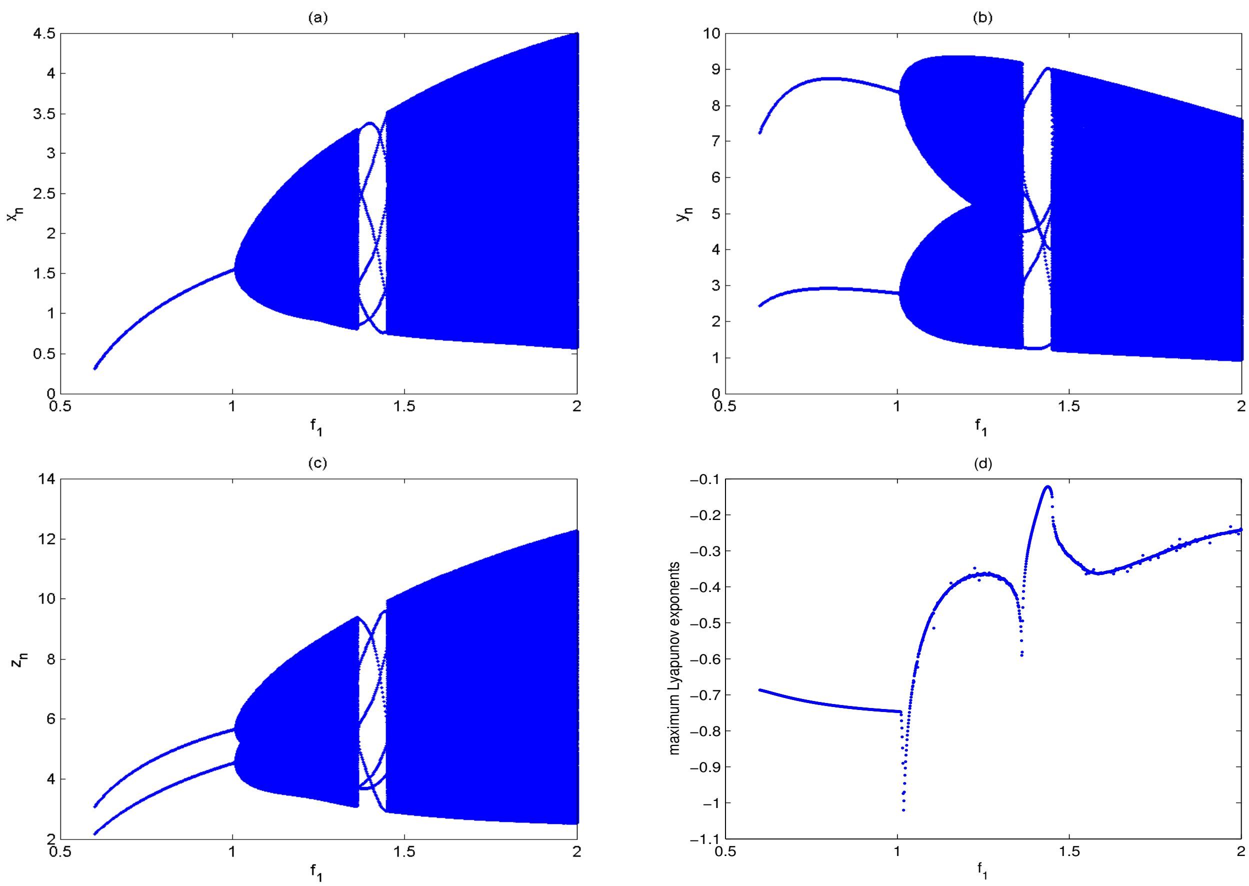


**Figure 7.5.** Bifurcation diagrams and MLE for system (1.2) with parameter values $r_1 = 2.98, r_2 = 2.2, r_3 = 2, c_1 = 0.2, c_2 = 1, h_1 = 1, h_2 = 1, d_1 = 1, d_2 = 1, E = 1, f_2 = 1, q = 0.2, a = 0.3, f_1 \in (0.6, 2)$ and initial point (0.5, 0.5, 0.5)

.

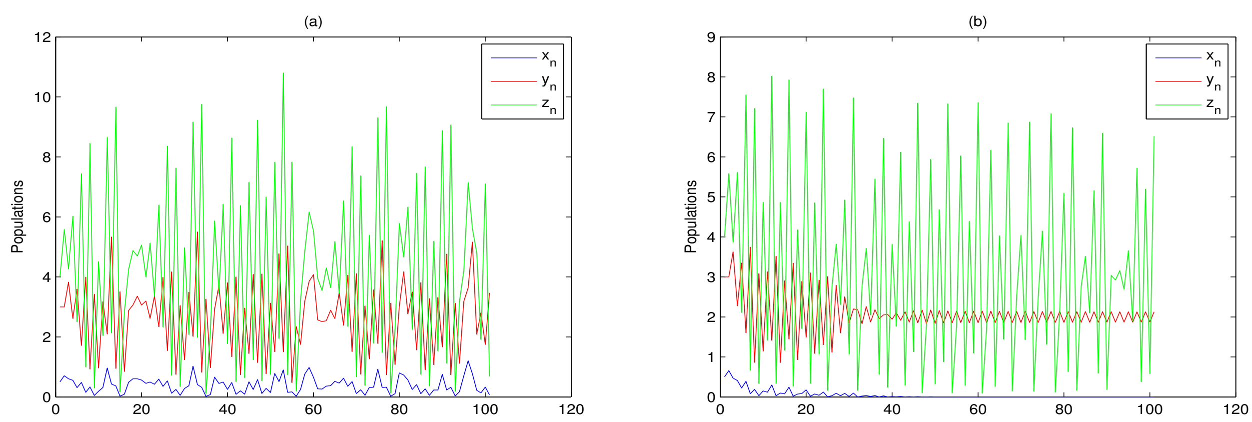


**Figure 7.6.** Time series plots of system (1.2) with parameter values $r_1 = 5.8, r_2 = 2, r_3 = 3, c_1 = 1, c_2 = 1, h_1 = 1, h_2 = 1, d_1 = 1, d_2 = 1, f_1 = 1, f_2 = 1, a = 1, q = 1$ for $E = 0.2$ and 0.28 respectively. initial point (0.5, 3, 4).

## 8. Discussion

In this article, a discrete-time Leslie-Gower two predator-one prey system with Michaelis-Menten type prey harvesting is investigated. To our knowledge, there are a few works that address the impact of non-linear harvesting on System (1.2). It is

shown that the system has at most twelve fixed points. Qualitative analysis shows that all the boundary fixed points, excepting $E_{23}$ are unstable. Under certain restrictions on the system parameters, $E_{23}$ may be stable, which in turn implies that that the prey population goes into extinction. As the trivial fixed point always exists and unstable, the three species cannot go to extinction together. It is established that multiple fixed points exist due to the presence of non-linear harvesting term. It is shown that Neimark-Sacker bifurcation occurs at the unique positive fixed point when the parameters $a, E, f_1$ are varied. The choice of these parameters is arbitrary, one may find similar type of bifurcations for other parameters also. Numerical simulations show that when the parameters $a$ and $E$ exceed a certain critical value, the system becomes stable (see Figs. 7.3 and 7.4) whereas the opposite holds $f_1$ is increased. In case of multiple fixed points, chaotic behaviour is observed. In particular, we observe when the predator population is chaotic, the prey population ultimately tends to extinct. This fact is clear when we increase the harvest rate from 0.2 to 0.28 (see Fig. 7.6 ). The proposed model admits more rich characteristics and more complicated dynamics than that exist in the continuous case. We have derived the condition for global stability of the positive fixed point by applying the iteration scheme and comparison principle of difference equations. Conditions of Theorem 4.4 indicate that when the intrinsic growth rate of the three species remains below one, the positive fixed point is globally asymptotically stable.
Sometimes bifurcation and chaotic behaviour are in fact unwanted situations in discrete dynamical systems, because there may be an extinction of the population due to chaos. So chaos control becomes a crucial issue. To prevent chaos, we have used the hybrid control method so that the stability of the system can be regained.
To our understanding, the dynamical study of discrete time model considering a Leslie-Gower two predator-one prey system with Michaelis-Menten type prey harvesting has not investigated yet.

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