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Certain Curvature Conditions on (k,μ)-Paracontact Metric Spaces

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Abstract
The aim of this paper is to classify (k,μ)-paracontact metric spaces satisfying certain curvature conditions. We present the curvature tensors of (k,μ)-Paracontact manifold satisfying the conditions $R\cdot W_6=0$, $R\cdot W_7=0$, $R\cdot W_8=0$ and $R\cdot W_9=0$. According these cases, (k,μ)-Paracontact manifolds have been characterized. Also, several results are obtained.

Keywords: (k,μ)-Paracontact Manifold, η-Einstein manifold, Riemannian curvature tensor
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1. Introduction

Paracontact manifolds are smooth manifolds of dimension $(2n+1)$ equipped with a 1-form η, a vector field ξ and a field of endomorphisms of tangent spaces ϕ such that $\eta(\xi)=1$, $\phi^2=I-\eta\otimes\xi$ and ϕ induces an almost paracomplex structure by kernel of η [1]. On the other hand, if the manifold is equipped with a pseudo-Riemannian metric g of signature $(n+1,n)$ satisfying

$$g(\phi X,\phi Y)=-g(X,Y)+\eta(X)\eta(Y),\ \ d\eta(X,Y)=g(X,\phi Y),$$

(M,η) becomes a contact manifold and (ϕ,ξ,η,g) is said to be a paracontact metric structure on M. In 1985, Kaneyuki and Williams initiated the perspective of paracontact geometry [5]. Zamkovoy performed a thorough study of paracontact metric Manifolds. [15]. Recently, B. Cappeletti-Montano, I. Küpeli Erken and C. Murathan introduced a new type of paracontact geometry so-called paracontact metric $(k,\mu)-$space, where k and μ are constant [4].

M. M. Tripathi and P. Gupta studied T-curvature tensors in semi-Riemannian manifolds. They defined T-conservative semi-Riemannian manifolds and give necessary and sufficient tensor on a Riemannian manifolds to be T-conservative. They proved that every T-flat semi-Riemannian manifold is Einstein. They also gave the conditions for semi-Riemannian manifold to be T-flat [8]. Since then several geometers studied curvature conditions and obtain various important properties [2, 6], [9]-[13].

The object of this paper is to study properties of the some certain curvature tensor in a $(k,\mu)-$paracontact metric manifold. In the present paper we survey $R\cdot W_6=0$, $R\cdot W_7=0$, $R\cdot W_8=0$ and $R\cdot W_9=0$, where W_6, W_7, W_8 and W_9 denote curvature tensors of manifold, respectively.

2. Preliminaries

An $(2n+1)$-dimensional manifold M is called to have an paracontact structure if it admits a $(1,1)-$tensor field ϕ, a vector field ξ and a 1-form η satisfying the following conditions [5]:

(i) $\phi^2 X = X - \eta(X)\xi$, for any vector field $X \in \chi(M)$, the set of all differential vector fields on M,

(ii) $\eta(\xi) = 1$, $\eta \circ \phi = 0$, $\phi\xi = 0$.

An almost paracontact structure is called to be normal if and only if the $(1,2)-$type torsion tensor $N_\phi = [\phi,\phi] - 2d\eta \otimes \xi$ vanishes identically, where $[\phi,\phi](X,Y) = \phi^2[X,Y] + [\phi X,\phi Y] - \phi[\phi X,Y] - \phi[X,\phi Y]$. An almost paracontact manifold equipped with a pseudo-Riemannian metric g so that

$$g(\phi X,\phi Y) = -g(X,Y) + \eta(X)\eta(Y), \quad g(X,\xi) = \eta(X) \tag{2.1}$$

for all vector fields $X,Y \in \chi(M)$ is said almost paracontact metric manifold, where signature of g is $(n+1,n)$. An almost paracontact structure is called to be a paracontact structure if $g(X,\phi Y) = d\eta(X,Y)$ with the associated metric g [15]. We now define a $(1,1)$ tensor field h by $h = \frac{1}{2}L_\xi \phi$, where L denotes the Lie derivative. Then h is symmetric and satisfies the conditions

$$h\phi = -\phi h, \qquad h\xi = 0, \qquad Trh = Tr.\phi h = 0. \tag{2.2}$$

If ∇ denotes the Levi-Civita connection of g, then we have the following relation

$$\widetilde{\nabla}_X \xi = -\phi X + \phi h X \tag{2.3}$$

for any $X \in \chi(M)$[15]. For a paracontact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$, if ξ is a killing vector field or equivalently, $h = 0$, then it is called a K-paracontact manifold.

An almost paracontact manifold is said to be para-Sasakian if and only if the following condition holds [15].

$$(\widetilde{\nabla}_X \phi)Y = -g(X,Y)\xi + \eta(Y)X$$

for all $X,Y \in \chi(M)$ [15]. A normal paracontact metric manifold is para-Sasakian and satisfies

$$R(X,Y)\xi = -(\eta(Y)X - \eta(X)Y) \tag{2.4}$$

for all $X,Y \in \chi(M)$, but this is not a sufficient condition for a para-contact manifold to be para-Sasakian. It is clear that every para-Sasakian manifold is K-paracontact. But the converse is not always true[3].

A paracontact manifold M is said to be η-Einstein if its Ricci tensor S of type $(0,2)$ is of the from $S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y)$,where a,b are smooth functions on M. If $b = 0$, then the manifold is also called Einstein and if $a = 0$, then it is called special type of η-Einstein manifolds [14].

A paracontact metric manifold is said to be a $(k,\mu)-$paracontact manifold if the curvature tensor $\widetilde{R}$ satisfies

$$\widetilde{R}(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY] \tag{2.5}$$

for all $X,Y \in \chi(M)$, where k and μ are real constants.

This class is very wide containing the para-Sasakian manifolds as well as the paracontact metric manifolds satisfying $R(X,Y)\xi = 0$ [16].

In particular, if $\mu = 0$, then the paracontact metric $(k,\mu)-$manifold is called paracontact metric $N(k)$-manifold . Thus for a paracontact metric $N(k)$-manifold the curvature tensor satisfies the following relation

$$R(X,Y)\xi = k\eta(Y)X - k\eta(X)Y \tag{2.6}$$

for all $X,Y \in \chi(M)$. Though the geometric behavior of paracontact metric $(k,\mu)-$spaces is different according as $k < -1$, or $k > -1$, but there are some common results for $k < -1$ and $k > -1$[4].

Lemma 2.1. *There does not exist any paracontact $(k,\mu)-$manifold of dimension greater than 3 with $k > -1$ which is Einstein whereas there exits such manifolds for $k < -1$ [4].*

In a paracontact metric $(k,\mu)-$manifold $M^{2n+1}(\phi,\xi,\eta,g)$, $n>1$, the following relation hold :

$$h^2=(k+1)\phi^2,\ \text{for}\ k\neq -1, \tag{2.7}$$

$$(\widetilde{\nabla}_X\phi)Y=-g(X-hX,Y)\xi+\eta(Y)(X-hX), \tag{2.8}$$

$$S(X,Y) = [2(1-n)+n\mu]g(X,Y)+[2(n-1)+\mu]g(hX,Y)+[2(n-1)+n(2k-\mu)]\eta(X)\eta(Y), \tag{2.9}$$

$$S(X,\xi)=2nk\eta(X), \tag{2.10}$$

$$QY = [2(1-n)+n\mu]Y+[2(n-1)+\mu]hY+[2(n-1)+n(2k-\mu)]\eta(Y)\xi, \tag{2.11}$$

$$Q\xi=2nk\xi, \tag{2.12}$$

$$Q\phi-\phi Q=2[2(n-1)+\mu]h\phi \tag{2.13}$$

for any vector fields X,Y on M^{2n+1} , where Q and S denotes the Ricci operator and Ricci tensor of (M^{2n+1},g), respectively[4].

The concept of W_6-curvature tensor was defined by [7]. W_6-curvature tensor, W_7-curvature tensor, W_8-curvature tensor and W_9-curvature tensor, of a $(2n+1)$-dimensional Riemannian manifold are, respectively, defined as

$$W_6(X,Y)Z=R(X,Y)Z-\frac{1}{2n}[S(Y,Z)X-g(X,Y)QZ], \tag{2.14}$$

$$W_7(X,Y)Z=R(X,Y)Z-\frac{1}{2n}[S(Y,Z)QX-g(Y,Z)QX], \tag{2.15}$$

$$W_8(X,Y)Z=R(X,Y)Z-\frac{1}{2n}[S(Y,Z)X-S(X,Y)Z], \tag{2.16}$$

$$W_9(X,Y)Z=R(X,Y)Z+\frac{1}{2n}[S(X,Y)Z-g(Y,Z)QX], \tag{2.17}$$

for all $X,Y,Z\in\chi(M)$ where, $\chi(M)$ is set of all vector spaces [7].

3. Certain Curvature Conditions on (k,μ)-Paracontact metric spaces

We will provide the significant themes of this work in this part.

Let M be $(2n+1)-$dimensional $(k,\mu)-$paracontact metric manifold and we explain W_6 curvature tensor from (2.14), we have

$$W_6(X,Y)\xi=k(g(X,Y)\xi-\eta(X)Y)+\mu(\eta(Y)hX-\eta(X)hY). \tag{3.1}$$

Putting $X=\xi$, in (3.1), we get

$$W_6(\xi,Y)\xi=k(\eta(Y)\xi-Y)-\mu hY. \tag{3.2}$$

In (2.15) choosing $Z=\xi$ and using (2.5), we obtain

$$W_7(X,Y)\xi = k\eta(X)Y + \frac{1}{2n}\eta(Y)QX + \mu(\eta(Y)hX - \eta(X)hY). \tag{3.3}$$

It follows

$$W_7(\xi,Y)\xi = k(\eta(Y)\xi - Y) - \mu hY. \tag{3.4}$$

In the same way, putting $Z=\xi$ in (2.16) and using (2.5), we have

$$W_8(X,Y)\xi = \frac{1}{2n}S(X,Y)\xi - k\eta(X)Y + \mu(\eta(Y)hX - \eta(X)hY). \tag{3.5}$$

In (2.16), choosing $X=\xi$, we get

$$W_8(\xi,Y)\xi = k(\eta(Y)\xi - Y) - \mu hY. \tag{3.6}$$

In (2.17), choosing $Z=\xi$, we obtain

$$W_9(X,Y)\xi \;=\; k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) + \frac{1}{2n}(S(X,Y)\xi - \eta(Y)QX). \tag{3.7}$$

In(3.7) it follows

$$W_9(\xi,Y)\xi = k(\eta(Y)\xi - Y) - \mu hY. \tag{3.8}$$

In (2.5), we arrive

$$R(\xi,Y)Z = k(g(Y,Z)\xi - \eta(Z)Y) + \mu(g(hY,Z)\xi - \eta(Z)hY), \tag{3.9}$$

choosing $Z=\xi$, in (3.9)

$$R(\xi,Y)\xi = k(\eta(Y)\xi - Y) - \mu hY. \tag{3.10}$$

Theorem 3.1. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a (k,μ)-paracontact space. Then M is a W_6 semi-symmetric if and only if M is an Einstein manifold.*

Proof. Suppose that M is a W_6 semi-symmetric. This implies that

$$\begin{aligned}(R(X,Y)W_6)(U,W)Z \;&=\; R(X,Y)W_6(U,W)Z - W_6(R(X,Y)U,W)Z\\ &\quad -W_6(U,R(X,Y)W)Z - W_6(U,W)R(X,Y)Z = 0,\end{aligned} \tag{3.11}$$

for any $X,Y,U,W,Z\in\chi(M)$. Taking $X=Z=\xi$ in (3.11), making use of (3.1) and (3.9), for $A=\frac{1}{2n}$, we have

$$\begin{aligned}(R(\xi,Y)W_6)(U,W)\xi \;&=\; R(\xi,Y)(k(g(Y,W)\xi - \eta(U)W) + \mu(\eta(W)hU\\ &\quad -\eta(U)hW)) - W_6(k(g(Y,U)\xi - \eta(U)Y)\\ &\quad +\mu(g(hY,U)\xi - \eta(U)hY),W)\xi\\ &\quad -W_6(U,k(g(Y,W)\xi - \eta(W)Y)\\ &\quad +\mu(g(hY,W)\xi - \eta(W)hY)\xi\\ &\quad -W_6(U,W)(k(\eta(Y)\xi - Y) - \mu hY) = 0.\end{aligned} \tag{3.12}$$

Taking into account (3.1) and (3.2) in (3.12), we have

$$\begin{aligned}&kW_6(U,W)Y + \mu W_6(U,W)hY + k\mu(\eta(W)g(Y,hU)\xi\\ &-g(Y,W)hU) + \mu^2(1+k)(\eta(W)g(Y,U)\xi\\ &-\eta(U)g(Y,W)\xi) + k\mu(g(hY,U)W - g(hY,W)hU)\\ &+\mu k(g(hY,U)hW - g(hY,W)U) + \mu^2(g(hY,U)hW\\ &-g(hY,W)hU) + k^2(g(Y,W)\eta(U)\xi - g(Y,W)U)\\ &+k\mu(g(Y,U)hW + g(U,W)hY) + k^2(g(Y,U)W\\ &-g(U,W)Y) = 0.\end{aligned} \tag{3.13}$$

Putting (2.10), (2.14), choosing $U=\xi$ and taking inner product with $\xi\in\chi(M)$ in (3.13), we arrive

$$AkS(W,Y)+A\mu S(W,hY)+k^2g(W,Y)+k\mu g(W,hY)=0. \tag{3.14}$$

Using (2.7) and replacing hY of Y in (3.14), we get

$$AkS(W,hY)+A\mu(1+k)S(W,Y)-2nkA(1+k)g(W,hY)+k\mu(1+k)g(W,Y)=0. \tag{3.15}$$

From (3.14) and (3.15), we have

$$S(W,Y)=2nkg(W,Y).$$

So, M is an Einstein manifold. Conversely, let $M^{2n+1}(\varphi,\xi,\eta,g)$ be an Einstein manifold, i.e. $S(W,Y)=2nkg(W,Y)$, then from equations (3.15), (3.14), (3.13), (3.12) and (3.11) we obtain M is a W_6 semi-symmetric. Which verifies our assertion. □

Theorem 3.2. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a (k,μ)-paracontact space. Then M is a W_7 semi-symmetric if and only if M is an $\eta-$Einstein manifold.*

Proof. Assume that M is a W_7 semi-symmetric. This yields to

$$\begin{aligned}(R(X,Y)W_7)(U,W)Z &= R(X,Y)W_7(U,W)Z-W_7(R(X,Y)U,W)Z\\ &\quad -W_7(U,R(X,Y)W)Z-W_7(U,W)R(X,Y)Z=0,\end{aligned} \tag{3.16}$$

for any $X,Y,U,W,Z\in\chi(M)$. Taking $X=Z=\xi$ in (3.16) and using (3.3), (3.9), (3.10), for $A=-\frac{1}{2n}$, we obtain

$$\begin{aligned}(R(\xi,Y)W_7)(U,W)\xi &= R(\xi,Y)(k\eta(U)W-A\eta(W)QU+\mu(\eta(W)hU\\ &\quad -\eta(U)hW))-W_7(k(g(Y,U)\xi-\eta(U)Y)\\ &\quad +\mu(g(hY,U)\xi-\eta(U)hY),W)\xi\\ &\quad -W_7(U,kg(Y,W)\xi-\eta(W)Y)\\ &\quad +\mu(g(hY,W)\xi-\eta(W)hY)\xi\\ &\quad -W_7(U,W)k(\eta(Y)\xi-Y)-\mu hY)=0.\end{aligned} \tag{3.17}$$

Taking into account that (3.4) and (3.9) in (3.17), we get

$$\begin{aligned}&kW_7(U,W)Y+\mu W_7(U,W)hY+k\mu(\eta(U)g(hY,W)\xi\\ &-g(Y,W)hU)+\mu^2(1+k)(\eta(W)g(Y,U)\xi\\ &-\eta(U)g(Y,W)\xi)-Ak(S(Y,U)\eta(W)\xi+\eta(W)\eta(U)QY)\\ &+A\mu(2nk\eta(W)\eta(U)hY-S(hY,U)\eta(W)\xi)\\ &+k^2(\eta(U)g(Y,W)\xi-\eta(W)g(Y,U)\xi)+k\mu(g(Y,U)hW\\ &-g(hY,W)U)+\mu^2(g(hY,U)hW-g(hY,W)hU)\\ &+\mu(kg(hY,U)W-A\eta(U)\eta(W)QhY)+k^2(g(Y,W)\eta(U)\xi\\ &+2nA\eta(U)\eta(W)Y)+k^2(g(Y,U)W-g(Y,W)U)=0.\end{aligned} \tag{3.18}$$

Putting $U=\xi$ and using (3.3) in (3.18), we get

$$AS(Y,W)+\mu S(W,hY)+2kg(Y,W)-2nkAg(Y,W)+\mu g(W,hY)=0. \tag{3.19}$$

Replacing hY of Y in (3.19) and making use of (2.7), we have

$$\begin{aligned}&AS(Y,hW)+\mu(1+k)S(Y,W)-2nk\mu(1+k)\eta(Y)\eta(W)\\ &-2nkAg(Y,hW)+\mu(1+k)g(Y,hW)-\mu(1+k)\eta(Y)\eta(W)=0.\end{aligned} \tag{3.20}$$

From (3.19), (3.20) and by using (2.9), for the sake of brevity, we set

$$\begin{aligned}p_1 &= (2nkA^2-2kA+\mu^2(1+k))[2(n-1)+\mu]+(A\mu+2nkA\mu-2k\mu)[2(1-n)+n\mu],\\ p_2 &= (A^2-\mu^2(1+k))[2(n-1)+\mu]+(2k\mu-2nkA\mu-A\mu),\\ p_3 &= (A\mu+2nkA\mu-2k\mu)[2(n-1)+n(2k-\mu)]-\\ &\quad (\mu^2(1+k)(2n+1))[2(n-1)+\mu]\end{aligned}$$

we conclude

$$p_2 S(Y,W) = p_1 g(Y,W) + p_3 \eta(Y)\eta(W).$$

Thus, M is an $\eta-$Einstein manifold. Conversely, let $M^{2n+1}(\varphi,\xi,\eta,g)$ be an $\eta-$Einstein manifold, i.e. $p_2 S(Y,W) = p_1 g(Y,W) + p_3 \eta(Y)\eta(W)$, then from equations (3.20), (3.19), (3.18), (3.17) and (3.16) we obtain M is a W_7 semi-symmetric. □

Theorem 3.3. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a (k,μ)-paracontact space. Then M is a W_8 semi-symmetric if and only if M is an $\eta-$Einstein manifold..*

Proof. Suppose that M is a W_8 semi-symmetric. This implies that

$$\begin{aligned}(R(X,Y)W_8)(U,W)Z &= R(X,Y)W_8(U,W)Z - W_8(R(X,Y)U,W)Z \\ &\quad -W_8(U,R(X,Y)W)Z - W_8(U,W)R(X,Y)Z = 0,\end{aligned} \tag{3.21}$$

for any $X,Y,U,W,Z \in \chi(M)$. Setting $X = Z = \xi$ in (3.21) and making use of (3.5), (3.9), (3.10), for $A = -\frac{1}{2n}$, we obtain

$$\begin{aligned}(R(\xi,Y)W_8)(U,W)\xi &= R(\xi,Y)(-k\eta(U)W - AS(U,W)\xi + \mu(\eta(W)hU \\ &\quad -\eta(U)hW)) - W_8(k(g(Y,U)\xi - \eta(U)Y) \\ &\quad +\mu(g(hY,U)\xi - \eta(U)hY),W)\xi \\ &\quad -W_8(U,k(g(Y,W)\xi - \eta(W)Y) \\ &\quad +\mu(g(hY,W)\xi - \eta(W)hY))\xi \\ &\quad -W_8(U,W)(k(\eta(Y)\xi - Y) - \mu hY) = 0.\end{aligned} \tag{3.22}$$

Inner product both sides of (3.22) by $Z \in \chi(M)$ and using of (3.5), (3.6) and (3.9), we get

$$\begin{aligned}&kg(W_8(U,W)Y,Z) + \mu g(W_8(U,W)hY,Z) + \mu^2(1+k)(\eta(W)\eta(Z)g(Y,U) \\ &-\eta(U)\eta(Z)g(Y,W)) + Ak(\eta(Y)\eta(Z)S(U,W) - \eta(Z)\eta(W)S(U,Y)) \\ &+A\mu(g(hY,Z)S(U,W) - \eta(W)\eta(Z)S(hY,U)) + Ak(S(U,W)g(Y,Z) \\ &-S(U,W)\eta(Y)\eta(Z)) + k^2(g(Y,U)g(W,Z) + g(Y,W)g(U,Z)) \\ &+\mu^2(g(hY,U)g(hW,Z) - g(hY,W)g(hU,Z)) + k\mu(g(hY,U)g(W,Z) \\ &-g(hY,W)g(U,Z)) - A(\mu S(hY,W)\eta(U)\eta(Z) + kS(Y,W)\eta(U)\eta(Z)) \\ &+k\mu(g(Y,U)g(hW,Z) - g(Y,W)g(hU,Z)) - k(\eta(W)\eta(Z)g(Y,U) \\ &+\eta(U)\eta(Z)g(Y,hW)) = 0.\end{aligned} \tag{3.23}$$

Making use of (2.7), (2.16) and choosing $W = Y = e_i,\ \xi,\ 1 \le i \le n$, for orthonormal basis of $\chi(M)$ in (3.23), we have

$$\begin{aligned}&kS(U,Z) + \mu S(U,hZ) + (kAr + 2nA\mu(1+k)[2(n-1)+\mu] \\ &-2nk^2 + \mu^2(1+k))g(U,Z) + k\mu(1-2n)g(U,hZ) \\ &-(2nk^2A + \mu^2(1+k)(2n+1) + k^2 + Akr \\ &+2nA\mu(1+k)[2(n-1)+\mu] + 2nkA\mu)\eta(U)\eta(Z) = 0.\end{aligned} \tag{3.24}$$

In (3.24), hZ of Z, we arrive

$$\begin{aligned}&kS(U,hZ) + \mu(1+k)S(U,Z) - 2nk\mu(1+k)\eta(U)\eta(Z) \\ &+(kAr + 2nA\mu(1+k)[2(n-1)+\mu] - 2nk^2 \\ &+\mu^2(1+k))g(U,hZ) + k\mu(1-2n)(1+k)g(U,Z) \\ &-k\mu(1-2n)(1+k)\eta(U)\eta(Z) = 0.\end{aligned} \tag{3.25}$$

From (3.24), (3.25) and by using (2.9), for the sake of brevity, we set

$$\begin{aligned}p_1 &= (kAr + 2nA\mu(1+k)[2(n-1)+\mu] - 2nk^2 + \mu^2(1+k)), \\ p_2 &= k\mu(1-2n), \\ p_3 &= -(2nk^2A + \mu^2(1+k)(2n+1) + k^2 + Akr + 2nA\mu(1+k)[2(n-1)+\mu] + 2nkA\mu),\end{aligned}$$

we conclude

$$
\begin{aligned}
q_1 &= (p_2\mu(1+k)-kp_1)[2(n-1)+\mu]+(kp_2-p_1\mu)[2(1-n)+n\mu],\\
q_2 &= (k^2-\mu^2(1+k))[2(n-1)+\mu]+(p_1\mu-kp_2),\\
q_3 &= (kp_2-p_1\mu)[2(n-1)+n(2k-\mu)]-(p_3k+2nk\mu^2(1+k)+p_2\mu(1+k))[2(n-1)+\mu],
\end{aligned}
$$

$$q_2S(U,Z)=q_1g(U,Z)+q_3\eta(U)\eta(Z),$$

So, M is an $\eta-$Einstein manifold. Conversely, let $M^{2n+1}(\varphi,\xi,\eta,g)$ be an $\eta-$Einstein manifold, i.e. $q_2S(U,Z)=q_1g(U,Z)+q_3\eta(U)\eta(Z)$, then from equations (3.25), (3.24), (3.23), (3.22) and (3.21) we get M is a W_8 semi-symmetric. □

Theorem 3.4. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a (k,μ)-paracontact space. Then M is a W_9 semi-symmetric if and only if M is an Einstein manifold.*

Proof. Assume that M is a W_9 semi-symmetric. This means that

$$
\begin{aligned}
(R(X,Y)W_9)(U,W,Z) &= R(X,Y)W_9(U,W)Z-W_9(R(X,Y)U,W)Z\\
&\quad -W_9(U,R(X,Y)W)Z-W_9(U,W)R(X,Y)Z=0,
\end{aligned}
\tag{3.26}
$$

for any $X,Y,U,W,Z\in\chi(M)$. Setting $X=Z=\xi$ in (3.26) and making use of (3.9), (3.7), for $A=\frac{1}{2n}$, we obtain

$$
\begin{aligned}
(R(\xi,Y)W_9)(U,W)\xi &= R(\xi,Y)(k(\eta(W)U-\eta(U)W)+\mu(\eta(W)hU\\
&\quad -\eta(U)hW)+A(S(U,W)\xi-\eta(W)QU))\\
&\quad -W_9(k(g(Y,U)\xi-\eta(U)Y)+\mu(g(hY,U)\xi\\
&\quad -\eta(U)hY,W)\xi-W_9(U,k(g(Y,W)\xi-\eta(W)Y)\\
&\quad +\mu(g(hY,W)\xi-\eta(W)hY))\xi\\
&\quad -W_9(U,W)(k(\eta(Y)\xi-Y)-\mu hY)=0.
\end{aligned}
\tag{3.27}
$$

Using (3.7), (3.8), (3.9) in (3.27), we get

$$
\begin{aligned}
&kW_9(U,W)Y+\mu W_9(U,W)hY+k\mu(\eta(W)g(Y,hU)\xi\\
&-\eta(U)g(Y,hW)\xi)+\mu^2(1+k)(\eta(W)g(Y,U)\xi\\
&-\eta(U)g(Y,W)\xi)+k^2(g(Y,U)W-g(Y,W)U)\\
&+kA(\eta(U)S(Y,W)\xi-\eta(W)\eta(U)QY)\\
&+A\mu(\eta(U)S(hY,W)\xi+2nk\eta(U)\eta(W)hY)\\
&+k\mu(g(Y,U)hW-g(Y,W)hU)+k\mu(g(hY,U)W\\
&-g(hY,W)U)+A\mu(S(U,hY)\eta(W)\xi-\eta(W)\eta(U)QhY)\\
&+\mu^2(g(hY,U)hW+g(hY,W)hU)-A\mu(S(U,W)hY\\
&+S(hY,U)\eta(W)\xi)+Ak(2nk\eta(W)\eta(U)Y-S(U,W)Y)=0.
\end{aligned}
\tag{3.28}
$$

Making use of (2.17), (2.1) and choosing $U=\xi$, in (3.28), we have

$$kS(Y,W)+\mu S(hY,W)-2nk^2g(Y,W)-2nk\mu g(hY,W)=0. \tag{3.29}$$

Replacing hY of Y in (3.29) and taking into account (2.7), we arrive

$$
\begin{aligned}
&kS(Y,hW)+\mu(1+k)S(Y,W)-2nk\mu(1+k)\eta(Y)\eta(W)\\
&-2nk^2g(W,hY)-2nk\mu(1+k)g(Y,W)\\
&+2nk\mu(1+k)\eta(W)\eta(Y)=0.
\end{aligned}
\tag{3.30}
$$

From (3.29), (3.30) and by using (2.7), we have

$$S(Y,W)=2nkg(Y,W).$$

This tell us, M is an Einstein manifold. Conversely, let $M^{2n+1}(\varphi,\xi,\eta,g)$ be an Einstein manifold, i.e. $S(Y,W)=2nkg(Y,W)$, then from equations (3.26), (3.27), (3.28) and (3.30), we obtain M is a W_9 semi-symmetric. Which verifies our assertion. □

Example 3.5. *We consider the 3-dimensional manifold $M = \{(x,y,z) \in \mathbb{R}^3,\ z \neq 0\}$, where (x,y,z) are standart coordinates of $\mathbb{R}^3$. The vector fields*

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = 4z^2\frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}.$$

Let g be the Riemannian metric defined by

$$\begin{aligned} g(e_1,e_2) &= g(e_1,e_3) = g(e_2,e_3) = 0, \\ g(e_1,e_1) &= g(e_2,e_2) = 1, \quad g(e_3,e_3) = -1 \end{aligned}$$

Let η be the 1-form defined by $\eta(X) = g(X,e_1)$ for any $X \in \chi(M)$. Let ϕ be the (1,1) tensor field defined by

$$\phi(e_1) = 0, \qquad \phi(e_3) = -e_2, \qquad \phi(e_2) = -e_3.$$

Let ∇ be the Levi-Civita connection with respect to the metric tensor g. Then we get

$$[e_3,e_1] = 0, \quad [e_1,e_2] = 0, \quad [e_2,e_3] = -8ze_1.$$

Then we have

$$\eta(e_1) = g(e_1,e_1) = 1, \quad \phi^2X = X - \eta(X)e_1, \quad g(\phi X,\phi Y) = -g(X,Y) + \eta(X)\eta(Y),$$

for any $X,Y \in \chi(M)$. Hence, (ϕ,ξ,η,g) defines a paracontact metric structure on M for $e_1 = \xi$.

The Levi-Civita connection ∇ of the metric g is given by the Koszul's formula

$$\begin{aligned} 2g(\nabla_X Y,Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &\quad - g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]). \end{aligned}$$

Using the above formula, we obtain.

$$\begin{aligned} \nabla_{e_1}e_1 &= 0, & \nabla_{e_2}e_1 &= -4ze_3, & \nabla_{e_3}e_1 &= -4ze_2, \\ \nabla_{e_1}e_2 &= -4ze_3, & \nabla_{e_2}e_2 &= 0, & \nabla_{e_3}e_2 &= 4ze_1, \\ \nabla_{e_1}e_3 &= -4ze_2, & \nabla_{e_2}e_3 &= -4ze_1, & \nabla_{e_3}e_3 &= 0. \end{aligned}$$

Comparing the above relations with $\nabla_X e_1 = -\phi X + \phi hX$, we get

$$he_2 = -(4z+1)e_2, \quad he_3 = -(4z+1)e_3, \quad he_1 = 0.$$

Using the formula $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$, we calculate the following:

$$\begin{aligned} R(e_2,e_1)e_1 &= \left[\frac{1}{(4z-1)^2} - 1\right]\{\eta(e_1)e_2 - \eta(e_2)e_1\} + \left[\frac{1}{(4z-1)^3} - \frac{16z^2+1}{4z+1}\right]\{\eta(e_1)he_2 - \eta(e_2)he_1\} \\ &= -16z^2e_2 \end{aligned}$$

$$\begin{aligned} R(e_3,e_1)e_1 &= \left[\frac{1}{(4z-1)^2} - 1\right]\{\eta(e_1)e_3 - \eta(e_3)e_1\} + \left[\frac{1}{(4z-1)^3} - \frac{16z^2+1}{4z+1}\right]\{\eta(e_1)he_3 - \eta(e_3)he_1\} \\ &= -16z^2e_3 \end{aligned}$$

$$\begin{aligned} R(e_2,e_3)e_1 &= \left[\frac{1}{(4z-1)^2} - 1\right]\{\eta(e_3)e_2 - \eta(e_2)e_3\} + \left[\frac{1}{(4z-1)^3} - \frac{16z^2+1}{4z+1}\right]\{\eta(e_3)he_2 - \eta(e_2)he_3\} \\ &= 0. \end{aligned}$$

By the above expressions of the curvature tensor and using (2.9), we conclude that M is a generalized $(k,\mu)-$paracontact metric manifold with $k = \left[\frac{1}{(4z-1)^2} - 1\right]$ and $\mu = \left[\frac{1}{(4z-1)^3} - \frac{16z^2+1}{4z+1}\right]$.

4. Conclusion

The aim of this paper is to classify (k,μ)-paracontact metric spaces satisfying certain curvature conditions. We present the curvature tensors of (k,μ)-Paracontact manifold satisfying the conditions $R \cdot W_6 = 0$, $R \cdot W_7 = 0$, $R \cdot W_8 = 0$ and $R \cdot W_9 = 0$. According these cases, (k,μ)-Paracontact manifolds have been characterized. Also, several results are obtained.

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