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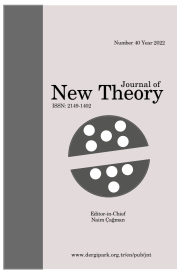

A New Transformation Method for Solving High-Order Boundary Value Problems

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Research Article

Abstract − The main purpose of this work is to provide a new approximation method, the so-called parameterised differential transform method (PDTM), for solving high-order boundary value problems (HOBVPs). Our method differs from the classical differential transform method by calculating the coefficients of the solution, which has the form of a series. We applied the proposed new method to fourth-order boundary value problems to substantiate it. The resulting solution is graphically compared with the exact solution and the solutions obtained by the classical DTM and ADM methods.

Keywords – *Differential transform method, boundary value problem, approximate solution*

Mathematics Subject Classification (2020) – 34B15, 65K05

1. Introduction

High-order boundary value problems (HOBVPs) for differential equations are essential for modelling a wide range of physical and chemical phenomena in all areas of natural science. In most cases, obtaining an exact solution to these boundary value problems is impossible. Therefore, there is growing interest in developing a new numerical method for finding a reliable approximate solution or understanding the exact solution's qualitative nature. This paper presents a new approximate method, parameterised differential transform, for solving HOBVPs. To show the applicability and effectiveness of PDTM, we will solve an illustrative fourth-order boundary value problem using this method.

Zhou firstly developed the classical differential transform method (DTM) in 1986 to solve some boundary value problems that arise when modelling electrical circuits [1]. In recent years, the application of classical DTM and its various generalisation to the solution of (HOBVPs) has attracted great interest. For example, Momami and Noor demonstrated different analytical techniques using the classical differential transform method, Adomian decomposition method (ADM) and homotopy perturbation method and gave a numerical comparison of these methods when solving a special fourth-order boundary value problem [2]. Hassan and Ertürk employed the DTM to solve HOBVPs [3]. Hussin and Kılıçman applied the DTM and ADM to solve linear and nonlinear HOBVPs [4]. Zahar used the DTM to obtain numerical solutions of singular perturbed fourth-order BVP's [5]. Ertürk and Momani used the DTM and ADM to get a numerical solution for fourth-

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order BVP [6]. They also gave a numerical comparison between these methods. Wazwaz used a modified decomposition method to obtain the numerical solution of special type HOBVPs [7]. In some papers, the DTM is generalised so that it can be used to study approximate solutions and the spectral properties of various types of Sturm-Liouville problems (see [8-11]).

In [12], Boukary et al. use the ADM to solve the fourth-order parabolic type partial differential equations. Kwami et al. [13] introduced a new modification of ADM for solving fourth-order ODEs. This modification is based on transforming the considered fourth-order ODE into an equivalent integral equation of the Volterra type. Mukhtarov and Yücel [14] utilised the ADM to investigate the eigenvalues and eigenfunctions of two-interval SLPs that arise in modelling many phenomena in physics and engineering. Rysak and Gregorczyk [15] have shown the effectiveness of the DTM in solving problems arising in fractional dynamical systems.

Yücel and Mukhtarov [16] developed a new generalisation of the DTM for solving nonclassical boundary value problems, which differs from the classical boundary value problems in that, the boundary conditions contain some internal points at which given additional conditions. Al-Saif and Harfash compare the reduced DTM and perturbation-iteration method in [17] solving two-dimensional Navier-Stokes equations. Duan et al. [18] provided an overview of the DTM and its applications in solving fractional-order differential equations. The applicability of DTM to a system of differential equations was investigated by Ayaz [19].

2. Parameterized DTM

Let $s\colon [c,d] \to R$ be a real-valued analytic function and $\alpha \in [0,1]$ be any real parameter.

Definition 2.1. [10] We say the sequence $\big(S_\alpha(c,d)\big)_n(s)$ is the parameterised differential transform of the original function $s(t)$ if

$$\big(S_\alpha(c,d)\big)_n(s) := \alpha\big(S_c(s)\big)_n + (1-\alpha)\big(S_d(s)\big)_n$$

where

$$\big(S_c(s)\big)_n := \frac{d^n s(c)}{n!} \text{ and } \big(S_d(s)\big)_n := \frac{d^n s(d)}{n!}$$

Definition 2.2. [10] We say the function $\mathrm{s(t)}$ is the inverse differential transform if

$$s_\alpha(t) := \sum_{n=0}^{\infty} \big(S_\alpha(c,d)\big)_n(s)\big(t-(\alpha c+(1-\alpha)d)\big)^n \tag{1}$$

provided that the series is convergent. The inverse differential transforms, we will denote by $(S_\alpha^{-1}(c,d)_n)(s)$.

Definition 2.3. [10] The N-th partial sum of the series (1) is said to be an N-th parameterised approximation of the original function $\mathrm{s(t)}$ and is denoted by $s_{\alpha,N}(t)$, that is

$$s_{\alpha,N}(t) := \sum_{n=0}^{N} \big(S_\alpha(c,d)\big)_n(s)\big(t-(\alpha c+(1-\alpha)d)\big)^n \tag{2}$$

By Definition 2.1., we can show that the parameterised differential transform has the following properties:

i. $\big(S_\alpha(c,d)\big)_n(\mu s) = \mu(S_\alpha(c,d)(s))_n$

ii. $\big(S_\alpha(c,d)\big)_n(f \pm s) = \big(S_\alpha(c,d)\big)_n(f) \pm \big(S_\alpha(c,d)\big)_n(s)$

iii. $\big(S_\alpha(c,d)\big)_n\left(\frac{d^m s}{dt^m}\right) = \frac{(n+m)!}{n!}\big(S_\alpha(c,d)\big)_n(s)$

3. Numerical Results

Example 3.1. Consider the following HOBVP

$$y^{(4)}(t) = 4e^t + y(t), 0 < t < 1 \tag{3}$$

subject to the boundary conditions

$$y(0) = 1, y''(0) = 3, y(1) = 2e, \text{ and } y''(1) = 4e \tag{4}$$

The exact solution for the problem is $y(t) = (1 + t)e^t$. A graph of the exact solution to the problem is presented in Fig. 1 as follows:

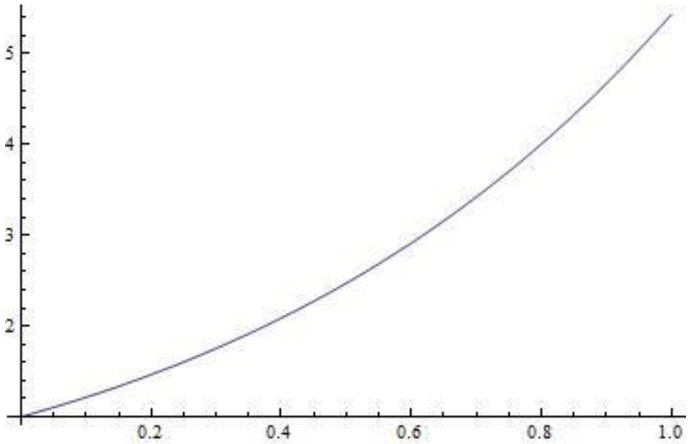

Fig. 1. Graph of the exact solution for the problem provided in (3)-(4)

If it is applied PDT to both sides of (3), then we obtain

$$\left(S_\alpha(0,1)\right)_{n+4}(y)(n+1)(n+2)(n+3)(n+4) = \left(\frac{4}{n!} + \left(S_\alpha(0,1)\right)_n(y)\right) \tag{5}$$

Therefore, from the definition of PDT,

$$y_\alpha(t) = \sum_{n=0}^{\infty} \left(S_\alpha(0,1)\right)_n(y)(t - t_\alpha)^n$$

and

$$y''_\alpha(t) = \sum_{n=0}^{\infty} \left(S_\alpha(0,1)\right)_n(y)n(n-1)(t - t_\alpha)^{n-2}$$

Moreover, for the boundary conditions $y(0) = 1, y''(0) = 3, y(1) = 2e, y''(1) = 4e$,

$$y_\alpha(0) = \sum_{n=0}^{N} \left(S_\alpha(0,1)\right)_n(y)(\alpha - 1)^n = 1$$

$$y''_\alpha(0) = \sum_{n=0}^{N} \left(S_\alpha(0,1)\right)_n(y)n(n-1)(\alpha - 1)^{n-2} = 3$$

$$y_\alpha(1) = \sum_{n=0}^{N} \left(S_\alpha(0,1)\right)_n(y)(\alpha)^n = 2e$$

$$y''_\alpha(1) = \sum_{n=0}^{N} \left(S_\alpha(0,1)\right)_n(y)n(n-1)(\alpha)^{n-2} = 4e$$

respectively. Here, let $\left(S_\alpha(0,1)\right)_0(y)=\rho$, $\left(S_\alpha(0,1)\right)_1(y)=\sigma$, $\left(S_\alpha(0,1)\right)_2(y)=\tau$, and $\left(S_\alpha(0,1)\right)_3(y)=\omega$, then substituting in the recursive relation (5), we can calculate the other terms of the PDT as

$$\left(S_\alpha(0,1)\right)_4(y)=\frac{1}{3!}+\frac{\rho}{4!},\left(S_\alpha(0,1)\right)_5(y)=\frac{4}{5!}+\frac{\sigma}{5!},\left(S_\alpha(0,1)\right)_6(y)=\frac{4}{6!}+\frac{\tau}{6!}$$

$$\left(S_\alpha(0,1)\right)_7(y)=\frac{4}{7!}+\frac{6\omega}{7!},\left(S_\alpha(0,1)\right)_8(y)=\frac{8}{8!}+\frac{\rho}{8!},\left(S_\alpha(0,1)\right)_9(y)=\frac{8}{9!}+\frac{\sigma}{9!}$$

$$\left(S_\alpha(0,1)\right)_{10}(y)=\frac{8}{10!}+\frac{2\tau}{10!},\left(S_\alpha(0,1)\right)_{11}(y)=\frac{8}{11!}+\frac{6\omega}{11!},\ \left(S_\alpha(0,1)\right)_{12}(y)=\frac{12}{12!}+\frac{\rho}{12!}$$

$$\left(S_\alpha(0,1)\right)_{13}(y)=\frac{12}{13!}+\frac{\sigma}{13!},\ \left(S_\alpha(0,1)\right)_{14}(y)=\frac{12}{14!}+\frac{2\tau}{14!}\ ,\ \left(S_\alpha(0,1)\right)_{15}(y)=\frac{12}{15!}+\frac{6\omega}{15!},\ldots$$

Hence, the parameterised series solution $y_\alpha(t)$ is evaluated up to $N=15$:

$$\begin{aligned} y_\alpha(t) = & \sum_{n=0}^{15}\left(S_\alpha(0,1)\right)_n(y)(t-t_\alpha)^n \\ = & \ \rho+\sigma(t-1+\alpha)+\tau(t-1+\alpha)^2+\omega(t-1+\alpha)^3+\left(\frac{1}{3!}+\frac{\rho}{4!}\right)(t-1+\alpha)^4 \\ & +\left(\frac{4}{5!}+\frac{\sigma}{5!}\right)(t-1+\alpha)^5+\left(\frac{4}{6!}+\frac{\tau}{6!}\right)(t-1+\alpha)^6+\left(\frac{4}{7!}+\frac{6\omega}{7!}\right)(t-1+\alpha)^7 \\ & +\left(\frac{8}{8!}+\frac{\rho}{8!}\right)(t-1+\alpha)^8+\left(\frac{8}{9!}+\frac{\sigma}{9!}\right)(t-1+\alpha)^9+\left(\frac{8}{10!}+\frac{2\tau}{10!}\right)(t-1+\alpha)^{10} \\ & +\left(\frac{8}{11!}+\frac{6\omega}{11!}\right)(t-1+\alpha)^{11}+\left(\frac{12}{12!}+\frac{\rho}{12!}\right)(t-1+\alpha)^{12}+\left(\frac{12}{13!}+\frac{\sigma}{13!}\right)(t-1+\alpha)^{13} \\ & +\left(\frac{12}{14!}+\frac{2\tau}{14!}\right)(t-1+\alpha)^{14}+\left(\frac{12}{15!}+\frac{6\omega}{15!}\right)(t-1+\alpha)^{15} \end{aligned}$$

Furthermore, the numbers ρ,σ,τ,ω are evaluated from the boundary conditions (4).

For $\alpha=\frac{1}{2},\alpha=\frac{1}{5},\alpha=\frac{999}{1000}$ the numerical PDT solutions are presented in Fig. 2-4 as follows:

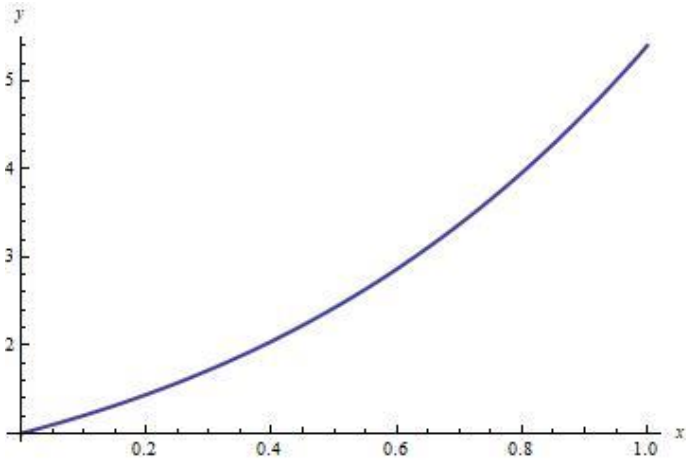

Fig. 2. Graph of the PDTM solution for the problem provided in (3)-(4) ($\alpha=\frac{1}{2}$)

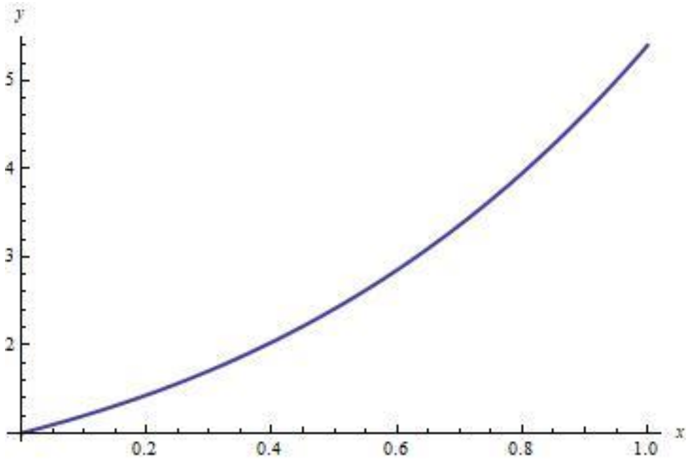

Fig. 3. Graph of the PDTM solution ($\alpha = \frac{1}{5}$)

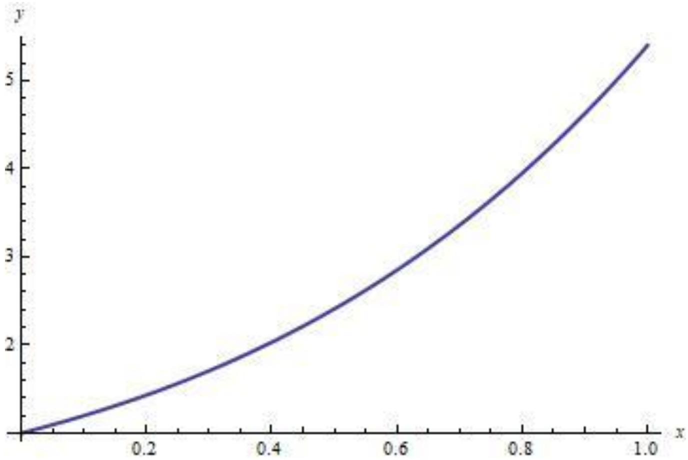

Fig. 4. Graph of the PDTM solution $\left(\alpha = \frac{999}{1000}\right)$

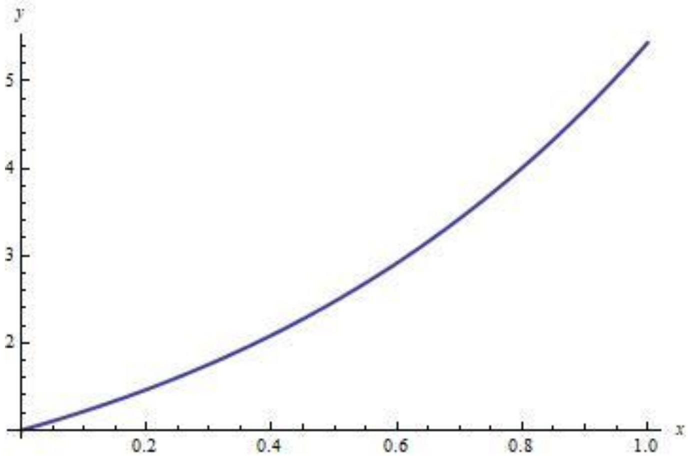

Fig. 5. Graph of the ADM solution for the problem provided in (3)-(4) (See, [7])

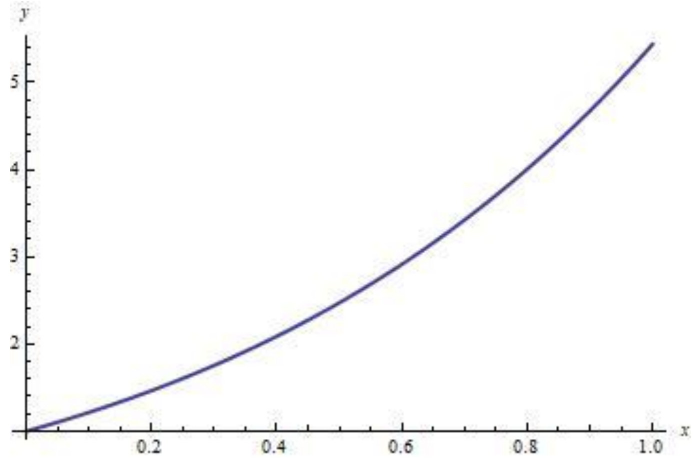

Fig. 6. Graph of the DTM solution for the problem provided in (3)-(4) (See, [6])

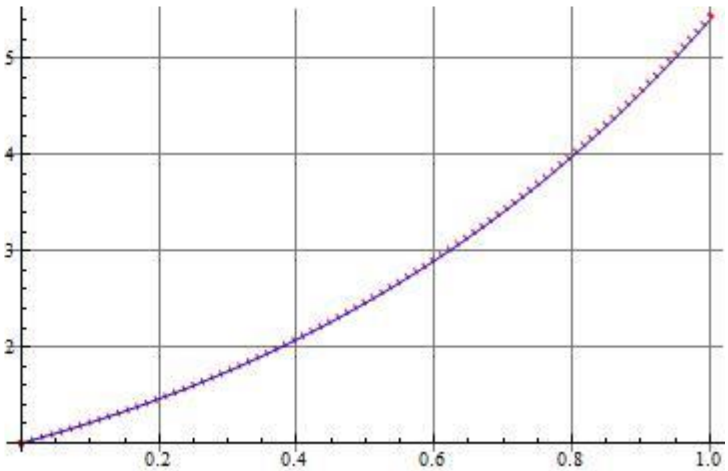

Fig. 7. Comparison of exact solution (red line) and the approximate solution for problem provided in (3)-(4) obtained by using DTM [6] (blue line)

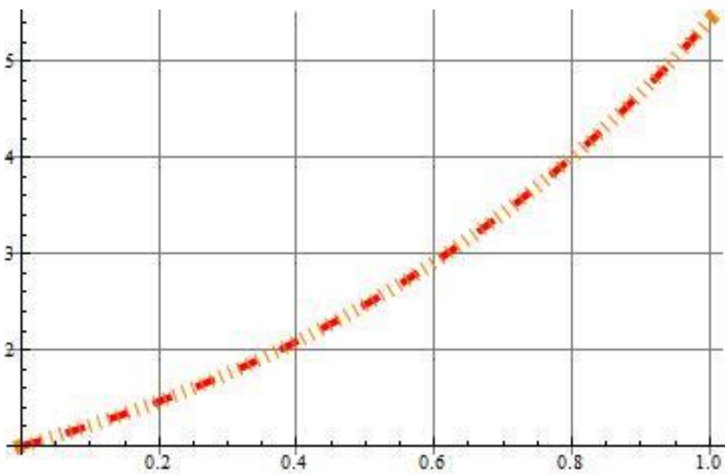

Fig. 8. Comparison of exact solution (red line) and the approximate solution for problem provided in (3)-(4) obtained by using PDTM (orange line)

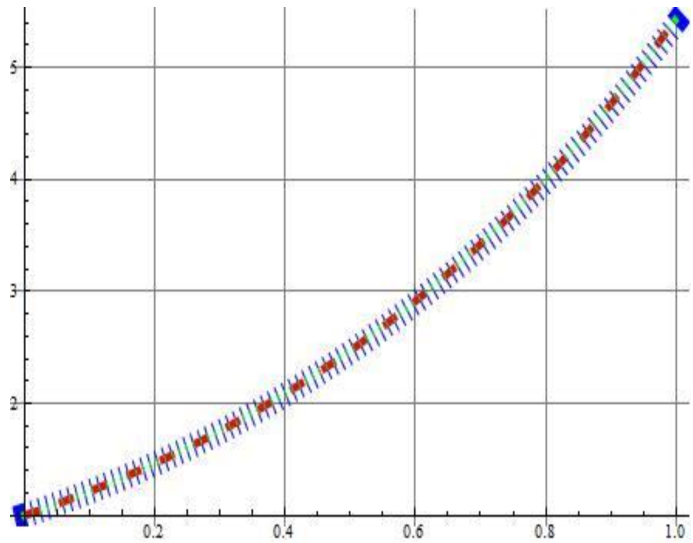

Fig. 9. Comparison of exact solution (red line) and the approximate solution for problem provided in (3)-(4) obtained by using ADM [7] (green line), PDTM (blue line)

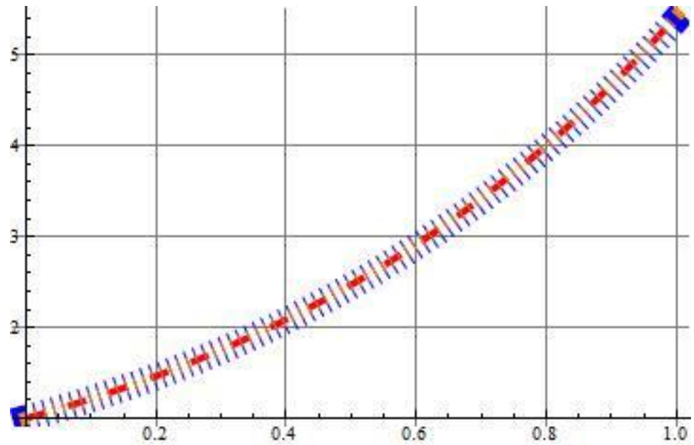

Fig. 10. Comparison of exact solution (red line) and the approximate solution for problem provided in (3)-(4) obtained by using DTM [6] (orange line), PDTM (blue line)

Example 3.2. Consider the boundary value problem

$$y^{(4)}(t) = y(t) + y''(t) + e^t(t-3) \tag{6}$$

subject to the boundary conditions

$$y(0) = 1, y'(0) = 0, y(1) = 0, y'(1) = -e \tag{7}$$

The exact solution to the problem is

$$y(t) = (1-t)e^t$$

If it is applied PDT to both sides of (6), then we obtain

$$\big(S_\alpha(0,1)\big)_{n+4}(y)(n+1)(n+2)(n+3)(n+4) = \big(S_\alpha(0,1)\big)_n(y) + \big(S_\alpha(0,1)\big)_{n+2}(y)(k+1)(k+2)$$

$$-\frac{3}{n!} + \sum_{n_1}^{n} \frac{\delta(n_1-1)}{(n-n_1)!} \tag{8}$$

Therefore, from the definition of PDT,

$$y_\alpha(t) = \sum_{n=0}^{\infty} \big(S_\alpha(0,1)\big)_n(y)(t - t_\alpha)^n$$

and

$$y'_\alpha(t) = \sum_{n=0}^{\infty} \big(S_\alpha(0,1)\big)_n(y)n(t - t_\alpha)^{n-1}.$$

Moreover, for the boundary conditions $y(0) = 1, y'(0) = 0, y(1) = 0, y'(1) = -e$,

$$y_\alpha(0) = \sum_{n=0}^{N} \big(S_\alpha(0,1)\big)_n(y)(\alpha - 1)^n = 1$$

$$y'_\alpha(0) = \sum_{n=0}^{N} \big(S_\alpha(0,1)\big)_n(y)n(\alpha - 1)^{n-1} = 0$$

$$y_\alpha(1) = \sum_{n=0}^{N} \big(S_\alpha(0,1)\big)_n(y)(\alpha)^n = 0$$

$$y'_\alpha(1) = \sum_{n=0}^{N} \big(S_\alpha(0,1)\big)_n(y)n(\alpha)^{n-1} = -e$$

respectively. Here, let $\big(S_\alpha(0,1)\big)_0(y) = \rho_1, \big(S_\alpha(0,1)\big)_1(y) = \rho_2, \big(S_\alpha(0,1)\big)_2(y) = \rho_3, \big(S_\alpha(0,1)\big)_3(y) = \rho_4$ and then substituting in the recursive relation (8), we can calculate the other terms of the PDT as

$$\big(S_\alpha(0,1)\big)_4(y) = \frac{1}{4!}(\rho_1 + 2\rho_3 - 3) \qquad \big(S_\alpha(0,1)\big)_5(y) = \frac{1}{120}(\rho_2 + 6\rho_4 - 2)$$

$$\big(S_\alpha(0,1)\big)_6(y) = \frac{1}{360}\left(\frac{\rho_1}{2} + 2\rho_3 - 2\right) \qquad \big(S_\alpha(0,1)\big)_7(y) = \frac{1}{840}\left(\frac{\rho_2 + 12\rho_4 - 2}{6}\right)$$

$$\big(S_\alpha(0,1)\big)_8(y) = \frac{1}{1680}\left(\frac{\rho_1}{12} + \frac{\rho_3}{4} - \frac{1}{4}\right) \qquad \big(S_\alpha(0,1)\big)_9(y) = \frac{1}{3024}\left(\frac{\rho_2}{60} + \frac{3\rho_4}{20} - \frac{1}{60}\right)$$

$$\big(S_\alpha(0,1)\big)_{10}(y) = \frac{1}{5040}\left(\frac{3\rho_1}{720} + \frac{5\rho_3}{360} - \frac{7}{720}\right)$$

Hence, the parameterised series solution $y_\alpha(t)$ is evaluated up to $N = 10$:

$$\begin{aligned}
y_\alpha(t) &= \sum_{n=0}^{10} \big(S_\alpha(0,1)\big)_n(y)(t - t_\alpha)^n \\
&= \rho_1 + \rho_2(t - 1 + \alpha) + \rho_3(t - 1 + \alpha)^2 + \rho_4(t - 1 + \alpha)^3 + \frac{1}{4!}(\rho_1 + 2\rho_3 - 3)(t - 1 + \alpha)^4 \\
&\quad + \frac{1}{120}(\rho_2 + 6\rho_4 - 2)(t - 1 + \alpha)^5 + \frac{1}{360}\left(\frac{\rho_1}{2} + 2\rho_3 - 2\right)(t - 1 + \alpha)^6 \\
&\quad + \frac{1}{840}\left(\frac{\rho_2 + 12\rho_4 - 2}{6}\right)(t - 1 + \alpha)^7 + \frac{1}{1680}\left(\frac{\rho_1}{12} + \frac{\rho_3}{4} - \frac{1}{4}\right)(t - 1 + \alpha)^8 \\
&\quad + \frac{1}{3024}\left(\frac{\rho_2}{60} + \frac{3\rho_4}{20} - \frac{1}{60}\right)(t - 1 + \alpha)^9 + \frac{1}{5040}\left(\frac{3\rho_1}{720} + \frac{5\rho_3}{360} - \frac{7}{720}\right)(t - 1 + \alpha)^{10}
\end{aligned}$$

A graph of the exact solution to the problem is presented in Fig. 11 as follows:

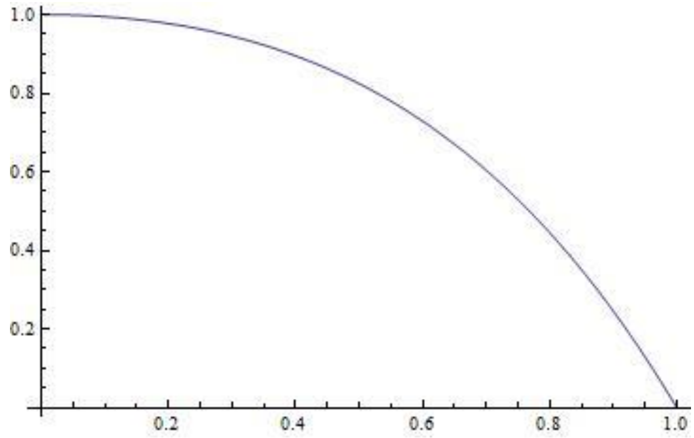

Fig. 11. Graph of the exact solution of the problem provided in (6)-(7)

For $\alpha = \frac{1}{2}$ the numerical PDT solution is presented in Fig. 12 as follows:

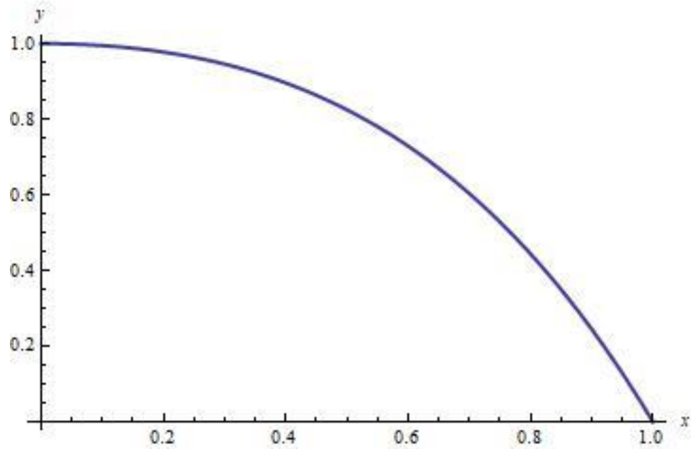

Fig. 12. Graph of the numerical PDT solution for $\alpha = \frac{1}{2}$ of the problem provided in (6)-(7)

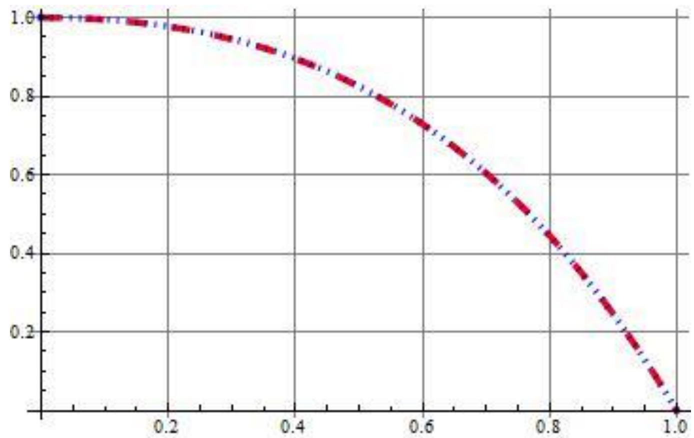

Fig. 13. Comparison of the exact solution (red dashing) with the PDT solution for $\alpha = \frac{1}{2}$ (blue line)

4. Conclusion

This paper provides a new semi-analytical method, the so-called parameterised differential transform method (P DTM), to find an exact solution in the series form or approximate various high-order boundary value problems. To show the reliability of our method, we solved an illustrative high-order boundary value problem. We compared the resulting solutions with the analytical solution and with the solutions obtained by the traditional DTM and ADM methods. Figures 5-10 and 13 show that the PDTM is the efficient method for solving high-order boundary value problems.

Author Contributions

All the authors contributed equally to this work. They all read and approved the last version of the manuscript.

Conflict of Interest

All the authors declare no conflict of interest.

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